Pricing Model with Credit Risk

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ABSTRACT

This paper presents a model for pricing derivatives with counterparty credit risk. Our study shows that the value of a defaultable interest rate swap is the sum of the values of individual defaultable swaplets. Each defaultable swaplet can be replicated by buying a risk-adjusted call option and selling a risk-adjusted put option. The risk-adjusting factors depend on hazard rates, recovery rates and settlement rules. The model gives us a better understanding of the impact of counterparty credit risk on swap value, credit value adjustment, swap rate and swap spread.

Key words: credit risk, derivative pricing, LIBOR market model, reduced-form model, credit valuation adjustment, swap spread.

1. Introduction

Interest rate swap (IRS) is one of the most popular financial derivatives. In the market, IRS's are quoted irrespective of credit ratings of counterparties. In another words, they are considered as default-free.

The valuation of default-free IRS's under a term structure of interest rates is a classical exercise. However, swap contracts are traded over-the-counter (OTC) where each party is exposed to the credit risk. Historical experience shows that credit risk often leads to significant losses. Therefore, one should incorporate the cost of the counterparty risk into the swap price.

Pricing defaultable derivatives or pricing the counterparty credit risk is a relatively new area of derivatives modeling and trading. Credit value adjustment (CVA) allows us to quantify counterparty credit risk as a single, measurable Profit & Loss number.

CVA is the difference between the risk-free trade value and the true (or risky or defaultable) trade value that takes into account the possibility of counterparty's default. Commonly, the CVA is not paid as a lump-sum upfront premium, but rather is structured into a funding spread.

Sundaresan (1991), Longstaff and Schwartz (1993), and Tang and Li (2007) simplify the problem by considering the IRS as a simple exchange of loans (receivable parts and payable parts). It is inappropriate to value a defaultable IRS by pricing the default risk of the promised gross payment from each party separately and then adding the two together. Because the promised cash flow exchange in an IRS is always netted. Another simplification consists of taking into account the presence of one risky counterparty only, as in Li (1998) and Arvanities and Gregory (2001). These approaches overlook the presence of bilateral default risk.

The first study on asymmetric defaultable IRS is conducted by Duffie and Huang (1996). They use a short rate interest rate model combined with a reduced-form default model and lead to numerical approximations by solving a recursive integral. Even the authors admit that it is a substantial complexity solution.

In this paper, we present an analytical model for valuing derivatives subject to default by both parties. While the general principles of the valuation model can be applied to other bilateral defaultable contingent claims, we focus on the valuation of defaultable IRS's in which both parties are exposed to credit risk.

The rest of this paper is organized as follows: Market models are discussed in Section 2. In section 3, the valuation of asymmetric defaultable IRS and CVA are elaborated. The results are presented in Section 4. The conclusions are given in Section 5.

2. Valuation Models

Consider an increasing maturity structure $0 \leq T_0 < T_1 < \dots < T_N$ from which expiry-maturity pairs of dates (T_{k-1}, T_k) for a family of spanning forward rates are taken. The simply compounded forward rate reset at t for forward period (T_{k-1}, T_k) is defined by

$$F_k(t) := F(t; T_{k-1}, T_k) = \frac{1}{\tau_k} \left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right) \quad (1)$$

where $P(t, T)$ denotes the time t price of a zero-coupon bond maturing at time T and $\tau_k := \tau(T_{k-1}, T_k)$ is the accrual factor or day count fraction for period (T_{k-1}, T_k) .

The name 'market model' refers to the modeling of market observable variables such as forward rates and swap rates. The explicit modeling market rates allows for natural formulas for interest rate option volatility, that are consistent with the market

practice of using the formula of Black for caps and swaptions. The typical market models are Black model, LIBOR market model (LMM) and swap market model (SMM).

