[white paper]

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[waiting peer review]

Categories in symbols

Open Mathematics Collaboration^{*†}

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Abstract

An introduction to the language of category theory is presented in a very minimalistic fashion, using the fewest number of symbols as possible. In this white paper, the focus is on the main concepts underlying categories.

keywords: category theory, functor, diagram, duality, categorical product

The most updated version of this white paper is available at

<https://osf.io/k9fbc/download>
<https://zenodo.org/record/5328443>

Introduction

1. This is the first paper of the “in symbols” *series* published in the *Open Journal of Mathematics and Physics*.
2. The mathematical content is presented in the Appendix, after the References.
3. Our purpose is pedagogical so that one can easily have an overview of the mathematical definitions and results around the main topic.

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4. Anyone familiarized with the basic mathematical symbols should be able to understand the material, with more or less effort depending on the mathematical maturity of the reader.
5. There are excellent introductory references about **Category Theory** in [1–4].
6. If you need further assistance, check [5, 6].

Electronic version

7. *Categories in symbols* as a mathematical (searchable) knowledge base is also available at

<https://omkb.notion.site/9fadacf7796640eba3a5eccf2dc30a2b>

or

<https://bit.ly/2Y7Zw2D>

Meta-linguistic symbols

8. $:=$ means that what is on the left is defined by what is on the right
9. $\equiv$ means that the strings on both sides are identical

Topics

10. *The following definitions are included in the Appendix.*
11. Category
12. Class of objects
13. Morphisms (maps, arrows)

14. Hom-set
15. Identity morphisms
16. Class of all morphisms
17. Small/Large category
18. Morphism leaving/entering an object
19. Examples of the categories of sets, monoids, (abelian) groups, R -modules, vector spaces over a field, (commutative) rings, fields, partially ordered sets (posets), relations, topological spaces, manifolds with smooth maps, and categories of the propositional calculus
20. Preordered (thin) category
21. Covariant/Contravariant functor
22. Local arrow part (restriction) of a functor
23. Set valued functor
24. Parallel/Antiparallel functors
25. Composition of functors
26. Special types of functors: full, faithful, and fully faithful
27. Embedding of a category in other category
28. (Contravariant) Power set functor
29. Forgetful functor
30. Underlying-set functor
31. (Large) Category of all small categories

- 32. Concrete/Abstract category
- 33. Subcategory
- 34. Diagrams
- 35. Directed graph (digraph)
- 36. Class of vertices (nodes)
- 37. Set of arcs (leaving/entering)
- 38. Loop
- 39. In/Out-degree
- 40. Labeled digraph
- 41. Directed path
- 42. Length of a path
- 43. Underlying digraph
- 44. Commutative diagrams
- 45. Morphisms left and right-invertible
- 46. Category isomorphism (invertible, i.e., with two-sided inverse)
- 47. Morphisms left (monic) and right-cancellable (epic)
- 48. Initial, terminal, and zero objects
- 49. Zero morphism
- 50. Dual (opposite) category
- 51. Dual property

- 52. Self-dual
- 53. Dual statement
- 54. Product category
- 55. Bifunctor
- 56. Category of arrows
- 57. Comma category (slice, coslice)
- 58. Source/Target object
- 59. Comma objects
- 60. Category of elements
- 61. Hom-set category
- 62. Product of objects
- 63. Mediating morphism
- 64. Projection maps
- 65. Poset category (supremum)
- 66. Binary products for a category
- 67. Product morphism
- 68. Categorical product
- 69. Mediating morphism trick

Open Invitation

Review, add content, and co-author this white paper [7, 8].

*Join the **Open Mathematics Collaboration**.*

Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [9, 10].

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<https://osf.io>

+ **Zenodo**

<https://zenodo.org>

Agreement

All authors agree with [8].

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Categories in symbols

FIVE BASIC CONCEPTS

- Categories
 - Functors
 - Natural transformations
 - Universality
 - Adjoints
-

DEFINITION

$\mathcal{C} :=$ **CATEGORY**

1. OBJECTS

$\text{Obj}(\mathcal{C}) :=$ **class of objects**

$A \in \text{Obj}(\mathcal{C}) \equiv A \in \mathcal{C}$

2. MORPHISMS

$\text{hom}_{\mathcal{C}}(A, B) :=$ **hom-set** for the pair (A, B)

$A, B \in \mathcal{C}$ (A and B are not necessarily distinct pair of objects)

elements of $\text{hom}_{\mathcal{C}}(A, B) :=$ **morphisms, maps** or **arrows** from A to B

$f \in \text{hom}_{\mathcal{C}}(A, B) \equiv f : A \rightarrow B \equiv f_{AB}$

$A, B :=$ objects of $\mathcal{C}$

$A = \text{dom}(f) :=$ **domain** of f

$B = \text{codom}(f) :=$ **codomain** of f

morphisms are not necessarily functions

3. DISJUNCTION

$\text{hom}_{\mathcal{C}}(A, B) \cap \text{hom}_{\mathcal{C}}(C, D) = \emptyset$ unless $A = C$ and $B = D$

4. COMPOSITION

$g \circ f \in \text{hom}_{\mathcal{C}}(A, C) :=$ **morphism** $:=$ **composition** of g with f

$f_{AB} \equiv f \in \text{hom}_{\mathcal{C}}(A, B)$

$g_{BC} \equiv g \in \text{hom}_{\mathcal{C}}(B, C)$

$\exists g \circ f, f \circ (g \circ h) = (f \circ g) \circ h$, associative

$g \circ f$ tacitly means $\text{dom}(g) = \text{codom}(f)$

5. IDENTITY MORPHISMS

$$\forall A \in \mathcal{C} \quad \exists 1_A \in \text{hom}_{\mathcal{C}}(A, A)$$

$$(f_{AB} \in \text{hom}_{\mathcal{C}}(A, B)) \rightarrow (1_B \circ f_{AB} = f_{AB} \wedge f_{AB} \circ 1_A = f_{AB})$$

