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Lambert W Function in Hydraulic Problems*

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Darcy's flow friction factor is expressed in implicit form in some of the relations such as Colebrook's and have to be solved by iteration procedure because the unknown friction factor appears on both sides of the equation. Lambert W function is implicitly elementary but is not, itself, an elementary function. Implicit form of the Lambert W function allows us to transform other implicit functions in explicit form without any kind of approximations or simplifications involved. But unfortunately, the Lambert W function itself cannot be solved easily without approximation. Two original transformations in explicit form of Colebrook's relation using Lambert W function will also be shown. Here will be shown efficient procedure for approximate solutions of the transformed relations.

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1. Introduction

Flow friction factor can be expressed in implicit form in some of the relations such as Colebrook's and hence has to be solved using iteration procedure because the unknown friction factor appears on both sides of such equation [1,2]. Many, more or less accurate, explicit approximations of the implicit Colebrook's equations for the flow friction were developed [3]. These equations are valuable for hydraulically 'smooth' pipe region of partial turbulence and even for fully turbulent regime [4]. The Lambert W function proposed by J.H. Lambert [5] in 1758 and refined by L. Euler can be used for explicit and exact transformation of the Colebrook's equation. Lambert W function is implicitly elementary but is not, itself, an elementary function [6]. Note that name "W" for Lambert

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function is not as old as the related function. The modern history of Lambert W began in the 1980s, when a version of the function was built into the Maple computer-algebra system and given the name W [7]. Implicit form of the Lambert W function allows us to transform other implicit functions in explicit form without any kind of approximations or simplifications involved. But unfortunately, the Lambert W function itself cannot be solved without approximation [8,9]. How well Colebrook's equation, itself, fits the experimental data is beyond the scope of this article. Some original transformations in explicit form of Colebrook's relation using Lambert W function will also be shown. These reformulated Colebrook's equations are now explicit in friction factor but with Lambert W function involved. Hence, these are not approximations, but the procedure for exact mathematical solution for the solution of the Lambert W function has not been developed yet and even more, probably will not be.

2. On the Colebrook's equation for flow friction factor

For the partial turbulence regime where the friction coefficient diverges to different straight lines for different constant relative roughness, Colebrook proposed adding Prandtl's (2.1) and von Karman's (2.2) equations for smooth, smoothing the contact between these two lines. How well the Colebrook equation's smoothing contact among the von Karman's and Prandtl's (NPK) relation can be best seen in graphical interpretation (Figure 1).

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg\left(\frac{2.51}{Re \cdot \sqrt{\lambda}}\right) = 2 \cdot \lg\left(Re \cdot \sqrt{\lambda}\right) - 0.8 \tag{2.1}$$

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg\left(\frac{\epsilon}{3.71 \cdot D}\right) = 1.14 - 2 \cdot \lg\left(\frac{\epsilon}{D}\right) = 1.74 - 2 \cdot \lg\left(\frac{2 \cdot \epsilon}{D}\right) \tag{2.2}$$

From strictly mathematical point of view, what Colebrook had done [2] is incorrect, i.e. log(A+B) is not log(A)+log(B), but physically this relation gives good results (smoothing the contact between two lines). So, finally Colebrook's relation can be noted as (2.3):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg\left(\frac{2.51}{Re \cdot \sqrt{\lambda}} + \frac{\epsilon}{3.71 \cdot D}\right) \tag{2.3}$$

Problem can be treated as inverse; according to logarithm's rules it is equally incorrect to split the Colebrook's relation into two pieces. Colebrook equation belong to so called log-law relationship. Other approach, not shown in this paper is so called power-law [10]. For example, Blasius form or power-law relationships is more suitable for flow regimes typical for plastic (polyethylene,