The forward rate dynamics under the Black model is

$$dF_k(t) = \sigma_k F_k(t) dW_k(t) \quad (2)$$

where σ_k is the Black caplet volatility or the spot volatility of the forward rate F_k ; $W_k(t)$ is a Brownian motion

The solution to equation (2) can be expressed as

$$\begin{aligned} F_k(T) &= F_k(t) \exp \left(-\frac{\sigma_k^2}{2} (T-t) + \sigma_k (W_k(T) - W_k(t)) \right) \\ &= F_k(t) \exp \left(-\frac{\sigma_k^2}{2} (T-t) + \sigma_k W_k(T-t) \right) \end{aligned} \quad (3)$$

The LMM, in contrast to the Black model, describes the dynamic of a whole family of forward rates under a common measure.

Under the lognormal assumption and the forward measure Q^i , the forward rate $F_k(t)$ of LMM follows the following dynamics:

$$\begin{aligned} \text{Case 1.} \quad i < k, t \leq T_i: \quad dF_k(t) &= v_k(t) F_k(t) \sum_{j=i+1}^k \frac{\rho_{kj} \tau_j v_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt \\ &\quad + v_k(t) F_k(t) dW_k(t) \end{aligned} \quad (4)$$

$$\text{Case 2.} \quad i = k, t \leq T_{k-1} \quad dF_k(t) = v_k(t) F_k(t) dW_k(t) \quad (5)$$

$$\begin{aligned} \text{Case 3.} \quad i > k, t \leq T_{k-1} \quad dF_k(t) &= -v_k(t) F_k(t) \sum_{j=k+1}^i \frac{\rho_{kj} \tau_j v_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt \\ &\quad + v_k(t) F_k(t) dW_k(t) \end{aligned} \quad (6)$$

where $v_k(t)$ is the instantaneous volatility. We further assume that the Brownian motions $W_k(t)$ and $W_j(t)$ of different forward rates $F_k(t)$ and $F_j(t)$ are instantaneously correlated according to ρ_{kj} , i.e.,

$$dW_k(t)dW_j(t) = \rho_{kj}dt \quad (7)$$

For a vanilla instrument like swap or cap, the payoff can be decomposed additively into a sum of sub-payoffs (each is involved with a single forward rate and associated with a contract called swaplet or caplet).

Our closed-form solution shows that the value of a bilateral defaultable IRS is the sum of the values of individual bilateral defaultable swaplets. Each bilateral defaultable swaplet can be replicated by buying a risk-adjusted call option and selling a risk-adjusted put option. The risk-adjusting factors depend on hazard rates, recovery rates and settlement rules.

3. CVA

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions, where Ω denotes a sample space; $\mathcal{F}$ denotes a σ -algebra; $\mathcal{P}$ denotes a probability measure. Let valuation date be t . Consider a fixed-for-floating swap. Two counterparties are denoted as A and B. Counterparty A pays a fixed rate, while counterparty B pays a floating-rate.

Assume the IRS has the first reset date T_0 and payment dates $T_1, \dots, T_n$. There are total n future cash flows X_i . From the perspective of counterparty A, The X_i is given by

$$X_i = (F_i(T_{i-1}) - K) \tau_i$$

We are considering that fixed rate payments and floating-rate payments occur at the same payment dates and with the same day-count conventions, and ignoring the swap funding spread.

Risk-Free Valuation

The discounted payoff of the IRS is

$$payoff^{Free}(t) = \sum_{i=1}^n D(t, T_i) (F_i(T_{i-1}) - K) \tau_i \quad (8)$$

where $D(t, T_i)$ is the discount factor. In general, bond price is deterministic but discount factor is stochastic. If interest rates are deterministic, the bond price $P(t, T_i)$ and the discount factor $D(t, T_i)$ are equivalent.

The pricing of the IRS can be obtained by considering the risk-neutral expectation E of its discounted payoff:

$$\begin{aligned} pv^{Free}(t) &= E_t \left\{ \sum_{i=1}^n D(t, T_i) (F_i(T_{i-1}) - K) \tau_i \right\} \\ &= \sum_{i=1}^n P(t, T_i) [E_t^i (F_i(T_{i-1}) - K) \tau_i] = \sum_{i=1}^n P(t, T_{i-1}) (F_i(t) - K) \tau_i \end{aligned} \quad (9)$$

Where $E_t := E \{ \cdot | \mathcal{F}_t \}$ is the expectation conditional on the $\mathcal{F}_t$; E_t^i is the expectation under forward measure Q^i conditional on the $\mathcal{F}_t$.