1_A := **identity morphism** for A

$\text{Mor}(\mathcal{C})$:= **class of all morphisms** of $\mathcal{C}$

NOTATIONS

$\text{hom}_{\mathcal{C}}(A, B) := (A, B) := [A, B] := \mathcal{C}(A, B) := \text{Mor}(A, B)$

- not all authors require property 3 in the definition of a category
 - for some authors, hom-sets are classes
-

locally small category := the **hom-classes** are sets

small category := $\text{Obj}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ are sets

large category := not a small category

$(h_1, h_2 \in \text{hom}_{\mathcal{C}}(A, B)) \rightarrow (h_1 \text{ and } h_2 \text{ are **parallel**})$

$A = \text{dom}(f) := f$ is a morphism **leaving** A

$B = \text{codom}(f) := f$ is a morphism **entering** B

EXAMPLES OF CATEGORIES

Set := category of *sets*

Obj := class of all *sets*

$\text{hom}(A, B)$:= set of all *functions* from A to B

Mon := category of *monoids*

Obj := class of all *monoids*

$\text{hom}(A, B)$:= set of all *monoid homomorphisms* from A to B

Grp := category of *groups*

Obj := class of all *groups*

$\text{hom}(A, B)$:= set of all *group homomorphisms* from A to B

AbGrp := category of *abelian groups*

Obj := class of all *abelian groups*

$\text{hom}(A, B)$:= set of all *group homomorphisms* from A to B

Mod_R := category of *R-modules* (R := ring)

Obj := class of all *R-modules*

$\text{hom}(A, B)$:= set of all *R-maps* from A to B

Vect_F := category of *vector spaces over a field F*

Obj := class of all *vector spaces over F*

$\text{hom}(A, B)$:= set of all *linear transformations* from A to B

Rng := category of *rings*

Obj := class of all *rings*

$\text{hom}(A, B)$:= set of all *ring homomorphisms* from A to B

CRng := category of *commutative rings*

Obj := class of all *commutative rings* (with identity)

$\text{hom}(A, B)$:= set of all *ring homomorphisms* from A to B

Field := category of *fields*

Obj := class of all *fields*

$\text{hom}(A, B)$:= set of all *ring embeddings* from A to B

Poset := category of *partially ordered sets*

Obj := class of all *partially ordered sets*

$\text{hom}(A, B)$:= set of all *monotone functions* from A to B

$f : P \rightarrow Q$ such that $p \leq q \Rightarrow f(p) \leq f(q)$

Rel := category of *relations*

Obj := class of all *sets*

$\text{hom}(A, B)$:= set of all *binary relations* from A to B

Top := category of *topological spaces*

Obj := class of all *topological spaces*

$\text{hom}(A, B)$:= set of all *continuous functions* from A to B

SmoothMan := category of *manifolds with smooth maps*

Obj := class of all *manifolds*

$\text{hom}(A, B)$:= set of all *smooth maps* from A to B

$$\mathcal{P} := \mathbf{Poset}(P, \leq)$$

$\mathcal{P} :=$ **category poset**

$(P, \leq) :=$ partially ordered set (poset)

$a, b, c \in P; a, b, c \in \mathbf{Poset}(P, \leq)$

$a, b, c :=$ objects of $\mathcal{P}$

$a > b \rightarrow \text{hom}(a, b) = \emptyset$

$a \leq b \rightarrow \text{hom}(a, b) = \{ab\}$

hom-sets specify $\leq$ on P

$(ab : a \rightarrow b) \wedge (bc : b \rightarrow c) \rightarrow a \leq b \leq c \rightarrow a \leq c \rightarrow \text{hom}(a, c) \neq \emptyset$

COMPOSITION: $bc \circ ab = ac$

$\text{hom}(a, a) = \{1_a\}; 1_a :=$ identity morphism for a

$$\mathbf{n} := \text{category } \mathbf{Poset}(\mathbb{N}, \leq)$$

$m, n \in \mathbb{N}; n = \{0, 1, \dots, n-1\}; 0 = \emptyset; m < n \rightarrow m \in n$

objects of $\mathbf{n} =: 0, 1, \dots, n-1 \in n$

the class of all morphisms of $\mathcal{C} =: \mathbf{Mor}(n) = \{\leq\}$

preordered (thin) category := there is at most one morphism between every pair of objects

$\exists f_{AB} \rightarrow$ we can define $A \preceq B \rightarrow \mathcal{C} :=$ **thin category**

$\exists! f_{AB} \in \mathcal{P} \leftrightarrow A \preceq B$

$\preceq$ **preorder** relation (reflexive, transitive) on the objects of $\mathcal{C}$

reflexivity $\leadsto$ identity; transitivity $\leadsto$ composition

$\mathcal{P} \equiv (P, \preceq) :=$ preordered class (category)

$p \in \mathcal{P} \rightarrow p \in P$

$\mathcal{C} :=$ category; $A, B \in \mathcal{C}$

$\forall \mathcal{C} : \exists f_{AB} \rightarrow A \preceq B$

$$\exists f_{\alpha\beta} \leftrightarrow \alpha \vdash \beta$$

$\alpha \vdash \beta := \alpha$ deduces β

$f_{\alpha\beta} \in \mathfrak{P}_1; \quad f'_{\alpha\beta} \in \mathfrak{P}_2$

$\mathfrak{P}_1, \mathfrak{P}_2 :=$ **categories** of the **propositional calculus**

wffs := well-formed formulas (of the system) := objects of $\mathfrak{P}_c$

$f'_{\alpha\beta} :=$ a *specific deduction* of β from α

$$F : \mathcal{C} \Rightarrow \mathcal{D}$$

(object part) $F : \mathbf{Obj}(\mathcal{C}) \rightarrow \mathbf{Obj}(\mathcal{D})$

