Forward Model of Electrical Impedance GeoTomography with Point Electrode Model Based on Mixed Finite Element Method: MATLAB Programming Notes

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Abstract

We address the need of forward model of electrical impedance geotomography using Mixed Finite Element Method. We use the mixed finite-element method with the standard P1- P0 pair for triangular decompositions for calculating model responses, which makes it possible to compute both the electric potential and the electric field vector economically.

For MATLAB implementation, the routine available is easy to modify without hassle.

Electrical impedance geotomography (GeoEIT) is a low-cost non-invasive imaging modality. It has the potential to be of great value in clinical diagnosis. One of the major problems in GeoEIT with complex geometry shape is its high demand in computation capability, power, and memory. A mixed finite element method (MFEM) is proposed to calculate the forward problem accurately. Compared with the traditional FEM, a smaller number of nodes and elements with the proposed method are required to achieve the same accuracy in our numerical computation model. The results demonstrate the efficiency of the MFEM in GeoEIT simulation. In the forward solution, it is capable of achieving better accuracy using less computational time and memory with MFEM.

1 Electrical Impedance GeoTomography as parameter identification problem

In the study of a direct electrical current in a bounded object Ω , the governing equation is the following elliptic partial differential equation:

$$-\nabla(\sigma \nabla u) = q \quad \text{in } \Omega \quad (1)$$

where $\Omega \in R^d, \{d = 2, 3\}$ is a bounded domain, $\sigma : \Omega \rightarrow R$ denotes the conductivity coefficient, $u : \Omega \rightarrow R$ is electrical potential, $q := I(\delta(r - s_i) - \delta(r - s_j))$ is current injection pair through point electrode at $s_{i,j}, i \neq j, 1 \leq i, j, \leq N$, where N denoting the number of electrodes. When the current source term q and the coefficient σ is given, with an appropriate prescribed boundary information for solution u so that the direct (forward) problem of u is uniquely solvable in Ω . Conversely, the recovery information about the coefficient of conductivity σ , given the pair (q, g) is given, where $g(r_k) := u(r_k)$ is measured potential at point electrodes at r_k due to current injection pattern q , is an inverse problem with σ being sensitive due to perturbation in u and q . The recovery of σ from point electrodes measurements u and q , utilizing the equation (1), that is called *Electrical Impedance GeoTomography* with point electrode models.

2 Governing Equation

Let us restrict the whole space to conductive half space Ω , where Ω is the bounded and open subset of R^2 having a piecewise continuous boundary $\Gamma (= \partial\Omega)$. denotes the ground surface ($z = 0$), and $\Gamma_2 = \partial\Omega \setminus \Gamma_1$, write $\rho = 1/\sigma$. The governing first-order system for electric resistivity problem in the domain Ω is expressed by the following Poisson equation with mixed boundary condition ([2]).

$$\nabla u(x, z) = -\rho(x, z)\mathbf{E}(x, z) \text{ in } \Omega \quad (2)$$

$$\nabla \cdot \mathbf{E}(x, z) = \frac{I}{\Delta S}[(\delta(x - x_s^+)\delta(z - z_s^+)) - (\delta(x - x_s^-)\delta(z - z_s^-))] \text{ in } \Omega \quad (3)$$

$$\mathbf{n} \cdot \mathbf{E}(x, z) = 0 \text{ on } \Gamma_1, \quad (4)$$

$$u(x, z) = 0 \text{ on } \Gamma_2 \quad (5)$$

where I is the current in ampere, (x_s^+, z_s^+) and (x_s^-, z_s^-) is the coordinate of point sources pair generated by an injected charge, ΔS is the elemental area about the charge-injected point and $\mathbf{n}$ is the outer normal vector at the boundary of Ω .

3 Mixed finite element method ([1],[3])

Given a triangulation $\mathcal{T}_h$ by simplexes T of a domain $\bar{\Omega}$, we introduce a finite-dimensional subspace W_h of $(L^2(\Omega))^2$ and a finite-dimensional subspace M_h of $H^1(\Omega)$ expressed as

$$W_h = \{q \in W : \forall T \in \mathcal{T}_h, q|_T \in (P_0(T))^2\}, \quad (6)$$

