Default Dependency and Correlation Study

Tim Xiao

ABSTRACT

This article studies the impact of default dependence and correlation on derivative pricing. We find that a fixed-rate payer with lower credit quality (higher credit risk) pays a higher fixed rate. We also find a credit spread of about 100 basis points translates into a swap spread of about 1.3 basis points. Our study shows that the credit impact on swap rates is approximately linear within the range of normally encountered credit quality. Finally, the model shows that a fully collateralized swap is risk-free, whereas a fully collateralized CDS is not equivalent to a risk-free one.

Key Words: asset pricing; credit risk modeling; unilateral, bilateral, multilateral credit risk; collateralization; comvariance; comrelation; correlation.

1 Introduction

A broad range of financial instruments bear credit risk. Credit risk may be unilateral, bilateral, or multilateral. Some instruments such as, loans, bonds, etc, by nature contain only unilateral credit risk because only the default risk of one party appears to be relevant, whereas some other instruments, such as, over the counter (OTC) derivatives, securities financing transactions (SFT), and credit derivatives, bear bilateral or multilateral credit risk because two or more parties are susceptible to default risk. This paper mainly discusses bilateral and multilateral credit risk modeling, with a particular focus on default dependency, as correlated credit risk is one of the greatest threats to global financial markets.

Central to the reduced-form models is the assumption that multiple defaults are independent conditional on the state of the economy. In reality, however, the default of one party might affect the default probabilities of other parties. Collin-Dufresne et al. (2003) and Zhang and Jorion (2007) find that a major credit event at one firm is associated with significant increases in the credit spreads of other firms. Giesecke (2004), Das et al. (2006), and Lando and Nielsen (2010) find that a defaulting firm can weaken the firms in its network of business links. These findings have important implications for the management of credit risk portfolios, where default relationships need to be explicitly modeled.

The endogenous approaches include the contagion (or infectious) models and frailty models. The frailty models (see Duffie et al. (2009), Koopman et al. (2011), etc) describe default clustering based on some unobservable explanatory variables. In variations of contagion or infectious type models (see Davis and Lo (2001), Jarrow and Yu (2001), etc.), the assumption of conditional independence is relaxed and default intensities are made to depend on default events of other entities. Contagion and frailty models fill an important gap but at the cost of analytic tractability. They can be especially difficult to implement for large portfolios.

Collateralization is one of the most important and widespread credit risk mitigation techniques used in derivatives transactions. According the ISDA (2012), 71% of all OTC derivatives transactions are subject to collateral agreements. The use of collateral in the financial markets has increased sharply over the past

decade, yet the research on collateralized valuation is relatively sparse. Previous studies seem to turn away from direct and detailed modeling of collateralization (see Fuijii and Takahashi (2012)). For example, Johannes and Sundareshan (2007), and Fuijii and Takahashi (2012) characterize collateralization via a cost-of-collateral instantaneous rate (or stochastic dividend or convenience yield). Piterbarg (2010) regards collateral as a regular asset in a portfolio and uses the replication approach to price collateralized contracts.

IRSs and CDSs are two of the largest segments of the OTC derivatives market, collectively accounting for around two-thirds of both the notional amount and market value of all outstanding derivatives. Given this framework, we are able to analyze the value of IRSs with bilateral credit risk and look at how swap rates are affected by correlated default risk. Our study shows that counterparty default correlations have a relatively small impact on swap rates. Furthermore, we find that the value of a fully collateralized IRS is equal to the risk-free value. This conclusion is consistent with the current market practice in which market participants commonly assume fully collateralized swaps are risk-free.

The standard CDS pricing model in the market assumes that there is no counterparty risk. Although this oversimplified model may be accepted in normal market conditions, its reliability in times of distress has recently been questioned. In fact, counterparty risk has become one of the most dangerous threats to the CDS market. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account (see ECB (2009)).

There is a significant increase in the use of collateral for CDS after the recent financial crises. Many people believe that, if a CDS is fully collateralized, there is no risk of failure to pay. Collateral posting regimes are originally designed and utilized for bilateral risk products, e.g., IRS, but there are many reasons to be concerned about the success of collateral posting in offsetting the risk of CDS contracts. First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movement of *IRS* contracts is far smoother and less volatile than *CDS*. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization cannot eliminate counterparty risk completely for a CDS.