(arrow part) $F : \mathbf{Mor}(\mathcal{C}) \rightarrow \mathbf{Mor}(\mathcal{D})$

$F := \mathbf{functor}$ (pair of functions) from $\mathcal{C}$ to $\mathcal{D}$

$\mathcal{C}, \mathcal{D} := \text{categories}$

COVARIANT, CONTRAVARIANT FUNCTOR

covariant functor := *preserves the direction of arrow*

contravariant functor := *reverses the direction of arrow*

$(F := \text{covariant}) \rightarrow (\forall A, B \in \mathcal{C}, F : \text{hom}_{\mathcal{C}}(A, B) \rightarrow \text{hom}_{\mathcal{D}}(FA, FB))$
 $F : \text{hom}_{\mathcal{C}}(A, B) \rightarrow \text{hom}_{\mathcal{D}}(FA, FB) \equiv F \text{ maps } f : A \rightarrow B \text{ in } \mathcal{C} \text{ to } Ff : FA \rightarrow FB \text{ in } \mathcal{D}$

$(F := \text{contravariant}) \rightarrow (\forall A, B \in \mathcal{C}, F : \text{hom}_{\mathcal{C}}(A, B) \rightarrow \text{hom}_{\mathcal{D}}(FB, FA))$
 $F : \text{hom}_{\mathcal{C}}(A, B) \rightarrow \text{hom}_{\mathcal{D}}(FB, FA) \equiv F \text{ maps } f : A \rightarrow B \text{ in } \mathcal{C} \text{ to } Ff : FB \rightarrow FA \text{ in } \mathcal{D}$

$f : A \rightarrow B \equiv A \xrightarrow{f} B := \text{morphism}$

$Ff : FA \rightarrow FB \equiv FA \xrightarrow{Ff} FB := \text{morphism}$

$F := \text{covariant functor}$ (maps one-arrow diagrams in $\mathcal{C}$ to one-arrow diagrams in $\mathcal{D}$, "identity loops" and "triangles" are preserved)

(similarly for contravariant functor)

$$F1_A = 1_{FA} \quad (\text{identity is preserved})$$

composition is preserved

$$(F := \text{covariant}) \rightarrow (F(g \circ f) = Fg \circ Ff)$$

$$(F := \text{contravariant}) \rightarrow (F(g \circ f) = Ff \circ Fg)$$

$F|_{\text{hom}_{\mathcal{C}}(A,B)} := \text{local arrow part of } F := \text{restriction of } F \text{ to } \text{hom}_{\mathcal{C}}(A, B)$

$(F : \mathcal{C} \Rightarrow \mathcal{C}) := \text{functor on } \mathcal{C}$

$$F : \mathcal{C} \Rightarrow \mathbf{Set}$$

$F := \text{set valued functor}$

$\mathcal{C} := \text{category}$

$$(F, G : \mathcal{C} \Rightarrow \mathcal{D}) \rightarrow (F \text{ and } G \text{ are parallel})$$

$$(F : \mathcal{C} \Rightarrow \mathcal{D} \wedge G : \mathcal{D} \Rightarrow \mathcal{C}) \rightarrow (F \text{ and } G \text{ are antiparallel})$$

$F, G := \text{functors}$

$\mathcal{C}, \mathcal{D} := \text{categories}$

COMPOSITION OF FUNCTORS

$$(G \circ F)(A) = G(FA) \quad \text{for } A \in \mathcal{C}$$
$$(G \circ F)(f) = G(Ff) \quad \text{for } f \in \text{hom}_{\mathcal{C}}(A, B)$$

$$G \circ F := GF$$

$$F : \mathcal{C} \Rightarrow \mathcal{D}; \quad G : \mathcal{D} \Rightarrow \mathcal{E}$$

$$F, G := \text{functors}$$

$$\mathcal{C}, \mathcal{D}, \mathcal{E} := \text{categories}$$

SPECIAL TYPES OF FUNCTORS

$$\forall \text{lap} := \text{surjective} \rightarrow F := \mathbf{full}$$

$$\forall \text{lap} := \text{injective} \rightarrow F := \mathbf{faithfull}$$

$$\forall \text{lap} := \text{bijective} \rightarrow F := \mathbf{fully faithfull}$$

$$\text{object part of } F := \text{injective} \rightarrow F := \mathbf{embedding} \text{ of } \mathcal{C} \text{ in } \mathcal{D}$$

$$\text{lap} := \text{local arrow parts (restrictions)}$$

$$F : \mathcal{C} \Rightarrow \mathcal{D}$$

$$F := \text{functor}$$

$$\mathcal{C}, \mathcal{D} := \text{categories}$$

$$\wp : \mathbf{Set} \Rightarrow \mathbf{Set}$$

$\wp :=$ **power set functor**

(sends A to its power set $\wp(A)$)

(sends each $f : A \rightarrow B$ to $f : \wp(A) \rightarrow \wp(B)$)

$f : A \rightarrow B :=$ **set function**

$f : \wp(A) \rightarrow \wp(B) :=$ **induced function** (sends X to fX)

(same notation $\leadsto$ function + induced function)

$$F : \mathbf{Set} \Rightarrow \mathbf{Set}$$

$F :=$ **contravariant power set functor**

(sends A to its power set $\wp(A)$)

(sends each $f : A \rightarrow B$ to $f^{-1} : \wp(B) \rightarrow \wp(A)$)

