Open library for space elevator design and engineering. Analyses of tether tapering functions

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Abstract

Tapering of a space elevator's tether from the planet surface to the altitude of the synchronous orbit and further to the counterweight altitude can reduce the mass of the whole system, as well as provide more or less even distribution of stress along the tether rather than in the case of using uniform tether. This document presents the analyses of some tether tapering functions and shows that exponentially tapered tether is not an optimal solution for the space elevator with climbers or other point loads like transitional stations. To handle these analyses ClarkeLib – an open library for calculation of basic space elevator parameters and loads has been developed. Estimation of Earth Elevator and Mars Elevator parameters has been made using this library, and results are represented in this paper.

Keywords: Space Elevator, Earth Elevator, Mars Elevator, Lunar Elevator, Tether Tapering, Carbon Nanotubes, Copernicus, ClarkeLib.

I. INTRODUCTION

Space elevator, being an extremely complex system, depends not only on the ultimate materials, but also on the general design approaches that can significantly reduce construction time and cost, increase efficiency and effect many aspects related to operation and maintenance.

One of the main design parameters of a space elevator is tether tapering law. The cross section of the tapered tether increases from the ground level to the altitude of the planet's synchronous orbit and then decreases afterwards. This is done to avoid excessive and exiguous stresses that arise in tether of uniform cross section.

The common approach, aimed to achieve uniform stress distribution along the tether is to use exponential tapering law. This is a good

solution for the tether without extra loads produced by climbers or other space elevator elements like transitional stations. In general case exponentially tapered tether is not the optimal solution as these point loads will break uniformity of stress distribution. So, alternative tapering functions must be considered. This paper provides a fresh glance at the tether tapering issue and concentrates on the approach of alternative tether tapering functions' analyses via computer simulation. Rather simple solution for the tether tapering function is proposed, which can be used as the initial point for further research.

To support all analyses provided within this paper with computer simulation data, ClarkeLib - an open source library for space elevator mathematical modeling was developed. This is an object-oriented library for SageMath - an open source system of computer algebra based

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on Python programming language [3].

The library is accessible through GitHub, and can be used to estimate general space elevator parameters like mass, counterweight height, tether cross sections, etc., as well as to analyze alternative tether tapering functions, and their influence on stress distribution along the tether.

Considering the fact that space elevator can be built not only on Earth but on other planets of the Solar System, like Mars, general terminology and considerations are used in math model description.

II. TETHER STATIC LOAD MODEL

According to the classic space elevator concept all loads applied to the tether and caused by gravity of the host planet must be counter-vailed by the centrifugal forces in the tether and centrifugal force of the counterweight placed above the synchronous orbit. The mass of the counterweight m_{cw} depends on the total loads and height of the counterweight h_{cw} . The force, produced by the counterweight can be represented as a sum of gravitational and centrifugal forces:

$$F_{cw} = -m_{cw} \cdot \frac{GM}{h_{cw}^2} + m_{cw} \cdot \omega^2 \cdot h_{cw} \quad (1)$$

where ω and M are angular velocity and mass of the host planet respectively, G - gravitational constant.

Section ratios are used to set relations between the tether cross sections in the anchor point (point on the ground level) – S_1 , at the altitude of the planet's synchronous orbit – S_2 , and at the altitude of the counterweight – S_3 .

$$S_2 = k_{syn} \cdot S_1 \quad (2)$$

$$S_3 = k_{cw} \cdot S_1 \quad (3)$$

An elementary force $dF = dF_g - dF_a$ acts in each cross section of the tether at the altitude h , where dF_g – an elementary force caused by the gravity of the host planet, dF_a – an elementary

force caused by the rotation of the planet and the space elevator.

$$dF_g = f \cdot \frac{GM}{h^2} dh \quad (4)$$

$$dF_a = f \cdot \omega^2 h dh \quad (5)$$

Parameter of linear mass density f is used to integrate elementary forces along the tether in order to get total forces:

$$f = S \cdot \rho \quad (6)$$

where S – is the cross section area of the tether and ρ – is the volume density of the tether material.

