Quantum theory within the probability calculus: a there-you-go theorem and partially exchangeable models

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“Ever since the advent of modern quantum mechanics in the late 1920’s, the idea has been prevalent that the classical laws of probability cease, in some sense, to be valid in the new theory. [...] The primary object of this presentation is to show that the thesis in question is entirely without validity and is the product of a confused view of the laws of probability” (Koopman, 1957). The secondary objects are: to show that quantum inferences are cases of partially exchangeable statistical models with particular prior constraints; to wonder about such constraints; and to plead for a dialogue between quantum theory and the theory of exchangeable models.

1 Introduction

Ever since the advent of modern quantum mechanics in the late 1920’s, the idea has been prevalent that the classical laws of probability cease, in some sense, to be valid in the new theory. More or less explicit statements to this effect have been made in large number and by many of the most eminent workers in the new physics [...]. Some authors have even gone farther and stated that the formal structure of logic must be altered to conform to the terms of reference of quantum physics [...].

Such a thesis is surprising, to say the least, to anyone holding more or less conventional views regarding the positions of logic, probability, and experimental science: many of us have been apt – perhaps too naively – to assume that experiments can lead to conclusions only when worked up by means of logic and probability, whose laws seem to be on a different level from those of physical science.

The primary object of this presentation is to show that the thesis in question is entirely without validity and is the product of a confused view of the laws of probability.

(B. O. Koopman, 1957)

Koopman's lucid presentation stands today as it did sixty years ago. Perhaps it can be made even clearer if we consider quantum systems with a finite number of energy levels – qubits, qutrits, etc., very important in today's chase for quantum computers (Nielsen et al. 2010) – and if adopt the so-called “operational approach” to quantum theory, which can be glimpsed in Koopman's presentation itself.

2 Operational approach

A pseudohistorical presentation:

After decades of inconclusive debates about what quantum systems “really” are, the measurement problem, Schödinger's cats, and similar questions, the operational approach (see e.g.: Ludwig 1983; Segal 1959; Mielnik 1968; 1969; Lamb 1969; Davies et al. 1970; Foulis et al. 1972; Randall et al. 1973; 1979; Wright 1978; Haag 1982; Kraus 1983; Wootters 1986; Busch et al. 1995; for more recent elaborations: Hardy 2001; Porta Mana 2003; 2004a; Barnum et al. 2006; Barrett 2007; Harrigan et al. 2007) emerged as a way to sidestep or postpone answering them and get (blindly?) on with experiments and technology.

The starting points of this approach are these:

- I. “all well-defined experimental evidence, even if it cannot be analysed in terms of classical physics, must be expressed in ordinary language making use of common logic [. . .]. This is a simple logical demand, since the word «experiment» can in essence only be used in referring to a situation where we can tell others what we have done and what we have learned” (Bohr 1948).
- II. This verbalization of every experiment is conveniently divided into three parts: the descriptions of a *preparation*, of a *measurement*, and of several possible *outcomes*; each represented by a proposition: S , M , O_i . The outcomes, implicit in the description of the measurement, can be probabilistically predicted; including deterministic, unit-probability predictions. For each quantum system we have a set of possible preparations and a set of possible measurements; elements from the two sets can be freely combined, at least in principle.
- III. To a preparation S we can associate a unit-trace, positive-definite Hermitean matrix ρ usually called *density matrix*; and to the outcomes $\{O_i\}$ of a measurement M , a set of positive-definite Hermitean

matrices $\{E_i\}$ summing up to the identity matrix; this set is called a *positive-operator-valued measure*. The dimension of these matrices depends on the system.

- IV. The probability of obtaining outcome O_i of the measurement M when the preparation is S is given by

$$P(O_i | M \wedge S) = \text{tr } E_i \rho, \quad (1)$$

usually called the *trace formula*. The properties of the matrices guarantee that the probability distribution for the outcomes $\{O_i\}$ is non-negative and normalized.

