Open Journal of Mathematics and Physics | Volume 2, Article 106, 2020 | ISSN: 2674-5747
https://doi.org/10.31219/osf.io/kxnt8 | published: 2 Jun 2020 | https://ojmp.org

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3-cycle commutator

Open Mathematics Collaboration*†

June 13, 2020

Abstract

We show within the maximum number of steps that if only one point is moved by two permutations, the commutator is a 3-cycle.

keywords: permutations, group theory, abstract algebra

The most updated version of this paper is available at
https://osf.io/kxnt8/download

Introduction

1. This is an open science experiment.

2. The experiment aims to foster reproducibility and transparency in pure mathematics by increasing the number of pedagogical papers and presenting as many steps as possible to prove a given mathematical proposition/theorem.

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Theorem

3. If g and h are permutations with the property that only one point is moved by both g and h, then the *commutator* $g^{-1}h^{-1}gh$ is a 3-cycle [1–3].

Algorithm

4. We will show/calculate the following:

   (a) $(g \neq h) \rightarrow (\alpha g \neq \alpha h)$;

   (b) α, αg, and αh are three distinct points, $\alpha h \neq \alpha \neq \alpha g \neq \alpha h$;

   (c) $\beta \neq \alpha, \alpha g, \alpha h$;

   (d) $(\beta g^{-1} = \beta h^{-1} = \beta) \rightarrow (\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1})$;

   (e) $\beta g \neq \beta$;

   (f) $\beta h^{-1} = \beta$;

   (g) $(\beta g^{-1} \neq \beta h^{-1}) \rightarrow (\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1})$;

   (h) $\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1}$;

   (i) $\beta[g,h] = \beta$;

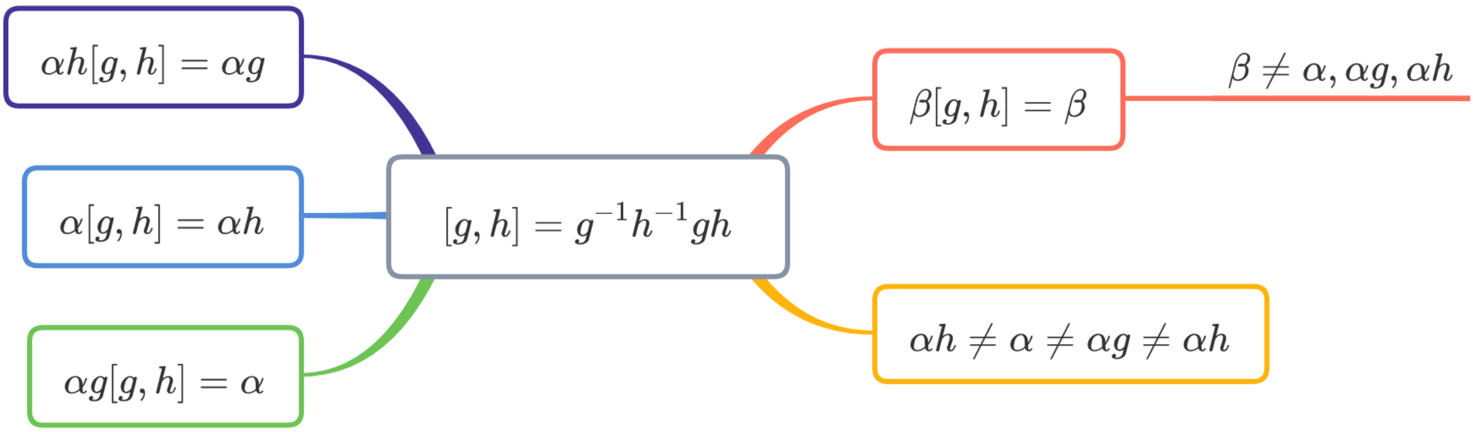
   (j) $\alpha g[g,h] = \alpha$;

   (k) $\alpha[g,h] = \alpha h$;

   (l) $\alpha h[g,h] = \alpha g$.