The rest of this paper is organized as follows: Pricing bilateral defaultable instruments is elaborated on in Section 2; valuing multilateral defaultable instruments is discussed in Section 3; the conclusions are presented in Section 4. All proofs and some detailed derivations are contained in the appendices.

2 Pricing Financial Instruments Subject to Bilateral Credit Risk

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

Dependence between the default times is only introduced by the dependence of the intensity $h_i(t)$ on a common process Z_t . Consequently, conditional on the path of Z_t , defaults are independent, which is the reason why this setup is also often called the conditional independence setup.

Three different recovery models exist in the literature. The default payoff is either i) a fraction of par (Madan and Unal (1998)), ii) a fraction of an equivalent default-free bond (Jarrow and Turnbull (1995)), or iii) a fraction of market value (Duffie and Singleton (1999)). The whole course of the recovery proceedings under the Bankruptcy and Insolvency act is a complex process that typically involves extensive negotiation and litigation. No model can fully capture all aspects of this process so, in practice, all models involve trade-offs between different perspectives and views. In general, the choice for a certain recovery assumption is based on the legal structure of an instrument to be priced. For example, the recovery of market value (RMV) assumption is well matched to the legal structure of an IRS contract where, upon default close-out, valuation will in many circumstances reflect the replacement cost of the transaction, whereas the best default recovery assumption for a CDS is that the claim made in the event of the reference default equals a fraction of the face value of the underlying bond.

Two counterparties are denoted as A and B . The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for party j ($j=A, B$) follows a Bernoulli

distribution, which takes value 1 with default probability q_j , and value 0 with survival probability p_j , i.e., $P\{Y_j = 0\} = p_j$ and $P\{Y_j = 1\} = q_j$. The marginal default distributions can be determined by the reduced-form models. The joint distributions of a multivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \quad (3a)$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \quad (3b)$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \quad (3c)$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \quad (3d)$$

where $E(Y_j) = q_j$, $\sigma_j^2 = p_j q_j$, and $\sigma_{AB} := E[(Y_A - q_A)(Y_B - q_B)] = \rho_{AB} \sigma_A \sigma_B = \rho_{AB} \sqrt{q_A p_A q_B p_B}$ where ρ_{AB} denotes the default correlation coefficient, and σ_{AB} denotes the default covariance.

1.1 Risky valuation without collateralization

We divide the time period (t, T) into n very small time intervals (Δt) and use the approximation $\exp(y) \approx 1 + y$ provided that y is very small. The survival and the default probabilities for the period $(t, t + \Delta t)$ are given by

$$\hat{p}(t) := p(t, t + \Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \quad (4a)$$

$$\hat{q}(t) := q(t, t + \Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \quad (4b)$$

Suppose that the value of the instrument at time $t + \Delta t$ is $V(t + \Delta t)$ that can be an asset or a liability.

There are a total of four ($2^2 = 4$) possible states shown in Table 1.

The risky value of the instrument at time t is the discounted expectation of all the payoffs and is given by

$$\begin{aligned}
V(t) &= E\left\{\exp(-r(t)\Delta t)\left[1_{V(t+\Delta t)\geq 0}\langle p_{00}(t)+\varphi_B(t)p_{01}(t)+\bar{\varphi}_B(t)p_{10}(t)+\varphi_{AB}(t)p_{11}(t)\rangle V(t+\Delta t)|\mathcal{F}_t\right.\right. \\
&\quad \left.\left.+1_{V(t+\Delta t)<0}\langle p_{00}(t)+\bar{\varphi}_A(t)p_{01}(t)+\varphi_A(t)p_{10}(t)+\varphi_{AB}(t)p_{11}(t)\rangle V(t+\Delta t)|\mathcal{F}_t\right]\right\} \\
&\approx E\left\{\exp\left[-\left(r(t)+1_{V(t+\Delta t)\geq 0}l_B(t)+1_{V(t+\Delta t)<0}l_A(t)\right)\Delta t\right]V(t+\Delta t)|\mathcal{F}_t\right\}=E\left\{\exp(-g(t)\Delta t)V(t+\Delta t)|\mathcal{F}_t\right\}
\end{aligned} \tag{5a}$$