In the verbalization of an experiment we can also include the description of a *transformation*, possibly parameterized by time; this is where Schrödinger's equation appears. Transformations are briefly discussed in appendix A. For the moment let's keep the description of this operational approach to a minimum. Appendix B shows how this approach comprises the old-fashioned quantum formalism with Hermitean operators & Co.

The operational approach favours the view of probability as an extension of the propositional truth calculus (Keynes 1957; Johnson 1924; Ramsey 1926; Cox 1946; Pólya 1949; Jaynes 2003; Hailperin 1996; 2011; Terenin et al. 2017). Sure, we can translate all this in terms of "random variables" about physical quantities, but the verbal and propositional character of this approach is fundamental. It works because quantum physicists usually agree on the coarsest, protocol-like verbal description of an experiment, even if they may disagree on what is "really" going on microscopically. They agree on how to divide the experiment into preparation, measurement, and outcomes. They agree on which density matrices and positive-operator-valued measures to associate with those divisions. Each physicist can add his or her own personal interpretation R_{personal} of what is "really" going on, but it becomes irrelevant when the coarse preparation is specified; we could write this irrelevance as

$$P(O_i | M \wedge S \wedge R_{\text{personal}}) = P(O_i | M \wedge S). \quad (2)$$

The operational approach will thus still be valid if we'll eventually agree on a microscopic interpretation of quantum phenomena.

We shall now find additional reasons for the propositional view of probability in this approach.

3 Convexity of preparations

The operational approach was accompanied by several developments in the mathematical formalism of quantum theory (e.g., the use of positive-operator-valued measures), recruiting from subjects like C^* -algebras, lattice theory, convex spaces. The latter I find most insightful.

The set of positive-definite Hermitean matrices associated with preparations and measurement outcomes in points III–IV above can be seen as a subset of a *real* vector space of dimension n^2 , where n is the dimension of these matrices. The trace product in eq. (1) is just a scalar multiplication of such vectors (or better, the contraction of a vector and a dual 1-form, without scalar products). This means that we can associate a *real*-valued vector s with each preparation, and a set of real-valued vectors $\{o_i\}$ with each set of measurement outcomes, and the trace formula (1) becomes

$$P(O_i | M \wedge S) = o_i \cdot s. \quad (3)$$

A brilliant paper by Hardy (2001), foreshadowed by Wootters (1986), showed that this formula is true for any physical theory: quantum, classical, or otherwise. In fact, it holds for any collection of three kinds of propositions satisfying points I–II, whether they be about physical theories or not (Porta Mana 2003; 2004a).

The sets of vectors $\{s\}$ and $\{\{o_i\}\}$ satisfy constraints that guarantee the positivity and normalization of the probabilities; these constraints say that these sets are convex spaces. Classical and quantum systems differ in the convex properties of their sets of vectors $\{s\}$; let's call these *states*, and let's call *extremal* the preparations that are represented by extremal states of these convex sets. The set of states of a classical system is a simplex; that of a quantum system is the convex hull of complex projective space $\mathbb{CP}^{n-1}$ (Bengtsson et al. 2006). See figure on the right for an example with $n = 3$ (qutrit). Because of these differing convex structures, for a classical system

3D sections of the convex hull of $\mathbb{CP}^2$
(Månsson et al. 2006)

there is always a measurement that allows us to infer with probability 1 which of two extremal preparations was made:

there is M such that $P(S|M \wedge O_i) = 0$ or 1 ,
 $S \in \{\text{extremal preparations}\}; \quad (4)$

whereas for a quantum system this is possible for particular extremal preparations only. These different behaviours under inference aren't foreign to the probability calculus, however. Kirkpatrick (2003a,b) showed that analogous inferential characteristics appear in some games with cards, for example; similar examples are easily constructed with urn-drawing (Porta Mana 2004b § IV).

Still today nobody knows why the sets of states of quantum systems have a projective-space convex structure. This is the “*only* mystery” (Feynman et al. 1965 § 1.1) of quantum theory.

