Embedding Topological n -Manifolds into Compact Nonstandard Topological n -Manifolds with Boundary

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Abstract

We show as a main message that there is a simple dimension-preserving way to openly and densely embed every topological manifold into a compact “nonstandard” topological manifold with boundary. This class of “nonstandard” topological manifolds with boundary contains the usual topological manifolds with boundary. In particular, the Alexandroff one-point compactification of every given topological n -manifold is a “nonstandard” topological n -manifold with boundary.

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By a manifold we always mean a topological manifold, namely a second-countable Hausdorff space where every point has some neighborhood homeomorphic to some open subset of some (fixed) Euclidean space $\mathbb{R}^n$.

By a *nonstandard manifold with boundary* we always mean a second-countable Hausdorff space where every point has some neighborhood homeomorphic to some subset of some (fixed) upper-half plane $\mathbb{H}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_n < +\infty\}$ such that this subset less the topological boundary $\partial\mathbb{H}_+^n$ of $\mathbb{H}_+^n$ is an open set in $\mathbb{R}^n$; thus every manifold (with boundary or not) is a nonstandard manifold with boundary. In any given topological space, we will correspondingly refer to a pair of a neighborhood of a given point and a homeomorphism of the neighborhood onto a subset of $\mathbb{H}_+^n$ of the type specified in the definition of a nonstandard manifold with boundary as a *nonstandard chart* of the given space about the given point.

The phrase “openly and densely embeddable” always means that an in-context topological embedding has range being an open and dense subset of the in-context codomain.

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We should like to show with a simple elementary proof that one can openly and densely embed any given n -manifold into a compact nonstandard n -manifold with boundary:

Theorem 1. *Every n -manifold is openly and densely embeddable into some compact nonstandard n -manifold with boundary.*

Proof. Let M be an n -manifold. Since M is in addition locally compact, let M_∞ be the Alexandroff (one-point) compactification of M with ∞ denoting the additional element and with the only innocuous modification that $\{\infty\}$ is not open in M_∞ by considering only compact proper subsets of M . Since then M_∞ is the only closed set in M_∞ that includes M , the set M is dense (yet still open) in M_∞ . This modification is evidently redundant when M is not a compact space.

Choose a nonempty compact proper subset K of M and a precompact coordinate open subset G of $M \setminus K$; and let φ be a homeomorphism between G and some open subset of the Euclidean subspace $\mathbb{R}_{++}^n := \{(x_1, \dots, x_n) \mid 0 < x_i < +\infty \text{ for all } 1 \leq i \leq n\}$.

If $G_\infty := G \cup \{\infty\}$, let $\varphi_\infty : G_\infty \rightarrow \mathbb{H}_+^n$ with $\varphi_\infty|_G := \varphi$ and $\varphi_\infty(\infty) := 0$. Then φ_∞ is a bijection between G_∞ and the union of $\{0\}$ and the φ -image $\varphi^1(G)$ of G .

We first claim that φ_∞ is continuous. To this end, let $A \subset G$. Since a closed proper subset of M_∞ is by construction precisely either a compact set in M or the union of $\{\infty\}$ and a closed set in M , and since G is precompact, the closure $\text{cl}_M(G)$ of G in M is a closed subset of M_∞ that includes A (and that does not contain ∞); and therefore it follows from the definition of closure that

$$\text{cl}_{G_\infty}(A) = \text{cl}_G(A).$$

By this equality, we will then succinctly write $\text{cl}(A)$ for the closure of any subset A of G with respect to either topology. Moreover, the above equality implies that ∞ is never an element of the closure of any given subset of G .

Fix $A \subset G_\infty$. If $\infty \notin A$, then

$$\varphi_\infty^1(\text{cl}(A)) = \varphi^1(\text{cl}(A)) \subset \text{cl}_n(\varphi^1(A)) = \text{cl}_n(\varphi_\infty^1(A))$$

by the continuity of φ . Here $\text{cl}_n(\cdot)$ denotes the closure operation in $\mathbb{H}_+^n$. If $\infty \in A$, then, since $\{\infty\}$ is closed in M_∞ , we have

$$\text{cl}_{G_\infty}(A) = \text{cl}(A \setminus \{\infty\}) \cup \{\infty\};$$

and so

$$\varphi_\infty^1(\text{cl}_{G_\infty}(A)) = \varphi^1(\text{cl}(A \setminus \{\infty\})) \cup \{0\} \subset \text{cl}_n(\varphi^1(A \setminus \{\infty\})) \cup \text{cl}_n(\{0\}) = \text{cl}_n(\varphi_\infty^1(A)).$$

The continuity of φ_∞ then follows.

To show that φ_∞^{-1} is continuous, we claim that φ_∞ is a closed map. But, given any closed set F in G_∞ , if $\infty \notin F$ then

$$\varphi_\infty^1(F) = \varphi^1(F) \not\ni 0$$

is closed in $\varphi_\infty^1(G_\infty)$; whereas if $\infty \in F$, then, since $F \setminus \{\infty\}$ is a closed set in G as previously argued, it follows that

$$\varphi_\infty^1(F) = \varphi^1(F \setminus \{\infty\}) \cup \{0\}$$

is also closed in $\varphi_\infty^1(G_\infty)$.

Now $(G_\infty, \varphi_\infty)$ is a nonstandard chart of M_∞ about ∞ . Upon choosing an atlas of M where every coordinate map has range being an open subset of $\mathbb{R}_{++}^n$, we have proved that M_∞ is a nonstandard n -manifold with boundary. Moreover, since M_∞ is compact by construction, and since the inclusion map $\text{id}_{M_\infty}|_M : M \rightarrow M_\infty$ openly and densely embeds M into M_∞ , the proof is complete. $\square$

Remark. In the proof of Theorem 1, second-countableness evidently plays merely a nominal role; the proof serves every locally compact Hausdorff space having a coordinate open cover (with the evident corresponding verbal changes).

Moreover, the proof of Theorem 1 is evidently more informative, in asserting that every n -manifold is an open and dense subspace of a compact nonstandard n -manifold with boundary. $\square$

For ease of reference, we state the technically evident yet interesting implication of the proof of Theorem 1 in the following

Proposition 1. *The Alexandroff one-point compactification of every given n -manifold is a nonstandard n -manifold with boundary.* $\square$