where

$$g(t)=r(t)+1_{V(t+\Delta t)\geq 0}l_B(t)+1_{V(t+\Delta t)<0}l_A(t) \tag{5b}$$

$$l_B(t)=(1-\varphi_B(t))h_B(t)+(1-\bar{\varphi}_B(t))h_A(t)-(1-\varphi_B(t)-\bar{\varphi}_B(t)+\varphi_{AB}(t))\rho_{AB}(t)\sqrt{h_A(t)h_B(t)} \tag{5c}$$

$$l_A(t)=(1-\varphi_A(t))h_A(t)+(1-\bar{\varphi}_A(t))h_B(t)-(1-\varphi_A(t)-\bar{\varphi}_A(t)+\varphi_{AB}(t))\rho_{AB}(t)\sqrt{h_A(t)h_B(t)} \tag{5d}$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise, $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$, $r(t)$ is the risk-free short rate, and φ_i is the recovery rate.

The pricing equation above keeps terms of order Δt . All higher order terms of Δt are omitted.

Similarly, we have

$$V(t+\Delta t)=E\left\{\exp(-g(t+\Delta t)\Delta t)V(t+2\Delta t)|\mathcal{F}_{t+\Delta t}\right\} \tag{6}$$

Note that $\exp(-g(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t+\Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned}
V(t) &= E\left\{\exp(-g(t)\Delta t)V(t+\Delta t)|\mathcal{F}_t\right\}=E\left\{\exp(-g(t)\Delta t)E\left[\exp(-g(t+\Delta t)\Delta t)V(t+2\Delta t)|\mathcal{F}_{t+\Delta t}\right]|\mathcal{F}_t\right\} \\
&= E\left\{\exp\left(-\sum_{i=0}^1 g(t+i\Delta t)\Delta t\right)V(t+2\Delta t)|\mathcal{F}_t\right\}
\end{aligned} \tag{7}$$

By recursively deriving from t forward over T where $V(T)=X_T$ and taking the limit as Δt approaches zero, we obtain

$$V(t)=E\left\{G(t,T)X_T|\mathcal{F}_t\right\}=E\left\{\exp\left[-\int_t^T g(u)du\right]X_T|\mathcal{F}_t\right\} \tag{8}$$

We may think of $G(t,T)$ as the bilateral risk-adjusted discount factor and $g(u)$ as the bilateral risk-adjusted short rate. Equation (8) has a general form that applies in a particular situation where we assume that parties A and B have independent default risks, i.e. $\rho_{AB}=0$ and $\varphi_{AB}=0$. Thus, we have:

$$V(t) = E\{\bar{G}(t, T)X_T | \mathcal{F}_t\} = E\left\{\exp\left[-\int_t^T \bar{g}(u)du\right]X_T | \mathcal{F}_t\right\} \quad (9a)$$

where

$$\bar{g}(u) = r(u) + 1_{V(u) \geq 0} \bar{l}_B(u) + 1_{V(u) < 0} \bar{l}_A(u) \quad (9b)$$

$$\bar{l}_B(u) = (1 - \varphi_B(u))h_B(u) + (1 - \bar{\varphi}_B(u))h_A(u) \quad (9c)$$

$$\bar{l}_A(u) = (1 - \varphi_A(u))h_A(u) + (1 - \bar{\varphi}_A(u))h_B(u) \quad (9d)$$

Equation (9) is the same as equation (2.5') in Duffie and Huang (1996).

In theory, a default may happen at any time, i.e., a risky contract is continuously defaultable. This Continuous Time Risky Valuation Model is accurate but sometimes complex and expensive. For simplicity, people sometimes prefer the Discrete Time Risky Valuation Model that assumes that a default may only happen at some discrete times. A natural selection is to assume that a default may occur only on the payment dates.

