Reducing Hamiltonian path to a vertex label

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1. Introduction and notations

Given a graph $G = (V, E)$, we want to label all the vertices $v \in V$ with values from $\{1, 2, \dots, n\}$ where $|V| = n$ such that for all edges $(x, y) \in E$ such that $|label(x) - label(y)| \geq 2$. We have to determine whether such a labelling is possible for a graph.

Now we are going to prove that doing so is an **NP**-complete problem. While showing that it belongs to **NP** is straightforward, to prove the hardness of this problem would require a reduction from a known **NP**-complete problem. We choose to reduce from Hamiltonian path as shown in the next section.

2. Details of the proof

Sketch of proof: Given a graph G , consider its complement G' . The graph G' is a **YES** instance of our problem if its vertices can be ordered so that adjacent vertices are not connected by an edge in G' . In other words, G' is a **YES** instance of our problem if the vertices of G can be ordered so that adjacent vertices are connected. So G' is a **YES** instance of our problem if and only if G has a Hamiltonian path.

Proof: Indeed, for one direction, if G is semi-Hamiltonian, then given one Hamiltonian path of it, we easily label vertices of our G' . Just starting with label 1 for the starting vertex of the path, increment along the way until the ending vertex with label n . Intuitively, each vertex with label i in G' is connected with both $i - 1$ and $i + 1$. So in G' , the complement of G , every edge (x, y) must satisfy the required condition.

For the other direction, if we can label G' , we would now like to have G semi-Hamiltonian. So, we start from the vertex with label 1, and this is our current vertex in the algorithm constructing a Hamiltonian path for G . As our vertex labelling is feasible, by connectedness of G , the only possible adjacent vertex of our current vertex would be a vertex with label 2. For if this is not the case, infeasibility is the property of our vertex labelling for G' . So, our current vertex now has label 2. Repeat this procedure until we reach the vertex with label n . Clearly, we obtain a Hamiltonian path for G . ■

References

- [1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, *Introduction to Algorithms*