In an experiment with a physical system – quantum or otherwise – we can imagine a state of knowledge S' where we are unsure about which of two preparations S_1, S_2 was made, with corresponding probabilities:

$$P(S_1|S') = q_1, \quad P(S_2|S') = q_2, \quad q_1 + q_2 = 1. \quad (5)$$

In this state of knowledge our predictions for any measurement outcome will be, by the probability calculus,

$$P(O_i|M \wedge S') = P(O_i|M \wedge S_1) P(S_1|S') + P(O_i|M \wedge S_2) P(S_2|S') = \\ (o_i \cdot s_1) q_1 + (o_i \cdot s_2) q_2 = o_i \cdot (q_1 s_1 + q_2 s_2). \quad (6)$$

The first equality tacitly uses some logical-independence assumptions that are quite natural in an experimental setup; e.g., the choice of measurement doesn't tell us anything about the preparation.

The last equality says that we can associate the vector $q_1 s_1 + q_2 s_2$, a convex combination of the states s_1 and s_2 , with the state of knowledge S' . This state of knowledge, usually called a *mixture*, can therefore be considered a citizen of the set of preparations.

It is natural to assume that states of knowledge like S' exist with all possible values of the distributions (q_1, q_2) , and also involving more than two preparations. This assumption implies that the “domain of discourse” for our system, even if it initially has only a finite number of preparations $\{S\}$ represented by states $\{s\}$, can always be extended to an infinite number of preparations, corresponding to the convex hull of $\{s\}$.

4 Lattice structure of measurements

We can also consider two kinds of state of knowledge involving measurements and outcomes (cf. Peres et al. 1998).

The first, as with preparations, is a state of knowledge M' where we are unsure whether measurement M_1 or M_2 was made, with probabilities

$$P(M_1|M') = q_1, \quad P(M_2|M') = q_2, \quad q_1 + q_2 = 1. \quad (7)$$

The set of outcomes allowed by this state of knowledge is $\{O_{1i}\} \cup \{O_{2j}\}$, and we have

$$P(O_{1i}|M' \wedge S) = P(O_{1i}|M_1 \wedge S) P(M_1|M') + P(O_{1i}|M_2 \wedge S) P(M_2|M') = \\ (\mathbf{o}_{1i} \cdot \mathbf{s}) q_1 + 0 = (q_1 \mathbf{o}_{1i}) \cdot \mathbf{s}, \quad (8)$$

and analogously for O_{2j} , assuming that the two original sets of outcomes are mutually exclusive. The state of knowledge M' , usually called a *mixture*, can thus be considered a measurement, associated with the vectors $\{q_1 \mathbf{o}_{1i}\} \cup \{q_2 \mathbf{o}_{2j}\}$.

The second is a state of knowledge M'' in which we are not interested in the outcomes $\{O_i\}$ of a particular experiment M , but in other events $\{O_j''\}$ which we can probabilistically infer from those outcomes, with

$$P(O_j''|O_i \wedge M'') = Q_{ji}, \quad \sum_j Q_{ji} = 1. \quad (9)$$

Then

$$P(O_j''|M'' \wedge S) = \sum_i P(O_j''|O_i \wedge M'') P(O_i|M \wedge S) = \\ \sum_i Q_{ji} (\mathbf{o}_i \cdot \mathbf{s}) = (\sum_i Q_{ji} \mathbf{o}_i) \cdot \mathbf{s}, \quad (10)$$

and we can consider M'' also as a measurement, associated with the vectors $\{\sum_i Q_{ji} \mathbf{o}_i\}$. We can call it a *dither*, but it includes *coarsenings*, i.e. situations where we aren't interested in distinguishing several outcomes; for example considering the set $\{O_1 \vee O_2, O_3\}$ instead of $\{O_1, O_2, O_3\}$.

Mixtures and dithers can also be combined. They make the set of measurement outcomes into something more than a convex set: it is a set of lattices that can be combined in the two ways just described.