$f : A \rightarrow B :=$ **set function**

$f^{-1} : \wp(B) \rightarrow \wp(A) :=$ **induced inverse function** (sends $X \subseteq B$ to $f^{-1}X \subseteq A$)

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$\begin{aligned} \forall A : A \in \mathbf{Obj}(\mathcal{C}) &\rightarrow A \in \mathbf{Obj}(\mathcal{D}) \\ \forall f : f \in \mathbf{hom}_{\mathcal{C}}(A, B) &\rightarrow f \in \mathbf{hom}_{\mathcal{D}}(A, B) \end{aligned}$$

$$F : \mathcal{C} \Rightarrow \mathcal{D}$$

F sends $A \in \mathcal{C}$ to itself (an object in $\mathcal{D}$)

F sends $f_{\mathcal{C}} : A \rightarrow B$ to itself (a morphism in $\mathcal{D}$)

$F :=$ **forgetful functor**

$U : \mathcal{C} \Rightarrow \mathbf{Set}$ on $\mathcal{C}$

$U :=$ **underlying-set functor**

$\mathcal{C}, \mathcal{D} :=$ categories

SmCat := (large) category of all *small* categories

$\#$ category of all categories

small categories $\in \text{Obj}(\text{SmCat})$

$F \in \text{Mor}(\text{SmCat})$

$F :=$ covariant functor

$(\exists F : \mathcal{C} \Rightarrow \mathbf{Set} \wedge \mathbf{F} := \text{faithful functor}) \rightarrow (\mathcal{C} := \text{concrete})$

1. $\forall A : A \in \mathcal{C} \rightarrow A$ can be thought of FA
2. $\forall f \in \mathcal{C}, (f : A \rightarrow B)$ can be thought of $(Ff : FA \rightarrow FB)$
3. 1_A can be thought of $F1 : FA \rightarrow FA$
 $(f \circ g) \in \mathcal{C}$ can be thought of $Ff \circ Fg$

abstract category := not concrete

$\mathcal{C} :=$ category

examples

Grp, Rng, Vect, Poset = concrete

Rel = abstract

All small categories are concrete.

$(\emptyset \neq \mathbf{Obj}(\mathcal{D}) \subseteq \mathbf{Obj}(\mathcal{C}) \wedge \emptyset \neq \mathbf{Mor}(\mathcal{D}) \subseteq \mathbf{Mor}(\mathcal{C})) \rightarrow (\mathcal{D} := \text{subcategory of } \mathcal{C})$

1. $\forall A, B \in \mathcal{D} : \text{hom}_{\mathcal{D}}(A, B) \subseteq \text{hom}_{\mathcal{C}}(A, B) \wedge (1_A)_{\mathcal{D}} = (1_A)_{\mathcal{C}}$
2. $(f_{\mathcal{D}} : A \rightarrow B \wedge g_{\mathcal{D}} : B \rightarrow C) \rightarrow (g \circ f)_{\mathcal{C}} = (g \circ f)_{\mathcal{D}}$

$(\text{hom}_{\mathcal{D}}(A, B) = \text{hom}_{\mathcal{C}}(A, B)) \rightarrow (\mathcal{D} := \text{full})$

$\mathcal{C}, \mathcal{D} :=$ category

$\subseteq$ subclass

$1_A :=$ identity map

$$(FC)_A = \{FA \mid A \in \mathcal{C}\}$$

$$(FC)_f = \{Ff \mid f \in \text{hom}_{\mathcal{C}}(A, B)\}$$

$$\neg \Box (FC := \text{subcategory of } \mathcal{D})$$

$FC :=$ **image** of $\mathcal{C}$

$(FC)_A :=$ set of **objects**

$(FC)_f :=$ set of **morphisms**

functor $=: F : \mathcal{C} \Rightarrow \mathcal{D}$

$\mathcal{C}, \mathcal{D} :=$ categories

$\Box :=$ it is necessary

example

$$\exists (g \circ f) \rightarrow (F(g) \circ F(f) = F(g \circ f) \in FC)$$

$$(F_{\text{obj}} := \text{injective}) \rightarrow (FC := \text{subcategory of } \mathcal{D})$$

$F_{\text{obj}} : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D}) :=$ object part of F

$F : \mathcal{C} \Rightarrow \mathcal{D}$

$\mathcal{C}, \mathcal{D} :=$ categories

$\mathcal{J}$ -**diagram** in $\mathcal{C}$ with **index category** $\mathcal{J} :=$ functor $J : \mathcal{J} \Rightarrow \mathcal{C}$

$\mathcal{J}, \mathcal{C} :=$ categories

$J(\mathcal{J}) :=$ image of $\mathcal{J}$

$\neg \square (J(\mathcal{J}) := \text{subcategory of } \mathcal{C})$

$\neg \square (J_\beta \circ J_\alpha \subseteq J(\mathcal{J}))$

$\square :=$ it is necessary

DIGRAPH-BASED VERSION OF A DIAGRAM

$\mathcal{J} :=$ category

$J :=$ functor

$J(\mathcal{J}) :=$ image of $\mathcal{J}$

$m, n, p, q \in \mathcal{J}$

$Jm, Jn, Jp, Jq \in J(\mathcal{J})$

$\alpha, \beta, J_\alpha, J_\beta :=$ morphisms

DIRECTED GRAPH

$$\mathcal{D} \supseteq \mathcal{V}(\mathcal{D}) \cup \left(\bigcup_{i,j} \mathcal{A}(v_i, w_j) \right)$$

$\mathcal{D} :=$ **directed graph** (**digraph**)

$\emptyset \neq \mathcal{V}(\mathcal{D}) :=$ class of **vertices** (**nodes**)