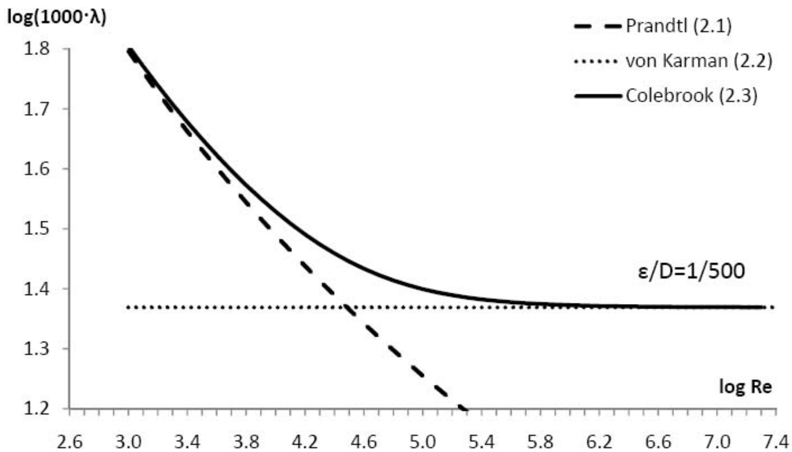

Figure 1: Implicit Colebrook relation and related approximations smoothed contact among relations for smooth and rough relations

PVC) pipes, while the Colebrook's relation (logarithmic law) with related approximations is better for flow regimes in steel pipes [11].

3. On the Lambert W-function

Lambert W-function can be noted as (3.1):

$$W(x) \cdot e^{W(x)} = x \tag{3.1}$$

For real values of the argument, x, the W-function has two branches, W_0 (the principal branch) and W_{-1} (the negative branch). W_0 is referred to as principal branch of the Lambert W function. Upper branch (noted as + in Figure 2) is valid for solution of our problem.

Formal solution of the Lambert W function can be defined as (3.2):

$$W \approx \ln \frac{x}{ln\left(\frac{x}{\ln\left(\frac{x}{\ln\left(\ddots\right)}\right)}\right)} \tag{3.2}$$

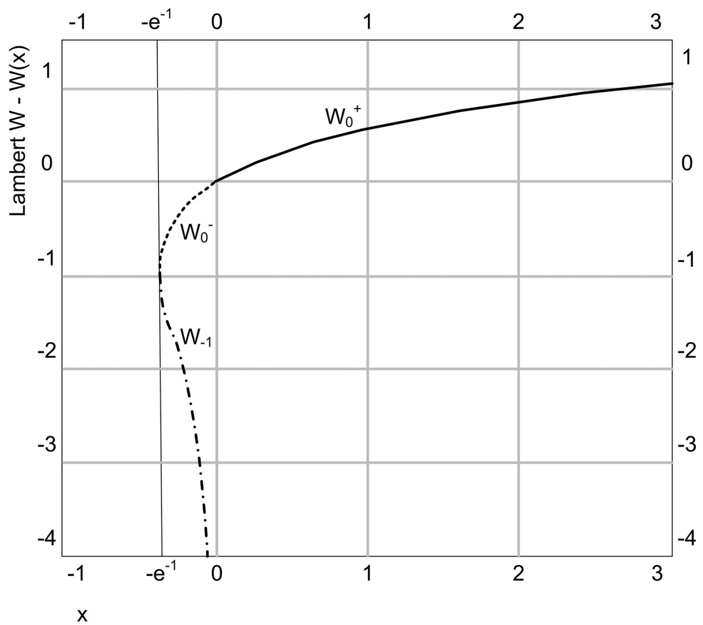

Figure 2: Real branches of the Lambert W function

But after Boyd [8], it is more convenient to define a new function and a new parameter such that both the domain and range are the nonnegative real axis as (3.3):

$$\omega = W + 1 \Leftrightarrow W = \omega - 1 \tag{3.3}$$

With also 'shifted' argument of the function y (3.4) [8]:

$$y = 1 + x \cdot e^1 \approx 1 + 2.71 \cdot x \Leftrightarrow x = \frac{y-1}{e^1} \approx \frac{y-1}{2.71} \approx 0.367 \cdot (y-1) \tag{3.4}$$

Then, the Lambert-W function can be transformed in shifted function (3.5) [8]:

$$(\omega - 1) \cdot e^{\omega} \approx y - 1 \tag{3.5}$$

Approximate solution of shifted function can be expressed as (3.6) [8]:

$$\omega_0 \approx (\ln(y+10) - ln(ln(y+10))) \cdot \tanh\left(\frac{\sqrt{2 \cdot y}}{ln(10) - ln(ln(10))}\right) \tag{3.6}$$