5 Conclusion

“Conclusion?” you might ask. “Wasn’t all the above just the preamble to a Proof that the standard probability calculus suffices for quantum theory? Where’s the proof?” The proof was under your eyes as you were reading. The mathematical formalism just presented covers that of quantum systems with a finite number of energy levels, and with some topological care can be extended to cover systems with continuous energy levels, the Schrödinger equation, and even quantum field theory as infinite limits. This mathematical formalism is so general that it can also describe exotic systems that have neither classical nor quantum inferential traits. Did we have to generalize the ordinary probability calculus for this formalism? did we use any exotic probability theory? No. There you go.

Neither complex-valued probabilities nor lattices of σ -algebras have been necessary. In fact, what we’ve done looks simply like a special application of the probability calculus. The only special feature is the vector-product formula (3), expressing some probabilities as the results of vector products. But the probability calculus doesn’t care where the numerical values of its probabilities come from, as long as they don’t break its basic rules. And as we’ve seen, its rules are not broken when dealing with quantum experiments. Moreover, the vector-product formula doesn’t have any physical significance: it appears when our domain of discourse involves three kinds of propositions logically related in a particular way (Porta Mana 2003; 2004a), but the propositions could be, say, about Donald Duck or parallel universes or other such things.

Only the particular convex structure of the preparations has physical significance, and it’s experimentally observed. But the usual probability calculus can accommodate every convex structure, including the one peculiar to quantum theory.

The notion that quantum systems require lattices of σ -algebras – and therefore a generalization of the probability calculus – arises when we ignore the specification of the measurement in the conditional of the probabilities $P(O_i | M \wedge S)$. We end up with several sets Ω_M and their σ -algebras, one for each M .

But this mathematical move does not make much sense, for at least three + one reasons.

First, in real applications we often need to consider uncertainties about measurement procedures (Leonhardt 1997; de Muynck 2002 ch. 7; Ziman et al. 2006; D’Ariano et al. 2004). To do so we must use the mixing, dithering, coarsening formulae (7)–(10) within the standard probability calculus. In other words, we must gather the flock of separate sets Ω_M and σ -algebras back together, as subsets and subalgebras of one set only.

Second, the lattice structure of these σ -algebras reflects the operations of measurement mixing and dithering described in § 4, operations clearly arising from conditionalization within one σ -algebra only.

Third, the game of wearing blinkers in order to see seemingly separate σ -algebras can be played in non-quantal, everyday contexts, like card or urn-drawing games (Kirkpatrick 2003a,b; Porta Mana 2004b § IV). Are these also “quantal”?

What’s worse, this mathematical move inhibits an already non-existent dialogue between quantum theory and the theory of statistical models based on exchangeability, sufficiency, symmetry (for a glimpse see Bernardo et al. 2000 ch. 4) that has been flourishing in probability and statistics since the 1930s, with many brilliant results and papers – e.g. those by Koopman & Pitman (Koopman 1936; Pitman 1936; Darmonis 1935), Diaconis & Freedman (Freedman 1962a,b; Diaconis 1977; 1988; 1992; Diaconis et al. 1980a,b,c; 1981; 1987; 1988; 1990), Martin-Löf (1974), Lauritzen (1974a,b; 1988; 1984; 2007), Ressel (1985), Aldous (1981; 1982; 1985; 2010), Kallenberg (1989; 2005), Cifarelli, Regazzini, Fortini, et al. (Cifarelli et al. 1979; 1980; 1981; 1982; Regazzini 1996; Fortini et al. 2000; 2002; 2012; 2014), to name very few besides those by de Finetti (1930; 1937; 1938), already known in the quantum literature. See Dawid’s review (2013) for a small glimpse. Such a dialogue would surely benefit both disciplines, as I hope the ideas presented in the next section show.