If we assume that a default may occur only on the payment date, the risky value of the instrument in a discrete-time setting is given by

$$\begin{aligned} V(t) &= E\left\{D(t, T)\left[1_{X_T \geq 0} \langle p_{00}(t, T) + \varphi_B(T)p_{01}(t, T) + \bar{\varphi}_B(T)p_{10}(t, T) + \varphi_{AB}(T)p_{11}(t, T) \rangle X_T | \mathcal{F}_t \right. \right. \\ &\quad \left. \left. + 1_{X_T < 0} \langle p_{00}(t, T) + \bar{\varphi}_A(T)p_{01}(t, T) + \varphi_A(T)p_{10}(t, T) + \varphi_{AB}(T)p_{11}(t, T) \rangle X_T | \mathcal{F}_t \right]\right\} \\ &= E\left[D(t, T)(1_{X_T \geq 0} k_B(t, T) + 1_{X_T < 0} k_A(t, T))X_T | \mathcal{F}_t\right] = E(K(t, T)X_T | \mathcal{F}_t) \end{aligned} \quad (10a)$$

where

$$\begin{aligned} k_B(t, T) &= p_B(t, T)p_A(t, T) + \varphi_B(T)q_B(t, T)p_A(t, T) + \bar{\varphi}_B(T)p_B(t, T)q_A(t, T) \\ &\quad + \varphi_{AB}(T)q_B(t, T)q_A(t, T) + \sigma_{AB}(t, T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T)) \end{aligned} \quad (10b)$$

$$\begin{aligned} k_A(t, T) &= p_B(t, T)p_A(t, T) + \varphi_A(T)q_A(t, T)p_B(t, T) + \bar{\varphi}_A(T)p_A(t, T)q_B(t, T) \\ &\quad + \varphi_{AB}(T)q_B(t, T)q_A(t, T) + \sigma_{AB}(t, T)(1 - \varphi_A(T) - \bar{\varphi}_A(T) + \varphi_{AB}(T)) \end{aligned} \quad (10c)$$

where $D(t, \tau)$ denotes the stochastic risk-free discount factor at t for the maturity T given by

$$D(t, T) = \exp\left[-\int_t^T r(u)du\right] \quad (10d)$$

We may think of $K(t, T)$ as the risk-adjusted discount factor, and $k_A(t, T)$ and $k_B(t, T)$ as the adjustment factors. Equation (10) tells us that the bilateral risky price of a single-payment instrument can be expressed as the present value of the payoff discounted by a risk-adjusted discount factor that has a switching-type dependence on the sign of the payoff.

Proposition 1: *The risky value of the multiple-payment instrument is given by*

$$V(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] \quad (11a)$$

where $t = T_0$ and

$$K(T_j, T_{j+1}) = D(T_j, T_{j+1}) \left(1_{(X_{j+1} + V(T_{j+1})) \geq 0} k_B(T_j, T_{j+1}) + 1_{(X_{j+1} + V(T_{j+1})) < 0} k_A(T_j, T_{j+1}) \right) \quad (11b)$$

where $k_A(T_j, T_{j+1})$ and $k_B(T_j, T_{j+1})$ are defined in Equation (10).

Proof: See the Appendix.

From Proposition 1, we can see that the intermediate values are vital to determine the final price. For a payment interval, the current risky value has a dependence on the future risky value. Only on the final payment date T_m , the value of the instrument and the maximum amount of information needed to determine the risk-adjusted discount factor are revealed.

1.2 Risky valuation with collateralization

Collateralization is the most important and widely used technique in practice to mitigate credit risk. The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount (MTA), etc.

In a typical collateral procedure, a financial instrument is periodically marked-to-market and the collateral is adjusted to reflect changes in value. The collateral is called as soon as the mark-to-market (MTM) value rises above the given collateral threshold, or more precisely, above the threshold amount plus the minimum transfer amount. Thus, the collateral amount posted at time t is given by

$$C(t) = \begin{cases} V(t) - H(t) & \text{if } V(t) > H(t) \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

where $H(t)$ is the collateral threshold. In particular, $H(t) = 0$ corresponds to full-collateralization¹; $H > 0$ represents partial/under-collateralization; and $H < 0$ is associated with over-collateralization. Full collateralization becomes increasingly popular at the transaction level. In this paper, we focus on full collateralization only, i.e., $C(t) = V(t)$.