The swap rate is the fixed rate that makes the market value of a given swap at initiation zero. The risk free swap rate is given by

$$S_{0,n}^{Free}(t) = \frac{\sum_{i=1}^n F_i(t) \tau_i P(t, T_i)}{\sum_{i=1}^n \tau_i P(t, T_i)} \quad (10)$$

Risky Valuation

There are two approaches to price defaultable financial instruments: Structural Approach and Reduced-Form Approach. The structural approach regards default as an endogenous event by focusing on the capital structure of the firm. Rendleman (1992) applies the structural model based on knowledge of the asset-liability structure of the

swap counterparties to the swaps market. The development of a structural model seems impractical given the data on generic market swap rates.

The reduced-form approach proposed by Duffie and Singleton (1999), and Jarrow and Turnbull (1994) did not explain the event of default endogenously, but characterized it exogenously by a jump process.

The stopping (default) time of party j ($j = A, B$) is modeled as a Poisson arrival with probability density function:

$$f_j(t) = h_j(t) \exp(-h_j(t)t) \quad (11)$$

where $h_j(t)$ is the hazard rate or the arrival intensity of a Poisson process whose first jump occurs at default. The stopping time for the IRS is defined as $\tau = \tau_A \wedge \tau_B$. It is well-known that the survival probability from t to T in this framework is given by

$$PS_j(t, T) = E \left[\exp \left(- \int_t^T h_j(u) du \right) \right]$$

In some previous studies, such as Duffie and Huang (1996), the bond spreads are used to determine credit quality. Both bond spreads and CDS spreads are closely related to each other. Conceptually a CDS spread is different from a bond spread and that they can differ in magnitude for technical reason. There is some empirical evidence that the CDS market typically leads the bond market, in particular during crisis situation.

A critical ingredient of the pricing of a defaultable IRS is the rules for settlement in default. There are two rules in the swap market. The “one-way payment (fault)” rule was specified by the early International Swap Dealers Association (ISDA) master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the IRS is positive for the defaulting party. The “two-way payment (no fault)” rule is based on the current ISDA documentation. In the event of default, if the contract has positive value to the non-defaulting party, the

defaulting party pays a fraction of the pre-default market value of the IRS to the non-defaulting party. If the contract has positive value to the defaulting party, the non-defaulting party will pay the full pre-default market value of the IRS to the defaulting party.

Consider any swaplet i . According to Duffie and Huang (1996), we can get the discounted defaultable payoff as

$$\begin{aligned} \text{payoff}_i^D(t) &= D(t, T_i) \left[X_i \exp \left(- \int_t^{T_i} R(u) du \right) \right] \\ &= D(t, T_i) \left[X_i \exp \left(- \int_t^{T_i} (1_{X_i \geq 0} s_B(u) + 1_{X_i < 0} s_A(u)) du \right) \right] \end{aligned} \quad (12)$$

where

$$\begin{aligned} R(u) &= 1_{X_i \geq 0} s_B(u) + 1_{X_i < 0} s_A(u) \\ s_B(u) &= (1 - \varphi_B(u)) h_B(u) + (1 - \bar{\varphi}_B(u)) h_A(u) \\ s_A(u) &= (1 - \varphi_A(u)) h_A(u) + (1 - \bar{\varphi}_A(u)) h_B(u) \end{aligned}$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise; φ_j represents the fraction of the cash flow X_i paid by the defaulting counterparty j when the cash flow is negative for j . The recovery rate φ_j is normally considered as deterministic, although it can be time-varying. $\bar{\varphi}_j$ represents the fraction of the cash flow X_i paid by non-defaulting counterparty j when the cash flow is negative for j . $\bar{\varphi}_j = 0$ represents the one-way settlement rule, while $\bar{\varphi}_j = 1$ represents two-way settlement rule. h_j is the hazard rate of counterparty j .