Figure 1: Space elevator static load model

Tapering laws can be different for the under-synchronous (lower) and above-synchronous (higher) strips of the tether, so these laws considered separately and marked with suffixes 1 and 2 respectively. Thus, tapering functions f_1 and f_2 describe change of the linear mass density in the lower and higher tether strips with altitude, and total gravity and centrifugal

forces can be written as

$$F_g^1 = \int_R^{h_{syn}} f_1 \cdot \frac{GM}{h^2} dh \quad (7)$$

$$F_a^1 = \int_R^{h_{syn}} f_1 \cdot \omega^2 h dh \quad (8)$$

for the lower tether strip and

$$F_g^2 = \int_{h_{syn}}^{h_{cw}} f_2 \cdot \frac{GM}{h^2} dh \quad (9)$$

$$F_a^2 = \int_{h_{syn}}^{h_{cw}} f_2 \cdot \omega^2 h dh \quad (10)$$

for the higher tether strip respectively. Here R – is radius of the planet, h_{cw} – altitude of the synchronous orbit and h_{cw} – is the altitude of the counterweight.

Linear mass density at the anchor point altitude $f_0 = S_1 \cdot \rho$ can be found from the expression for stress in the cross section of the tether

$$f_0 = \frac{k_{safe} \cdot \rho F_{anchor}}{\sigma_b} \quad (11)$$

where F_{anchor} – is the force that is applied to the free end of the tether near the surface of the planet to ensure it stays stretched, k_{safe} – is the safety factor and σ_b – is the tensile strength of the tether material.

As we are interested in the unstrained tether, tapering coefficient k_{syn} can be found in the same way - from the expression for stress in the cross section at the altitude of the synchronous orbit

$$\frac{F_{pull}}{S_2} = \frac{\sigma_b}{k_{safe}} \quad (12)$$

where $F_{pull} = F_g^1 - F_a^1 + F_{anchor} + F_{climbers}$ – is the total pull force that acts in the cross section at the altitude of the synchronous orbit, and $F_{climbers}$ – is the total force produced by the weight of climbers.

Other unknown parameters can be found by solving the next system of equations:

$$\begin{cases} F^1 + F^2 + F_{anchor} + F_{climbers} = F_{cw} \\ \frac{F_{cw}}{S_3} = \frac{\sigma_b}{k_{safe}} \end{cases} \quad (13)$$

where $F^1 = F_g^1 - F_a^1$ and $F^2 = F_g^2 - F_a^2$. The first equation in the system above – is the equation of the equilibrium, while the second one – is the equation for stress in the tether cross section at the altitude of the counterweight. This system can be solved against k_{cw} and m_{cw} for a given value of the counterweight height h_{cw} .

III. TETHER TAPERING FUNCTIONS

The common solution for the unstrained tether is to use an exponential tapering all the way along the tether so that a cross section increases up to the altitude of the synchronous orbit and then decreases afterwards [2].

$$f = f_0 \cdot e^{(GM\rho^{\frac{k_{safe}}{\sigma}}) \cdot (\frac{1}{R} + \frac{R^2}{2h_{syn}^3} - \frac{1}{h} - \frac{h^2}{2h_{syn}^3})} \quad (14)$$

This approach is applicable for a space elevator without climbers and other point loads along the tether. Indeed, we can see uniform stress distribution on the diagram below for Earth Elevator without climbers, anchor force of 10 kN and tether material as strong as 50 MYuri and as dense as $1300 \frac{kg}{m^3}$. Safety factor of 1.3 has been considered.

Figure 2: Exponentially tapered tether without climbers

But if any additional point load appears on the tether like climbers, the stress distribution loose its uniformity, and some part of the tether close to the planet's surface appears to be underloaded.

Figure 3: Exponentially tapered tether with climbers

Such results can be achieved if we add total force from all climbers to the anchor force in the first equation in (13). Other way most part of the tether will be overloaded. 7×20 -ton climbers are uniformly distributed along the lower strip of the tether up to the altitude of the synchronous orbit.