6 Quantum theory as a partially exchangeable model

Many inferences in physics are instances of infinitely exchangeable statistical models (Bernardo et al. 2000 §§ 4.2–3):

$$p(D^{(1)}, D^{(2)}, \dots | H) = \int \left[\prod_i p(D^{(i)} | \theta, H) \right] p(\theta | H) d\theta, \quad (11)$$

where $D^{(i)} \in \{O_j\}$ are observed outcomes of a set of experiments (1), (2), ... made in identical conditions and each $p(D^{(i)} | \theta, H)$ is a

categorical distribution (i.e. one-trial multinomial). This expression can be interpreted as a mixture of product probabilities $p(D^{(i)} | \theta, H)$ indexed by the vector parameter θ , weighted by the distribution $p(\theta | H)$. The integration is defined over a simplex, but the distribution $p(\theta | H)$ can effectively restrict it to a subset thereof. The distribution on the left side is usually called the *predictive* distribution.

The integral formula above results automatically when we assume that the joint probability of any number of outcomes is invariant under their permutations, no matter how many outcomes we consider. This assumption is called *infinite exchangeability*, and this result is de Finetti's representation theorem (1930; Hewitt et al. 1955). The theorem leaves undetermined the distribution $p(\theta | H)$ only, usually called the *prior*. All infinitely exchangeable distributions over the outcomes are in one-one correspondence with all distributions $p(\theta | H)$.

The exchangeability assumption can in turn be motivated by the identical condition in which the experiments were made. The remarkable part of this representation is that it automatically introduces mathematical objects analogous to the statistical states (Liouville distributions) $\{s\}$ of a discrete classical system: $\{\theta\} \equiv \{s\}$. We can interpret it as saying that each experiment was independently made with the same – but unknown – preparation S . Hence the integral, with the prior $p(\theta | H) \equiv p(s | H)$ representing our knowledge H about the preparation. The features of this prior, like its support and maxima, may thus be motivated by physical laws or constraints.

Many authors (see the list at the end of § 5) later proved various generalizations of this representation theorem, extending it to predictive distributions invariant under other symmetry groups, or possessing sufficient statistics.

Inference for quantum systems does not quite fit within the simple statistical model above, however. As we saw in the previous sections, quantum systems allow for a set $\{M\}$ of distinct measurements that cannot be obtained by marginalization from one another. Inferences for such systems therefore require that the conditional of the predictive distribution above specify which measurements $M^{(i)} \in \{M\}$ are performed, as shown in the probabilities of the previous sections. If we again interpret these experiments as independently made with the same but unknown preparation, we arrive (Porta Mana et al. 2006) at the

expression

$$p(D^{(1)}, D^{(2)}, \dots | M^{(1)}, M^{(2)}, \dots, H) = \int_{\text{conv } \mathbb{CP}^{n-1}} \left[\prod_i p(D^{(i)} | M^{(i)}, s, H) \right] p(s | H) ds. \quad (12)$$

The integration is over the convex hull of complex projective space $\mathbb{CP}^{n-1}$ (Bengtsson et al. 2006), as explained in § 3, where n is the number of quantum states completely distinguishable with a single measurement.

The expression above is a particular case of a *partially exchangeable model* (Bernardo et al. 2000 §§ 4.6; Gelman et al. 2014 ch. 5). The assumption of *partial* exchangeability states that the predictive distribution is invariant under permutations of outcomes of the same kind of measurement, but not across different kinds of measurement. This makes sense also because different measurements may have different numbers of outcomes. For examples of when and why this kind of assumption arises see Bernardo et al. (2000 ch. 4).