$(v, w) :=$ ordered pair of nodes

$\mathcal{A}(v, w) :=$ set of **arcs** from v to w (**leave** v , **enter** w)

$a_1, a_2 \in \mathcal{A}(v, w) \rightarrow a_1 \parallel a_2$

$\ell \in \mathcal{A}(v, v) \rightarrow \ell :=$ **loop**

$v_1 \in \mathcal{V}(\mathcal{D})$

ind(v_1) := **in-degree** of $v_1 :=$ cardinal number of *arcs*
entering v_1

outd(v_1) := **out-degree** of $v_1 :=$ cardinal number of *arcs*
leaving v_1

deg(v_1) = **ind**(v_1) + **outd**(v_1) := **degree** of v_1

LABELED DIGRAPH, DIRECTED PATH

$\mathcal{D}_\ell :=$ **labeled digraph** := all nodes and arcs are labeled
(parallel arcs have distinct labels)

no distinct nodes and no distinct arcs have the same label $\rightarrow$
 $\mathcal{D}_\ell$ is **uniquely labeled**

$e_1 \in \mathcal{A}(v_1, v_2), e_2 \in \mathcal{A}(v_2, v_3), \dots, e_{n-1} \in \mathcal{A}(v_{n-1}, v_n) :=$ **directed path** (sequence of *arcs*)

length of a path := *number of arcs* in the path

$A \xrightarrow{f} B \xrightarrow{f} C :=$ path

$g \circ f :=$ label of $A \xrightarrow{f} B \xrightarrow{f} C$

underlying digraph for the diagram := *nodes* and *arcs* are
labeled (from $\mathcal{J}$)

NOTATION

$\mathbb{D}, \mathbb{E}, \mathbb{F}, \dots :=$ diagrams

$\mathbb{D} := \mathbb{D}(J : \mathcal{J} \Rightarrow \mathcal{C})$

$$(\forall A, B : (A, B) \in \mathbb{D} \wedge A \xrightarrow{f} B \wedge \exists(A, B) \text{len}(A \xrightarrow{f} B) \geq 2) \implies \mathbb{D} \text{ **commutes**}$$

$\mathbb{D} :=$ diagram in $\mathcal{C}$

$\mathcal{C} :=$ category

$A, B :=$ objects

$f :=$ morphism

$A, B, f \in \mathcal{C}$

len := length of a directed path

example 1: the following diagram commutes since $\beta_1 \circ \gamma = \alpha_1$ and $\beta_2 \circ \gamma = \alpha_2$

example 2: $\mathbb{E} := E \xrightarrow{e} A \xrightarrow{f} B$ and $A \xrightarrow{g} B$

$\mathbb{E} :=$ **commutative** diagram; $f \parallel g$

commutative condition: $f \circ e = g \circ e$

$$(\exists f_L : B \rightarrow A) \implies (f : A \rightarrow B := \mathbf{left-invertible})$$

$$f_L \circ f = 1_A$$

$$(\exists f_R : B \rightarrow A) \implies (f : A \rightarrow B := \mathbf{right-invertible})$$

$$f \circ f_R = 1_B$$

$$(\exists f^{-1} : B \rightarrow A) \implies (f : A \rightarrow B := \mathbf{invertible (isomorphism)})$$

$$f^{-1} \circ f = 1_A$$

$$f \circ f^{-1} = 1_B$$

$$A \approx B$$

$$\neg \square (\text{category isomorphism} = (f := \text{injective}) \vee (f := \text{surjective}))$$

$f_L := \mathbf{left\ inverse}$

$f_R := \mathbf{right\ inverse}$

$f^{-1} := (\mathbf{two-sided})\ \mathbf{inverse\ of}\ f$

$A \approx B := A\ \text{and}\ B\ \text{are}\ \mathbf{isomorphic}$

$\mathcal{C} := \text{category}$

$\square := \text{it is necessary}$

$\neg \square := \text{not necessarily}$

$$\exists f^{-1} \rightarrow \exists ! f^{-1}$$

$$\exists (i_1 \circ i_2) \implies (i_1 \circ i_2) := \text{isomorphism} \wedge (i_1 \circ i_2)^{-1} = i_2^{-1} \circ i_1^{-1}$$

$f := \text{morphism}$

$i_1, i_2 := \underline{\text{isomorphisms}}$

$$(\forall g, h : C \rightarrow A . g \parallel h \wedge f \circ g = f \circ h \rightarrow g = h) \implies (f := \textbf{left-cancellable} \text{ (monic or mono)})$$

$$(\forall g, h : B \rightarrow C . g \parallel h \wedge g \circ f = h \circ f \rightarrow g = h) \implies (f := \textbf{right-cancellable} \text{ (epic or epi)})$$

$f : A \rightarrow B :=$ morphism

$\mathcal{C} :=$ category

$$f := \text{left-invertible} \implies f := \text{left-cancellable (monic)}$$

$$f := \text{right-invertible} \implies f := \text{right-cancellable (epic)}$$

$$f := \text{invertible} \implies f := \text{monic and epic}$$

$$f := \text{monic (left-cancellable) and right-invertible} \implies f := \text{isomorphism}$$

$$f := \text{epic (right-cancellable) and left-invertible} \implies f := \text{isomorphism}$$