The default-adjusted spreads s_B and s_A in (12) have general forms that apply in several particular situations. Under the one-way settlement rule, we have

$$s_B(u) = (1 - \varphi_B(u)) h_B(u) + h_A(u)$$

$$s_A(u) = (1 - \varphi_A(u))h_A(u) + h_B(u)$$

Under the two-way settlement rule, we have

$$s_B(u) = (1 - \varphi_B(u))h_B(u)$$

$$s_A(u) = (1 - \varphi_A(u))h_A(u)$$

Equation (12) can be rewritten as

$$\begin{aligned} payoff_i^D(t) &= D(t, T_i) \left[X_i 1_{X_i \geq 0} \exp\left(-\int_t^{T_i} s_B(u) du\right) + X_i 1_{X_i < 0} \exp\left(-\int_t^{T_i} s_A(u) du\right) \right] \\ &= D(t, T_i) [C_B(t, T_i) X_i 1_{X_i \geq 0} + C_A(t, T_i) X_i 1_{X_i < 0}] \end{aligned} \quad (13)$$

where $C_j(t, T_i) = \exp\left(-\int_t^{T_i} s_j(u) du\right)$ is a deterministic function under a deterministic

default intensity model and a deterministic recovery assumption.

An IRS is a linear product. The total discounted defaultable payoff of the IRS is just the sum of the discounted defaultable payoffs of the future swaplets, that is,

$$payoff^D(t) = \sum_{i=1}^n D(t, T_i) [C_B(t, T_i) X_i 1_{X_i \geq 0} + C_A(t, T_i) X_i 1_{X_i < 0}] \quad (14)$$

The price of the defaultable IRS can be obtained by considering the risk-neutral expectation E of its discounted defaultable payoff:

$$\begin{aligned} pv^D(t) &= E_t \left\{ \sum_{i=1}^n D(t, T_i) [C_B(t, T_i) X_i 1_{X_i \geq 0} + C_A(t, T_i) X_i 1_{X_i < 0}] \right\} \\ &= \sum_{i=1}^n P(t, T_i) E_t^i [C_B(t, T_i) X_i 1_{X_i \geq 0} + C_A(t, T_i) X_i 1_{X_i < 0}] \\ &= \sum_{i=1}^n P(t, T_i) [C_B(t, T_i) E_t^i(X_i 1_{X_i \geq 0}) + C_A(t, T_i) E_t^i(X_i 1_{X_i < 0})] \end{aligned} \quad (15)$$

The key to value an asymmetric defaultable IRS is to calculate the expectations:

$$E_t^i(X_i 1_{X_i \geq 0}) \text{ and } E_t^i(X_i 1_{X_i < 0}).$$

According to (3), we can calculate

$$\begin{aligned}
E_t^i(X_i 1_{X_i \geq 0}) &= E_t^i[(F_i(T_{i-1}) - K) \tau_i 1_{F_i(T_{i-1}) - K \geq 0}] \\
&= \tau_i \int_{-d_{i2}}^{\infty} \left[F_i(t) \exp\left(-\frac{\sigma_i^2(T_{i-1} - t)}{2} + \sigma_i \sqrt{T_{i-1} - t} x\right) - K \right] \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (16) \\
&= \tau_i (F_i(t) \Phi(d_{i1}) - K \Phi(d_{i2}))
\end{aligned}$$

where x is a standard normal variable; Φ is the standard normal cumulative distribution function; $d_{i1,i2} = \frac{\ln(F_i(t)/K) \pm \sigma_i^2(T_i - t)/2}{\sigma_i \sqrt{T_i - t}}$. Equation (16) is a standard

Black formula on a call option.

Similarly, we have

$$E_t^i(X_i 1_{X_i < 0}) = E_t^i[(F_i(T_{i-1}) - K) \tau_i 1_{F_i(T_{i-1}) - K < 0}] = -\tau_i [K \Phi(-d_{i2}) - F_i(t) \Phi(-d_{i1})] \quad (17)$$

This is a standard Black formula on a put option.