Improved solution after Boyd is also available (3.7) [8]:

$$\overline{\omega_0} \approx \omega_0 \cdot (1 + \frac{\left(ln(y) - \frac{7}{5}\right) \cdot e^{-\frac{3}{40}\cdot\left(\ln(y)-\frac{7}{5}\right)^2}}{10}) \tag{3.7}$$

This solution can be used as the first guess for Newton's iteration scheme to reduce the relative error (3.8) [8]:

$$\omega_{i+1} \approx \omega_i - \frac{(\omega_i - 1) - e^{(-\omega_i)\cdot(y-1)}}{\omega_i} \tag{3.8}$$

Barry et al [9] give different way for calculation of approximate solution of the Lambert W function (3.9):

$$W_0^+(x) \approx \ln\left(\frac{6 \cdot x}{5 \cdot ln\left(\frac{5}{12} \cdot \left(\frac{x}{\ln\left(1+\frac{12\cdot x}{5}\right)}\right)\right)}\right) \tag{3.9}$$

This solution (3.9) is used in this paper to develop approximation of Colebrook's equation based on the Lambert-W function.

4. Transformations of Colebrook's equation based on Lambert-W function

Transformation of Colebrook's equation based on Lambert-W function already exist in scientific literature (4.1) [12]:

$$\lambda = \left(O_1 \cdot W\left(\frac{e^{\frac{O_1}{O_2\cdot O_3}}}{O_2 \cdot O_3}\right) - \frac{O_1}{O_2}\right)^{-2} \tag{4.1}$$

where:

$$O_1 = \frac{\epsilon}{3.71 \cdot D}$$

$$O_2 = \frac{2.51}{Re}$$

$$O_3 = \frac{2}{ln(10)} \approx 0.868589$$

Note that previous equation (4.1) cannot be used for high values of Reynolds number and relative roughness, because numbers become too big even for capabilities of today available computers [13,14]. Papers of Goudar and

Sonnad [13-17], Nandakumar [18] and Clamond [19] deal with the Lambert W function in hydraulic in a very successful way.

According to the original idea of Colebrook and White to unite the Prandtl's (NPK) and von Karman's equations in a one coherent model, their equation can be reformulated using the Lambert W-function in a manner presented as (4.2):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg \left(10^{\frac{-1}{\ln(10)} \cdot W\left(\frac{Re \cdot \ln(10)}{5.02}\right)} + \frac{\epsilon}{3.71 \cdot D} \right) \tag{4.2}$$

Or equivalent equation can be given in similar form (4.3):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg \left(\frac{5.02 \cdot W\left(\frac{Re \cdot \ln(10)}{5.02}\right)}{Re \cdot \ln(10)} + \frac{\epsilon}{3.71 \cdot D} \right) \tag{4.3}$$

Note that previous two equations are not approximations of Colebrook's relation (2.3). They are exact mathematical transformed, but unfortunately they contain Lambert W-function which cannot be solved without approximations.

5. Approximation of Colebrook's equation based on Lambert-W function

Using equation (4.2) and approximate solution after Boyd (3.9), approximate formula for calculation of Darcy's friction factor after Colebrook can be written as (5.1):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \lg \left(10^{-0.4343 \cdot S} + \frac{\epsilon}{3.71 \cdot D} \right) \approx 2 \cdot \lg \left(\frac{2.18 \cdot S}{Re} + \frac{\epsilon}{3.71 \cdot D} \right) \tag{5.1}$$

where:

$$S = \ln \frac{Re}{1.816 \cdot \ln\left(\frac{1.1 \cdot Re}{\ln(1+1.1 \cdot Re)}\right)}$$

Comparison of accuracy of different approximations of implicit Colebrook's equation (2.3) are available in literature [3].

6. Conclusion

It may be difficult for many to recall the time as recently as the 1970's when there were no personal computers or even calculators that could do much more than add or subtract. In that environment an implicit relationship such as Colebrook's (2.3), which was well-known then, was impractical and some simplification was essential. All available approximations of the Colebrook's equation are very accurate, but most of these explicit relations created to solve the implicit Colebrook's form have been made obsolete by advance in computing technology. The average error of almost all explicit approximations of the Colebrook's relation is less than 1%. Solution of this implicit equation today can be obtained easily and quickly by Microsoft Office Excel 2007 very accurately.

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