$f : A \rightarrow B :=$ morphism

$\mathcal{C} :=$ category

$$(\forall A \in \mathcal{C} \exists! f : I \rightarrow A) \implies (I := \mathbf{initial})$$

$$(\forall A \in \mathcal{C} \exists! f : A \rightarrow T) \implies (T := \mathbf{terminal})$$

$$(A := \mathbf{initial} \text{ and } \mathbf{terminal}) \implies (A := \mathbf{zero\ object})$$

$$C := \mathbf{initial} \vee \mathbf{terminal} \implies \mathbf{hom}(C, C) = \{1_C\}$$

objects $:= I, A, T, C \in \mathcal{C}$

$\mathcal{C} :=$ category

$$I_1, I_2 \in \mathcal{C} \implies I_1 \approx I_2$$

$$T_1, T_2 \in \mathcal{C} \implies T_1 \approx T_2$$

$I_1, I_2 :=$ initial objects

$T :=$ terminal object

$\mathcal{C} :=$ category

$\approx$ isomorphism

Set $:=$ category

$\exists! I \wedge I = \emptyset$

$\forall A \in \mathbf{Set} : A := \mathbf{singleton} \implies A := \mathbf{terminal}$

$I :=$ initial object

singleton set

terminal object

$$\forall f \in \mathcal{C} : f = h_{0B} \circ g_{A0} \implies f := \text{zero morphism}$$

$$0 \in \mathcal{C}$$

$$f : A \rightarrow B; \quad h : 0 \rightarrow B; \quad g : A \rightarrow 0$$

$$\mathcal{C} := \text{category}$$

any morphism entering or leaving 0 is a zero morphism

$$\forall A, B : A, B \in \mathcal{C} \implies \exists! (\text{zero morphism}) \text{ between } A \text{ and } B$$

$$z := \text{zero morphism} \implies (f \circ z) := \text{zero morphism} \wedge (z \circ g) := \text{zero morphism}$$

(zero morphisms “absorb” other morphisms)

$$0 \in \mathcal{C}$$

$$z : A \rightarrow B$$

$$\mathcal{C} := \text{category}$$

$$A \in \mathcal{C} \rightarrow A \in \mathcal{C}^{\text{op}}$$

$$\text{hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{hom}_{\mathcal{C}}(B, A)$$

$$\mathcal{C} := \text{category}$$

$\mathcal{C}^{\text{op}} := \text{opposite category (dual category)}$ (morphisms are reversed)

$$f \in \text{hom}_{\mathcal{C}^{\text{op}}}(A, B) \wedge g \in \text{hom}_{\mathcal{C}^{\text{op}}}(B, C) \implies g \circ_{\text{op}} f \in \text{hom}_{\mathcal{C}^{\text{op}}}(A, C) = f \circ g \in \text{hom}_{\mathcal{C}}(C, A)$$

$\circ_{\text{op}} := \text{composition in } \mathcal{C}^{\text{op}}$

$$g \circ_{\text{op}} f = f \circ g$$

$$(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}, \quad (\text{every category is a dual category})$$

$$\forall \mathcal{C} : \mathcal{C} \text{ has } p^{\text{op}} \leftrightarrow \mathcal{C}^{\text{op}} \text{ has } p \implies p^{\text{op}} := \text{dual property to } p$$

$p :=$ property of $\mathcal{C}$

$\mathcal{C} :=$ category

$\mathcal{C}^{\text{op}} :=$ dual category

isomorphism is self-dual,

$$A \approx B \text{ in } \mathcal{C} \leftrightarrow A \approx B \text{ in } \mathcal{C}^{\text{op}}$$

$\approx$ isomorphism

dual statement := same statement stated for $\mathcal{C}^{\text{op}}$ and expressed in terms of $\mathcal{C}$

$$(\mathcal{C} \text{ has } \Pi \rightarrow \mathcal{C} \text{ has } p) \equiv (\mathcal{C}^{\text{op}} \text{ has } \Pi \rightarrow \mathcal{C}^{\text{op}} \text{ has } p) \equiv (\mathcal{C} \text{ has } \Pi^{\text{op}} \rightarrow \mathcal{C} \text{ has } p^{\text{op}})$$

$\Pi = \{q_i \mid i \in I\} :=$ set of properties

$\Pi^{\text{op}} = \{q_i^{\text{op}} \mid i \in I\} :=$ set of dual properties

$p :=$ single property

PRINCIPLE OF DUALITY

$$\begin{aligned} \Pi \Rightarrow p &\leftrightarrow \Pi^{\text{op}} \Rightarrow p^{\text{op}} \\ (\mathcal{C} \text{ has } \Pi \rightarrow \mathcal{C} \text{ has } p) &\leftrightarrow (\mathcal{C} \text{ has } \Pi^{\text{op}} \rightarrow \mathcal{C} \text{ has } p^{\text{op}}) \end{aligned}$$

$$\begin{aligned} \Pi = \Pi^{\text{op}} \quad (\Pi \text{ is self-dual}) \\ \Pi \Rightarrow p &\leftrightarrow \Pi \Rightarrow p^{\text{op}} \end{aligned}$$

product category: $\mathcal{B} \times \mathcal{C}$

$$(f, g) \circ (h, k) = (f \circ h, g \circ k)$$

$$F : \mathcal{A} \times \mathcal{B} \Rightarrow \mathcal{C}$$

$$B, B' \in \mathcal{B}; \quad C, C' \in \mathcal{C}$$

$$(B, C) := \text{objects of } \mathcal{B} \times \mathcal{C}$$

$$f : B \rightarrow B'; \quad g : C \rightarrow C'$$

$$(f, g) := \text{morphism from } \mathcal{B} \times \mathcal{C} \text{ to } \mathcal{B}' \times \mathcal{C}'$$

$$F := \mathbf{bifunctor} := \text{functor from } \mathcal{A} \times \mathcal{B} \text{ to } \mathcal{C}$$

$$\text{Obj}(\mathcal{C}^\rightarrow) = \text{hom}_{\mathcal{C}}(A, B)$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \beta \\ A' & \xrightarrow{g} & B' \end{array}$$

$$(\alpha : A \rightarrow A', \beta : B \rightarrow B') :=$$

morphism in $\mathcal{C}^\rightarrow$ (pair of arrows in $\mathcal{C}$)

$$(g \circ \alpha = \beta \circ f) := \text{the diagram commutes}$$

$$(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta) \quad := \text{composition in } \mathcal{C}^\rightarrow$$

$$(1_A, 1_B) := \text{identity morphism}$$

$$A, B, A', B' \in \mathcal{C}$$

$$f : A \rightarrow B; \quad g : A' \rightarrow B'$$

$$\mathcal{C} := \text{category}$$

$$\mathcal{C}^\rightarrow := \text{category of arrows of } \mathcal{C}$$

$$\text{morphism in } \mathcal{C}^\rightarrow := \text{morphism between arrows}$$