Therefore, the value of a defaultable swaplet is a risk-adjusted call option minus a risk-adjusted put option. In other words, a defaultable swaplet can be replicated by buying a risk-adjusted call option and selling a risk-adjusted put option, that is:

$$\begin{aligned}
pv^D(t) &= P(t, T_i) C_B(t, T_i) E_t^{T_i} [1_{X_{T_i} \geq 0} X_{T_i}] + P(t, T_i) C_A(t, T_i) E_t^{T_i} [1_{X_{T_i} < 0} X_{T_i}] \\
&= P(t, T_i) \tau_i \{C_B(t, T_i) (F_i(t) \Phi(d_{i1}) - K \Phi(d_{i2})) - C_A(t, T_i) (K \Phi(-d_{i2}) - F_i(t) \Phi(-d_{i1}))\} \quad (18)
\end{aligned}$$

The price of the defaultable IRS is

$$\begin{aligned}
pv^D(t) &= \sum_{i=1}^n P(t, T_i) [C_B(t, T_i) E_t^i(X_i 1_{X_i \geq 0}) + C_A(t, T_i) E_t^i(X_i 1_{X_i < 0})] \\
&= \sum_{i=1}^n P(t, T_i) \tau_i \{C_B(t, T_i) (F_i(t) \Phi(d_{i1}) - K \Phi(d_{i2})) \\
&\quad - C_A(t, T_i) (K \Phi(-d_{i2}) - F_i(t) \Phi(-d_{i1}))\} \quad (19)
\end{aligned}$$

If both parties have the same credit quality (symmetric credit risk) and follow two-way settlement rule, i.e.,

$$C(t, T_i) = C_A(t, T_i) = C_B(t, T_i) = \exp\left(-\int_t^{T_i} s(u)du\right) = \exp\left(-\int_t^{T_i} h(u)(1 - \varphi(u))du\right),$$

equation (19) can be expressed as

$$\begin{aligned} pv^D(t) &= \sum_{i=1}^n P(t, T_i) \tau_i C(t, T_i) \{ (F_i(t) \Phi(d_{i1}) - K \Phi(d_{i2})) \\ &\quad - (K \Phi(-d_{i2}) - F_i(t) \Phi(-d_{i1})) \} \\ &= \sum_{i=1}^n P(t, T_i) \tau_i C(t, T_i) (F_i(t) - K) \end{aligned} \quad (20)$$

where $\Phi(x) + \Phi(-x) = 1$.

This is exactly the formula of Duffie and Singleton (1997). The defaultable swap rate (K) under asymmetric credit risk can only be solved numerically, because K is also contained in d_{i1} and d_{i2} . However, the analytic solution exists for the defaultable swap rate under symmetric credit risk. The symmetric defaultable swap rate that makes the market value of the IRS at initiation zero is

$$S_{0,n}^D(t) = \frac{\sum_{i=1}^n P(t, T_i) \tau_i F_i(t) C(t, T_i)}{\sum_{i=1}^n P(t, T_i) \tau_i C(t, T_i)} \quad (21)$$

4. Conclusion

In this paper we present an analytical model for pricing defaultable IRS's with asymmetric credit qualities. The model is based on the market models of interest rate dynamics and the reduced-form model of default time.

The object modeled under the market models is market-observable. It is also consistent with the market standard approach for pricing caps/floors using Black's formula. The market models have now become some of the most popular models for pricing such derivatives. They are generally considered to have more desirable theoretical calibration properties than short rate or instantaneous forward rate models.

Unlike structural models, reduced-form models do not condition default explicitly on the value of the firm, and parameters related to the firm's value need not be estimated to implement the model. For pricing and hedging, reduced-form models are the preferred methodology.

In the case where the floating-rate payer is a LIBOR party, we confirm findings in Duffie and Huang (1996) that the swap spreads are relatively less sensitive to credit quality comparing to the bond spreads or the CDS spreads. The credit impact on swap rates is approximately linear. In the case where both parties have spreads against the LIBOR that was not studied closely before, we find that the credit impact on swap rates is not linear any more. The swap funding spreads have a significantly impact on swap spreads as well.

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