The assumption of partial exchangeability leads to a representation theorem too (de Finetti 1938; Bruno 1964; Diaconis et al. 1980b), of the form

$$p(D^{(1)}, D^{(2)}, \dots | M^{(1)}, M^{(2)}, \dots, H) = \int \left[\prod_i p(D^{(i)} | M^{(i)}, \eta_{M^{(i)}}, H) \right] p[(\eta_M) | H] \prod_M d\eta_M, \quad (13)$$

where each outcome $D^{(i)}$ belongs to the set of possible outcomes of measurement $M^{(i)}$, these measurements belong to the system's set of possible measurements, $M^{(i)} \in \{M\}$, and each $p(D^{(i)} | M^{(i)}, \eta_{M^{(i)}}, H)$ is a categorical distribution. Just like the expression (11) for infinite exchangeability, also this expression is a mixture of products of distributions indexed by parameters (η_M) , one for each kind of measurement, weighted by the prior $p[(\eta_M) | H]$. The integration is defined over the *Cartesian product of simplices* $\prod_M \{(\eta_M)\}$, but the prior can effectively restrict it to a subset thereof. There is again a one-one correspondence between all partially exchangeable predictive distributions (left side) and all priors. The expression (11) for infinite exchangeability is a special case of the one above when $\{M\}$ comprises only one kind of measurement.

This generalized representation theorem is remarkable because it also automatically introduces a mathematical object, the parameter space

$\{(\eta_M)\}$, which is similar to a space of states. We can interpret it as saying that each experiment was independently made with the same – but unknown – preparation (η_M) . Also in this case the prior $p[(\eta_M)|H]$ expresses our knowledge H about the possible preparations, and its functional features can be motivated by physical laws or constraints.

We said that the inferential formula (12) for quantum systems is a particular case of the expression (13) for partial exchangeability. Let's see what additional features make it a particular case. First we have to identify the states s in the former with the parameters (η_M) in the latter. Then we see that it is a particular case because *the support of the prior $p[(\eta_M)|H]$ is restricted to a particular lower-dimensional convex subset of the Cartesian product*: the convex hull of a complex projective space. This restriction reflects the statistical properties of quantum systems, and is remarkably strong, because it reduces the support of the prior distribution from an infinite-dimensional manifold to a finite-dimensional one, for example from the function space $\{f \mid f: \mathbf{RP}^2 \rightarrow [0, 1]\}$ to the three-dimensional ball $\mathbf{CP}^1$ in the case of a qubit.

The full partially exchangeable model (13) allows for more general cases: less constrained than the quantum case, e.g. where each measurement has an outcome having probability 1; and more constrained than the quantum case, e.g. where no measurement can ever have a sure outcome.

For a quantum system, the a priori restriction on the prior of the partially exchangeable model reflects our empirical observation of measurement-outcome constraints typical of such systems: uncertainty relations, etc.. For example, if preparing an electron spin in a particular way we have probability 1 of obtaining $+z$ in a measurement along the z direction, then with the same preparation we cannot have probability 1 or 0 of obtaining $+x$ in a measurement along the x direction. We still don't know why such constraints exist. The mathematical formalism of quantum theory expresses and enforces these constraints, but doesn't explain why they exist either; just like the equation $dS/dt \geq 0$ (Truesdell 1984; Owen 1984) expresses and enforces the empirically found second law of thermodynamics, but doesn't explain why it must be so.

The exchangeability representation theorems (11) and (13) are welcome by many scientists, including yours faithfully, because they pull the notion of *state* out of the hat, thus also demoting it to a secondary, in principle avoidable notion. And they do so by promoting the notion

of repeated, reproducible experiments, which science indeed hinges on. This point of view has been forcefully promoted by some probabilists in recent years (see e.g. Cifarelli et al. 1982; Regazzini 1996; Fortini et al. 2000).

The partially exchangeable model, including quantum inferences as special cases, thus demotes quantum states too. But it does so without de-emphasizing the physical and inferential properties characteristic of quantum systems, properties still reflected in the peculiar constraints of the model's prior. And moreover it emphasizes the necessity of considering *several* distinct measurements when dealing with quantum systems. These emphases should be contrasted with the features of the "quantum" exchangeability representation theorem by Hudson et al. (Hudson et al. 1976; Hudson 1981), neatenened by Caves et al. (2002). This representation is surely useful in applications (van Enk et al. 2002; Fuchs et al. 2004). But it's tailor-made for quantum systems and therefore partially veils their peculiarity, if only by not openly showing the whole infinite-dimensional space of "unknown states" allowed by the full model; and it also veils the fact that quantum inferences need an assumption of *partial* exchangeability.