$$M_h = \{v \in M : \forall T \in \mathcal{T}_h, v|_T \in P_1(T)\} \quad (7)$$

Therefore Galerkin finite-element method is defined as

$$\int_{\Omega} \rho(x, z)\mathbf{E}_h(x, z) \bullet \mathbf{p}(x, z) dx dz + \int_{\Omega} \mathbf{p}(x, z) \bullet \nabla u_h(x, z) dx dz = 0 \quad \text{for all } \mathbf{p} \in \mathbf{W}_h$$

$$\int_{\Omega} \mathbf{E}_h(x, z) \bullet \nabla v_h(x, z) \, dx dz = \int_{\Omega} \frac{I}{\Delta S} [(\delta(x - x_s^+) \delta(z - z_s^+)) - (\delta(x - x_s^-) \delta(z - z_s^-))] v_h \, dx dz \text{ for all } v_h \in M_h \quad (9)$$

and we find $(\mathbf{E}_h, u_h) \in \mathbf{W}_h \times M_h$ satisfying equations (8) and (9). In equation (8,9), we divide the region by triangular element and define the potential and electric field using basis functions of finite dimensional space $\mathbf{W}_h$ and M_h as

$$\mathbf{E}_h(x, z) = \sum_{i=1}^{2M} e_i \Psi_i, \quad (10)$$

$$u_h(x, z) = \sum_{i=1}^N u_i \phi_i(x, z), \quad (11)$$

where N is the number of nodal points of domain, M is the number of triangular elements, and $\Psi_i(x, z)$ and $\phi_i(x, z)$ are the basis functions of $\mathbf{W}_h$ and M_h , respectively. Then, equations (8,9) can be written in a matrix form as

$$\mathbf{K} \mathbf{U} = \mathbf{F} \quad (12)$$

with

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{O} \end{bmatrix} \\ \mathbf{U} &= [e_1 \dots e_{2M} \quad u_1 \dots u_N]^T \\ \mathbf{F} &= [0 \quad -\mathbf{f}]^T \end{aligned}$$

From equation (5), we can obtain solution vectors $\mathbf{u}_h = (u_1, \dots, u_N)$ and $\mathbf{e}_h = (e_1, \dots, e_{2M})$ using

$$\mathbf{C}^T \mathbf{B}^{-1} \mathbf{C} \mathbf{u}_h = \mathbf{f}, \quad (13)$$

$$\mathbf{e}_h = -\mathbf{A}^{-1} \mathbf{C} \mathbf{u}_h \quad (14)$$

Since Ψ_i is constant, $\mathbf{A}$ is a diagonal matrix. Therefore, computing time required for obtaining $\mathbf{u}_h$ and $\mathbf{e}_h$ is compatible with that required when we use other numerical methods (finite element or finite difference method).

Programming Note:

The MATLAB implementation in [1] is almost straightforward to implement, except the need to modify the following terms :

$$\int_{\Omega} \rho(x, z) \mathbf{E}_h(x, z) \bullet \mathbf{p}(x, z) \, dx dz.$$

Assuming that $\rho(x, z)$ is piece-wise continuous, and if we choose Triangulation of the domain such that $\rho(x, z)|_{T_i} = \rho_i, i = 1, \dots, Ne$, where $\mathcal{T} = \{T_i \subset \Omega; T_1, \dots, T_{Ne}\}$, is the triangulation of Ω .

76 4 Example

77 In order to examine the implementation of above setting, we use it for generating syn-
 78 thetic data, we take a simple 2 dimensional case, as shown in Figure 1.

79 A pair of electrodes, situated at nodes 7 and 8, and resistivity set at 1 Ohm, in the
 80 domain except at triangle 3 which have value 2 Ohm.

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81 coordinate = [0 0 ;0 -1 ;1/3 -1 ;2/3 -1 ;1 -1 ;1 0 ;2/3 0 ;1/3 0] ;
82 element = [1 2 3 ;1 3 8 ;8 3 4 ;8 4 7 ;7 4 5 ;7 5 6] ;
83 Dirichlet = [1 8; 8 7; 7 6];
84 Neumann = [1 2; 2 3; 3 4; 4 5; 5 6]
85 Dipole = [7 8]
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Figure 1: *Mesh* consists of six elements in Ω

86 5 Conclusions

87 Unlike in the conventional DC modeling, we are enforced to use the mixed finite-element
 88 method with $P_1 - P_0$ triangular element which guarantees to compute electric field
 89 and potential simultaneously. Off the shelf MATLAB library for Mixed Finite Element
 90 Method is straightforward to solve the Governing equations.

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103 MATLAB Scripts

104 The script for case 1 will be put in here