Overview

5. $[4, 5]$



Proof

6. g, h = distinct permutations on a set Ω
7. $\exists! \alpha \in \Omega$ moved by both g and h
8. $[g, h] = g^{-1}h^{-1}gh$ = commutator
9. Notation: $g(h(\alpha)) \equiv \alpha hg$.
10. Since both g and h move α, $\alpha \neq \alpha g$ and $\alpha \neq \alpha h$.
11. Since $g \neq h$, $\alpha g \neq \alpha h$.
12. Thus, from (10) and (11), α, αg, and αh are three distinct points, namely, $\alpha h \neq \alpha \neq \alpha g \neq \alpha h$.
13. Suppose $\beta \neq \alpha, \alpha g, \alpha h$.
14. There are two possibilities, either $\beta g^{-1} = \beta h^{-1} = \beta$ or $\beta g^{-1} \neq \beta h^{-1}$.
15. Case 1: $(\beta g^{-1} = \beta h^{-1} = \beta) \to (\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1})$.
16. Case 2: suppose g^{-1} moves β, $\beta g^{-1} \neq \beta$.
17. $(\beta g^{-1} \neq \beta) \leftrightarrow (\beta g^{-1}g \neq \beta g) \leftrightarrow (\beta \neq \beta g)$
18. Thus g moves β, $\beta g \neq \beta$.
19. Since $\beta g \neq \beta \neq \alpha$, due to (7), $\beta h^{-1} = \beta$.
20. Inserting $\beta = \beta h^{-1}$ in βg^{-1}, we have $\beta g^{-1} = \beta h^{-1}g^{-1}$.
21. Since $\beta g^{-1}g = \beta$, g moves βg^{-1}.
22. Since g moves βg^{-1}, $\beta g^{-1}g = \beta \neq \alpha$, h^{-1} fixes βg^{-1}, $\beta g^{-1}h^{-1} = \beta g^{-1}$.
23. Thus, from (20) and (22), $\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1}$.
24. Therefore, from (16) and (23), $(\beta g^{-1} \neq \beta h^{-1}) \to (\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1})$

25. Both cases 1 and 2 lead to the conclusion $\beta g^{-1}h^{-1} = \beta h^{-1}g^{-1}$.

26. It follows that $\beta[g,h] = \beta g^{-1}h^{-1}gh = \beta h^{-1}g^{-1}gh = \beta h^{-1}h = \beta$.

27. Then $\beta[g,h] = \beta$.

28. (27) shows that $[g,h]$ fixes every member of Ω other than α, αg, and αh.

29. Now let's calculate

    (a) $\alpha g[g,h]$,
    (b) $\alpha[g,h]$,
    (c) $\alpha h[g,h]$.

30. $\alpha g[g,h] = \alpha g g^{-1}h^{-1}gh = \alpha h^{-1}gh$

31. Note that $\alpha h^{-1}h = \alpha$, which means that h moves αh^{-1}.

32. Since h moves αh^{-1} and due to (7), g fixes αh^{-1}, i.e., $\alpha h^{-1}g = \alpha h^{-1}$.

33. Then, using (32) in (30), $\alpha g[g,h] = \alpha h^{-1}gh = \alpha h^{-1}h = \alpha$.

34. Therefore, $\alpha g[g,h] = \alpha$.

35. Interchanging g and h in (34), $\alpha h[h,g] = \alpha$.

36. Then $\alpha[h,g]^{-1} = \alpha h$.

37. Note that $[h,g]^{-1} = (h^{-1}g^{-1}hg)^{-1} = g^{-1}h^{-1}gh = [g,h]$.

38. Thus $\alpha[g,h] = \alpha h$.

39. <u>Definition</u>: a permutation of a set Ω is a bijective map $\Omega \to \Omega$.

40. <u>Proposition</u>: the commutator is a permutation.

41. Since (28), and due to (39) and (40), we have $\alpha h[g,h] = \alpha g$.

42. In summary, from (34), (38) and (41),

   (a) $\alpha g[g,h] = \alpha$;
   (b) $\alpha[g,h] = \alpha h$;
   (c) $\alpha h[g,h] = \alpha g$.

43. Therefore, the commutator $[g,h]$ is a 3-cycle. $\square$

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Open Science

The **latex file** for this paper together with other *supplementary files* are available [5].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [8], please cite it accordingly [9]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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