In the theory of exchangeable models it is known (Lauritzen 1988; Ressel 1985; Diaconis 1988; 1992; Kallenberg 2005; Dawid 2013) that the symmetries of a predictive distribution imply a particular form of its *likelihood* and the space of parameters – hence the support of the prior – in the integral representation. This leads to interesting questions. . .

- . . . for quantum theory: What kinds of symmetries or physical laws could cause the particular restrictions on the prior of the partially exchangeable model? The quantum literature offers studies of possible such symmetries (Wigner 1959; Haag 1996; Hølevø 2011), but their discussion disregards partially exchangeable models.
- . . . for the theory of exchangeable models: In which other contexts can or do analogous restrictions on the prior appear? Say, finance? biology? In which other contexts could they fruitfully be employed?

"Consent with both theories that they may enjoy each other: it shall be to our good" (Shakespeare 1623 scene V.II, paraphrased).

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Appendices

A Transformations

In the operational approach, the *transformation* of a preparation is some knowledge T , possibly dependent on a parameter like time (hence a collection of such propositions), that allows us to infer one preparation from another, possibly at different times, and possibly across different physical systems:

$$P(\hat{S} | T \wedge S) = q. \quad (14)$$

This notion can be shown to subsume the usual notions of deterministic evolution, stochastic evolution, and collapse, and can be naturally combined with all the probabilistic formulae we have seen so far. Every transformation can be associated with a linear map acting on the state vectors: $\hat{s} = Ts$. We can also consider mixtures of transformations, and so on.

It's important to note that Schrödinger's equation describes the time-dependence of *the linear map* T associated with a particular transformation T – not of the probability (14) itself.

B Traditional quantum theory from the operational approach

In the space of such matrices of positive-definite Hermitean matrices of dimension n we can always find n orthogonal projectors $\{\Pi_i\}$ such that $\Pi_i \Pi_j = \delta_{ij} \Pi_i$. Such projectors have unit trace; they can thus represent the density matrix associated with a preparation. They can also be written as $|\psi_i\rangle\langle\psi_i|$, where $\langle\psi_i|$ is a unit complex vector and $|\psi_i\rangle$ its dual. The sum of orthogonal projectors is the identity matrix; a set of orthogonal projectors can thus be associated with the outcomes of a measurement, too.

When we consider a preparation and a measurement outcome associated with orthogonal projectors E_k, E_i , the trace formula becomes the famous

$$P(O_i | M \wedge S) = \text{tr } E_i E_k \equiv |\langle \psi_i | \psi_k \rangle|^2. \quad (15)$$

If the outcomes $\{O_i\}$ of a measurement associated with orthogonal projectors are numerical values $\{\lambda_i\}$, the expected value is another famous formula:

$$\sum_i \lambda_i P(O_i | M \wedge S) = \sum_i \lambda_i \text{tr } E_i \rho = \langle \psi_k | \left[\sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \right] | \psi_k \rangle, \quad (16)$$

where the expression in brackets is a Hermitean operator with real spectrum.

Orthogonal projector matrices are associated with preparations and measurements which jointly lead to completely certain outcomes, in the sense of eq. (4). The more general density matrices and positive-operator-valued measures were introduced to describe experimental situations in which noise sources make the preparation uncertain, and interaction with other systems during measurement can lead to noise in the outcomes or even to their proliferation (Busch et al. 1989; de Muynck 2002 ch. 7). The more general measurements associated with positive-operator-valued measures also include *simultaneous* measurements of conjugate quantities like position and momentum (Arthurs et al. 1965; Busch et al. 1984; Appleby 1998), which are routine in fields like quantum optics (Leonhardt 1997 ch. 6).

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- (“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)
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