COSLICE CATEGORY (COMMA CATEGORY)

$(A \rightarrow \mathcal{C}) = \{(B, f : A \rightarrow B) \mid B \in \mathcal{C}\} := \mathbf{comma\ category}$
:= category of **arrows leaving** A (**coslice category**)

$$\overline{\alpha} : (B, f : A \rightarrow B) \Rightarrow (C, g : A \rightarrow C)$$

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow g \\ B & \xrightarrow{\alpha} & C \end{array}$$

$$\overline{\alpha} \circ \overline{\beta} = \overline{\alpha \circ \beta}$$

$$\overline{1_B} \circ \overline{\alpha} = \overline{1_B \circ \alpha} = \overline{\alpha}$$

$$\overline{\alpha} \circ \overline{1_B} = \overline{\alpha \circ 1_B} = \overline{\alpha}$$

$A :=$ **source object**

$B :=$ **target objects**

$(B, f : A \rightarrow B) :=$ **comma objects**

$\overline{\alpha} \in (A \rightarrow \mathcal{C})$

$\overline{1_B} :=$ identity morphism for $(B, f : A \rightarrow B)$

$A, B, C, \alpha \in \mathcal{C}$

$\alpha \circ f = g$

$\mathcal{C} :=$ category

actually, $(A \rightarrow \mathcal{C}) = (A \rightarrow I_{\mathcal{C}})$

$I_{\mathcal{C}} :=$ identity functor on $\mathcal{C}$

SLICE CATEGORY

$(\mathcal{C} \rightarrow A) = \{(B, f : B \rightarrow A) \mid B \in \mathcal{C}\} := \mathbf{comma\ category}$
:= category of **arrows entering** A (**slice category**)

$$\overline{\alpha'} : (B, f : B \rightarrow A) \Rightarrow (C, g : C \rightarrow A)$$

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & C \\ & \searrow f & \swarrow g \\ & A & \end{array}$$

$$\overline{\alpha'} \in (\mathcal{C} \rightarrow A)$$

$$g \circ \alpha = f$$

$$(A \rightarrow \mathcal{C}) \text{ is the } \underline{\text{dual}} \text{ of } (\mathcal{C} \rightarrow A)$$

COMMA CATEGORY: First Generalization

$$(A \rightarrow F) = \{(C, f : A \rightarrow FC) \mid C \in \mathcal{C}\}$$

$$\alpha : (C_1, f_1 : A \rightarrow FC_1) \Rightarrow (C_2, f_2 : A \rightarrow FC_2)$$

$F : \mathcal{C} \Rightarrow \mathcal{D} :=$ functor

source object $:= A \in \mathcal{D}$

$FC :=$ target objects (image of $\mathcal{C}$ under F)

$\alpha \in (A \rightarrow F) :=$ comma category

$(C, f : A \rightarrow FC) :=$ comma objects

$F\alpha' \circ f_1 = f_2$, $\alpha' : C_1 \rightarrow C_2$ in $\mathcal{C}$ between "pre-target" objects

$$(F \rightarrow A) = \{(C, f : FC \rightarrow A) \mid C \in \mathcal{C}\}$$

$$\alpha : (C_1, f_1 : FC_1 \rightarrow A) \Rightarrow (C_2, f_2 : FC_2 \rightarrow A)$$

$f_2 \circ F\alpha' = f_1$, $\alpha' : C_1 \rightarrow C_2$ between "pre-source" objects in $\mathcal{C}$

$(A \rightarrow F)$ is the dual of $(F \rightarrow A)$

$(\alpha : B \rightarrow B', \beta : C \rightarrow C') := \text{morphism from } (B, C, f : FB \rightarrow GC) \text{ to } (B', C', f' : FB' \rightarrow GC')$

$$G\beta \circ f = f' \circ F\alpha$$

$(B, C, f : FB \rightarrow GC) := \text{object of } (F \rightarrow G)$

$(F \rightarrow G) := \text{comma category}$

$F : \mathcal{B} \Rightarrow \mathcal{D}; \quad G : \mathcal{C} \Rightarrow \mathcal{D}$

$F, G := \text{functors}$

$B \in \mathcal{B}; \quad C \in \mathcal{C}; \quad f, g \in \mathcal{D}$

$\mathcal{B}, \mathcal{C}, \mathcal{D} := \text{categories}$

$f : (C, a) \rightarrow (D, b) := \text{morphism } f : C \rightarrow D \text{ such that } Ff(a) = b$