For a discrete one-period (t, u) economy, the posted collateral at time t is $C(t) = V(t)$. At time u , there are several possible states: i) Both A and B survive. The instrument value is equal to the market value $V(u)$; ii) Either or both counterparties A and B default. The instrument value is the future value of the collateral, i.e., $C(u) = V(t) / D(t, u)$ where we consider the time value of money only. Since the majority of the collateral is cash according to ISDA (2012), it is reasonable to consider the time value of money only for collateral assets. The large use of cash means that collateral is both liquid and not subject to large fluctuations in value. The value of the collateralized instrument at time t is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E \left[D(t, u) (p_{00}(t, u)V(u) + p_{01}(t, u)C(u) + p_{10}(t, u)C(u) + p_{11}(t, u)C(u)) \middle| \mathcal{F}_t \right] \\ &= E \left[D(t, u) p_{00}(t, u) V(u) \middle| \mathcal{F}_t \right] + [1 - E(p_{00}(t, u) \middle| \mathcal{F}_t)] V(t) \end{aligned} \quad (13a)$$

or

$$V(t) = E \left[D(t, s) p_{00}(t, s) V(s) \middle| \mathcal{F}_t \right] / E(p_{00}(t, s) \middle| \mathcal{F}_t) \quad (13b)$$

If we assume that default probabilities are uncorrelated with interest rates and payoffs², we have

$$V(t) = E \left[D(t, s) V(s) \middle| \mathcal{F}_t \right] \quad (14)$$

¹ There are three types of collateralization: Full-collateralization is a process where the posting of collateral is equal to the current MTM value. Partial/under-collateralization is a process where the posting of collateral is less than the current MTM value. Over-collateralization is a process where the posting of collateral is greater than the current MTM value.

² Moody's Investor's Service (2000) presents statistics that suggest that the correlations between interest rates, default probabilities, and recovery rates are very small and provides a reasonable comfort level for the uncorrelated assumption.

Equation (14) is the formula for the risk-free valuation. Thus, we have the following proposition.

Proposition 2: *If a bilateral risky instrument is fully collateralized, the risky value of the instrument is equal to the risk-free value, as shown in equation (14).*

Since an *IRS* is a typical bilateral risky contract, Proposition 2 squares with the results of Johannes and Sundaresan (2007), and is also consistent with the current market practice in which market participants commonly assume fully collateralized swaps are risk-free and it is common to build models of swap rates assuming that swaps are free of counterparty risk.

1.3 Numerical results

To study the impact of bilateral credit risk, we have selected a new 10-year fixed-for-floating *IRS* with a quarterly payment frequency. Two counterparties are denoted as *A* and *B*. Counterparty *A* pays a fixed rate and counterparty *B* pays a floating rate. All calculations are from the perspective of party *A*.

Since the payoff of an *IRS* is determined by interest rates, we need to model the evolution of the floating rates. Interest rate models are based on evolving either short rates, instantaneous forward rates, or market forward rates (e.g., the LIBOR Market Model (LMM)). Since both short rates and instantaneous forward rates are not directly observable in the market, the models based on these rates have difficulties in expressing market views and quotes in term of model parameters, and lack agreement with market valuation formulas for basic derivatives. On the other hand, the object modeled under the LMM is market-observable. It is also consistent with the market standard approach for pricing caps/floors using Black's formula. They are generally considered to have more desirable theoretical calibration properties than short rate or instantaneous forward rate models. Therefore, we choose the LMM lattice proposed by Xiao (2011) for pricing the defaultable *IRS*. We also implement the Hull-White trinomial tree to verify the results and ensure robustness of the valuation. This paper, however, only reports the results produced by the LMM lattice.

From Table 3, we derive the following conclusions: First, a fixed-rate payer with lower credit quality (higher credit risk) pays a higher fixed rate. Second, a credit spread of about 100 basis points translates into a swap spread of about 1.3 basis points. Finally, the credit impact on swap rates is

approximately linear within the range of normally encountered credit quality. This confirms the findings of Duffie and Huang (1996). Intuitively, a risk-free floating-rate payer demands a higher fixed rate if the fixed-rate payer has a lower credit score.

Assume $\varphi_{AB} = 0.5$. The effect of the default correlation ρ_{AB} on swap rate is shown in Figure 1. We can draw the following conclusions from the results: First, the counterparty default correlation and the swap rates have a negative relation, i.e., a negative sensitivity of swap rates to changes in counterparty default correlation is obtained. Second, the graph suggests an almost linear relationship between the swap rates and the default correlation. Finally, the impact of the default correlation is modest (e.g., in the range of $[-2, 2]$ basis points).