Now it becomes clear that other tapering function for the lower strip of the tether must be considered. As simple as it can be, cubic polynomial fits this purpose pretty well.

$$f_1 = f_0 \left((1 - k_{syn}) \frac{(h - h_{syn})^3}{(R - h_{syn})^3} + k_{syn} \right) \quad (15)$$

As higher strip of the tether stays uniformly strained, and no climbers are about to operate at this part of the tether (except interplanetary ships or service climbers, which is not the point of interest for the current paper), it is reasonable to leave the exponential law for it. As long

as the higher tether strip is free of climbers and transitional stations, exponential tapering can be used there. But if any regular vehicle flow will take place there in future, it's better to analyze and reconsider tapering function for the higher strip as well.

Figure 4: Tether with climbers, cubic tapering function on the lower strip and exponential tapering function on the higher strip

IV. CLARKELIB

To handle all analyses with computer simulation data an open source library for calculation of basic space elevator parameters and loads has been developed. The library is written for SageMath - an open source system of computer algebra [3] and can be found on GitHub¹. It provides a simple object-oriented interface to calculate basic parameters of multiplanet space elevator, analyze dependency of SE mass, counterweight mass and tether mass on the altitude of the counterweight. The simplest way to use the library is to add all files from the repository to the newly created project on SageMathCloud – a cloud computing platform that supports editing of Sage worksheets, LaTeX documents,

¹ <https://github.com/nomad-vagabond/ClarkeLib>

and IPython notebooks [4]. For instructions on how to use ClarkeLib see README file in the repository.

V. EARTH ELEVATOR

Analyses of the Earth Elevator mass dependency on the counterweight altitude has been made using Clarkelib. Interesting computational fact is that the Earth Elevator with exponentially tapered tether theoretically can be built with currently available materials. But on practice it is impossible. For example, a 100-Mm tether with specific strength of 3.5 MYuri (T1000G CNT fiber [5]) will require total space elevator mass of $16 \cdot 10^{10}$ tons! A tether's cross section of $2.8 \cdot 10^3 \text{ m}^2$ at GEO altitude drops down to 42 mm^2 at the anchor point. Mass of the counterweight needed is about 17 Mt.

So, it is obvious that humanity must dive into searching the ways to significantly increase tensile strength of the industrial CNT fibers. This, probably, can be achieved by using non-regular nanostructures, like nanobuds [6], instead of smooth carbon nanotubes. A fullerene parts, covalently bonded to the side of car-

bon nanotubes increase their friction. It means that aligned assembly of such molecules can be much stronger than the same assembly of CNTs.

Depending on the tensile strength of the future tether material, tether fabrication process and space elevator's construction scheme may vary. For example, adopted earlier material of 50 MYuri enables construction of 100 Mm-long space elevator with 1.97-kt tether, capable of handling 7×20 -ton climbers. 93 launches of Falcon Heavy LV are necessary to deliver this mass to GEO. To reduce tether mass we can use heavier counterweight, that will shorten tether but increase total mass of the space elevator. Even if we assume that counterweight is not launched from the Earth but is a space-born object (asteroid or collected space debris), massive enough to reduce tether's length twice, the mass of the tether stays too big to be launched from the Earth by chemical rockets. Alternative option is to produce tether on the Moon. Such venture can significantly reduce tether's launch cost, especially if Moon Elevator [7] is available by that time.

The Earth Elevator tether that could be de-

Figure 5: Space elevator computation using ClarkeLib and SageMathCloud via internet browser.

livered to GEO from the Earth's surface and is capable of handling 7×20 -ton climbers should be much stronger than 50 MYuri. In addition, reducing payload capacity of the initial space elevator tether is rational solution to decrease launch cost [1]. If we assume that initial 50-Mm tether will be capable of carrying 7×5 -ton buildup climbers and is made of 100-MYuri CNT fiber, than its weight goes down to 84 tons. The weight of the counterweight is 820 t, which is still high because of its low altitude. However this mass can be collected from dead satellites in GEO right now [8]. Such a "seed" space elevator tether can be launched to GEO in 4 launches.