3 Conclusion

This article presents a new valuation framework for pricing financial instruments subject to credit risk. In particular, we focus on modeling default relationships. Some well-known risky valuation models in the market can be viewed as special cases of this framework, when the default dependencies are ignored.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

The model shows that a fully collateralized swap is risk-free, while a fully collateralized CDS is not equivalent to a risk-free one. Therefore, we conclude that collateralization designed to mitigate counterparty risk works well for financial instruments subject to bilateral credit risk, but fails for ones subject to multilateral credit risk.

Appendix

Proof of Proposition 1. Let $t = T_0$. On the first cash flow payment date T_1 , let $V(T_1)$ denote the market value of the instrument excluding the current cash flow X_1 . According to Equation (10), we have

$$V(t) = E[K(T_0, T_1)(X_1 + V(T_1)) | \mathcal{F}_t] \quad (\text{A1})$$

Similarly, we have

$$V(T_1) = E[K(T_1, T_2)(X_2 + V(T_2)) | \mathcal{F}_{T_1}] \quad (\text{A2})$$

Note that $K(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E[K(T_0, T_1)(X_1 + V(T_1)) | \mathcal{F}_t] = E[K(T_0, T_1)X_1 | \mathcal{F}_t] \\ &\quad + E\left\{K(T_0, T_1)\left[E(K(T_1, T_2)X_2 | \mathcal{F}_{T_1}) + E(K(T_1, T_2)V(T_2) | \mathcal{F}_{T_1})\right] | \mathcal{F}_t\right\} \\ &= E[K(T_0, T_1)X_1 | \mathcal{F}_t] + E\left[\left(\prod_{j=0}^1 K(T_j, T_{j+1})\right)X_2 | \mathcal{F}_t\right] + E\left[\left(\prod_{j=0}^1 K(T_j, T_{j+1})\right)V(T_2) | \mathcal{F}_t\right] \end{aligned} \quad (\text{A3})$$

By recursively deriving from T_2 forward over T_m , where $V(T_m) = X_m$, we have

$$V(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1})\right)X_i | \mathcal{F}_t\right] \quad (\text{A4})$$

References

Arora, Navneet, Priyank Gandhi, and Francis A. Longstaff (2009), “Counterparty credit risk and the credit default swap market,” Working paper, UCLA.

Collin-Dufresne, P., R. Goldstein, and J. Helwege (2003), “Is credit event risk priced? Modeling contagion via the updating of beliefs,” Working paper, Haas School, University of California, Berkeley.

Das, S., L. Freed, G. Geng, N. Kapadia (2006), “Correlated default risk,” Journal of Fixed Income, 16: 7-32.

Deardorff, Alan V. (1982): “The general validity of the Heckscher-Ohlin Theorem,” American Economic Review, 72 (4): 683-694.

Duffie, D., and M. Huang (1996), “Swap rates and credit quality,” Journal of Finance, 51: 921-949.

ECB (2009), “Credit default swaps and counterparty risk,” European central bank.

FinPricing, 2019, Market data services, <https://finpricing.com/lib/EqSpread.html>

Fujii, M. and A. Takahashi (2012), “Collateralized CDS and default dependence – Implications for the central clearing,” Journal of Credit Risk, 8(3): 97-113.

Hull, J. and A. White (2004), “Valuation of a CDO and a nth to default CDS without Monte Carlo simulation,” Journal of Derivatives, 12: 8-23.

Jarrow, R., and S. Turnbull (1995), “Pricing derivatives on financial securities subject to credit risk,” Journal of Finance, 50: 53-85.

Johannes, M. and S. Sundaresan (2007), “The impact of collateralization on swap rates,” Journal of Finance, 62: 383-410.

Koopman, S., A. Lucas, and B. Schwaab (2011), “Modeling frailty-correlated defaults using many macroeconomic covariates,” Journal of Econometrics, 162: 312-325.

Laurent, J. and J. Gregory (2005), “Basket default swaps, CDOs and factor copulas,” Journal of Risk, 7: 03-122.

Longstaff, F., and E. Schwartz (2001): “Valuing American options by simulation: a simple least-squares approach,” *The Review of Financial Studies*, 14 (1): 113-147.

Moody’s Investor’s Service (2000): “Historical default rates of corporate bond issuers, 1920-99,”

Piterbarg, V. (2010), “Funding beyond discounting: collateral agreements and derivatives pricing,” *Risk Magazine*, 2010 (2): 97-102.