Specific strength of 100 MYuri is 130 GPa of tensile strength, which is close to the strength of the individual CNTs and, probably will never be reached by the industrial fibers. The value of 50 MYuri is more realistic forecast, so initial tether of 284 t will be required to handle 7×5 -ton build-up climbers. This is 11 launches of the Falcon Heavy (approx. \$ 1 billion). The counterweight of 1.4 kt will be required to keep 50-Mm tether stretched. This mass still can be collected from the space junk that is in abundance in space nowadays. Such initial space elevator can double its own payload capacity in 100 days, so just in a year or two it will be able to service 20-t climbers.

Table 1: Earth Elevator of 50 Mm length

Climbers	7×5 t		7×20 t	
Tether specific strength, MYuri	50	100	50	100
Tether mass, t	284	84	1 020	304
Counterweight mass, t	1 407	819	5 067	2 968

VI. MARS ELEVATOR

As the gravity of Mars is much weaker than the gravity of Earth and the period of Mars' rotation is pretty close to the Earth's one, requirements to the Mars Elevator's tether material is lower. What if it can be constructed with

materials available now?

On the Figure 6 we can see diagrams representing dependence of the counterweight's, tether's and total Mars Elevator's masses on the altitude of the counterweight. Calculations has been made for the material whith similar mechanical properties as T1000G CNT fiber [5], namely density of $1800 \frac{kg}{m^3}$ and tensile strength of 6.3 GPa. Anchor force of 10 kN and 4×5 -ton climbers each were considered as main loads.

Figure 6: Dependency of the Mars Elevator, its tether and counterweight masses on the counterweight altitude.

According to the calculation results, mass of the adopted Mars Elevator tether is too high to be launched from Earth for any altitude of the counterweight. On other hand, Mars has two small moons – Phobos (semi-major axis is approx. 9.5 Mm) and Deimos (semi-major axis is approx. 23.5 Mm), which we can mine for carbon to produce tether in Mars orbit. As the altitude of the areosynchronous orbit is about 17 Mm above the martian surface, Deimos is much better candidate for this purpose.

We can't use Deimos as the counterweight entirely because its orbital velocity is smaller than needed and its mass is too big (1.476210^{15}

kg) to be "pushed" by the means of the rocket propulsion technologies that we have today. But we still can mine it and use its regolith and stones to build up the counterweight of necessary mass and ship it to the right place. The biggest trouble of the Mars Elevator is massive Phobos in a low orbit. To deal with it tether must execute periodical avoidance maneuvers, as it is described in the Arthur Clarke's book "Fountains of Paradise" [9]. To eliminate the need to do the same with Deimos it is rational to place counterweight at the altitude just below Deimos' orbit, approximately at 20 Mm above the martian surface. The distance of only 63 km will separate Deimos from the counterweight. In this case tether of approximately 6 000 t and counterweight of approximately 29 000 tons will be required. All this mass can be extracted from Deimos by robotic equipment launched from Earth.

Table 2: Mars Elevator tethers of 20 Mm length

Tether specific strength, MYuri	3.5	10	30
Tether mass, t	5 942	254.5	41.3
Counterweight mass, t	28 731	2 941	1 298

Such venture will require development of new technologies to mine small celestial bodies like Deimos, produce high-quality CNT fiber in low-gravity environment, rocket propellant from mined material, etc. But if once built, Mars Elevator will significantly simplify discovery of the red planet without the need to send one-way suicidal missions to Mars. Moreover, approbation of the Space Elevator technology in real environment will boost development of this promising space transportation system on Earth.

VII. CONCLUSIONS

The revolutionary space transportation system called Space Elevator shouldn't be associated with our home planet only. The gain

of building space elevators on other planets of the Solar System isn't less than the one of building the Earth Elevator. Moreover, due to the high mass of the estimated Earth Elevator's tether, it is reasonable to start from building non-Earth elevators with a Lunar Elevator in the head of the list. Production of the Earth Elevator tether on the Moon and sending it to GEO by a Lunar Elevator may significantly reduce the total cost of this challenging project. The Mars Elevator can significantly boost the red planet's exploration and become the gate for future Mars colonists.

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