$F : \mathcal{C} \rightarrow \mathbf{Set} := \text{set-valued functor}$

$\text{Obj}(\mathbf{Elts}(F)) = (\mathcal{C}, a)$

$\mathbf{Elts}(F) := \text{category of elements}$

$C \in \mathcal{C}, \ a \in FC$

$\mathcal{C} := \text{category}$

$$\{\text{hom}_{\mathcal{C}}(A, X) \mid X \in \mathcal{C}\} = \text{Obj}(\mathcal{C}(A, -))$$

$$\forall f : X \rightarrow Y \text{ in } \mathcal{C} \quad \exists f^{\leftarrow} \in \mathcal{C}(A, -) : \text{hom}_{\mathcal{C}}(A, X) \rightarrow \text{hom}_{\mathcal{C}}(A, Y)$$

$$\forall \alpha \in \text{hom}_{\mathcal{C}}(A, X) : f^{\leftarrow}(\alpha) = f \circ \alpha$$

$$\begin{array}{ccc}
 & A & \\
 \text{hom}_{\mathcal{C}}(A, X) \swarrow & \xrightarrow{f^{\leftarrow}} & \searrow \text{hom}_{\mathcal{C}}(A, Y) \\
 X & \xrightarrow{f} & Y
 \end{array}$$

$$\mathcal{C}(A, -) := \text{hom-set category}$$

$$\text{"follow by } f \text{"} =: f^{\leftarrow} \in \mathcal{C}(A, -)$$

$$\mathcal{C} := \text{category}$$

$$\{\text{hom}_{\mathcal{C}}(X, A) \mid X \in \mathcal{C}\} = \text{Obj}(\mathcal{C}(-, A))$$

$$\forall f : X \rightarrow Y \text{ in } \mathcal{C} \quad \exists f^{\rightarrow} \in \mathcal{C}(-, A) : \text{hom}_{\mathcal{C}}(Y, A) \rightarrow \text{hom}_{\mathcal{C}}(X, A)$$

$$\forall \alpha \in \text{hom}_{\mathcal{C}}(A, X) : f^{\rightarrow}(\alpha) = \alpha \circ f$$

$$\begin{array}{ccc}
 & A & \\
 \text{hom}_{\mathcal{C}}(X, A) \nearrow & \xleftarrow{f^{\rightarrow}} & \nwarrow \text{hom}_{\mathcal{C}}(Y, A) \\
 X & \xrightarrow{f} & Y
 \end{array}$$

$$\mathcal{C}(-, A) := \text{hom-set category}$$

$$\text{"preced by } f \text{"} =: f^{\rightarrow} \in \mathcal{C}(-, A)$$

$$\begin{aligned}
 (\forall \mathcal{C}_1 : \mathcal{C}_1 \in \mathcal{C}) \Rightarrow \mathcal{C}' &:= (\mathcal{C}_1 \sqcup * \wedge f_A : * \rightarrow A) := \\
 \text{hom-set category} &\Rightarrow (\forall A : A \in \mathcal{C} \rightarrow A = \text{hom}_{\mathcal{C}'}(*, A))
 \end{aligned}$$

$$(f : A \rightarrow B \text{ in } \mathcal{C}) = (f^{\leftarrow} : \text{hom}_{\mathcal{C}'}(*, A) \rightarrow \text{hom}_{\mathcal{C}'}(*, B))$$

initial "object" (morphism) $:= * \notin \mathcal{C}$

$\sqcup :=$ adjoining

$A \in \mathcal{C}$

$$(A \times B, \rho_1 : A \times B \rightarrow A, \rho_2 : A \times B \rightarrow B) :=$$

product of A and B

$$\forall (X \in \mathcal{C}, f : X \rightarrow A, g : X \rightarrow B) \exists ! \theta : X \rightarrow A \times B$$

object $:= A \times B \in \mathcal{C}$

morphisms $:= \rho_1, \rho_2 \in \mathcal{C}$

$\theta :=$ **mediating morphism** (map)

$\rho_1, \rho_2 :=$ **projection maps**

the diagram commutes, $\rho_1 \circ \theta = f$ and $\rho_2 \circ \theta = g$

$A, B \in \mathcal{C} :=$ category

$$\alpha = \beta \leftrightarrow \rho_1 \circ \alpha = \rho_1 \circ \beta \wedge \rho_2 \circ \alpha = \rho_2 \circ \beta$$

$\alpha, \beta : X \rightarrow A \times B$

(application, uniqueness, mediating morphism)

$$P_1, P_2 \in \mathbf{Poset}(P) \rightarrow P_1 \times P_2 := \mathbf{supremum}$$

$$P_1, P_2 \in \mathbf{Poset}(P) := \mathbf{category\ poset}$$

examples

Set $\rightarrow$ product := cartesian product

Grp, Mod, Vect, Rng $\rightarrow$ product := usual direct product

$$\forall C_1, C_2 \in \mathcal{C} : \exists C_1 \times C_2 \rightarrow \mathcal{C} \text{ has binary products}$$

Field := does not have products

objects $=: C_1, C_2 \in \mathcal{C} :=$ category

$$\theta := (f_1 \times f_2 : (A_1 \times A_2, \alpha_1, \alpha_2) \rightarrow (B_1 \times B_2, \beta_1, \beta_2)) := \mathbf{product\ morphism}$$

$$f_1 : A_1 \rightarrow B_1; \quad f_2 : A_2 \rightarrow B_2$$

$\alpha_i, \beta_i :=$ projection maps for $A_1 \times A_2$ and $B_1 \times B_2$, respectively

morphisms $=: f_1, f_2 \in \mathcal{C}$

$\mathcal{C} :=$ category with binary products
categorical product

MEDIATING MORPHISM TRICK

goal: $X \rightarrow B_1 \times B_2$

define: $\delta_1 : X \rightarrow B_1, \delta_2 : X \rightarrow B_2$

use the definition of product

$\exists! \tau : X \rightarrow B_1 \times B_2$ such that $\beta_1 \circ \tau = \delta_1$ and $\beta_2 \circ \tau = \delta_2$

$\tau :=$ mediating morphism (map)

we need a pair of maps: $A_1 \times A_2 \rightarrow B_1$ and $A_1 \times A_2 \rightarrow B_2$

$$\alpha_i : A_1 \times A_2 \rightarrow A_i$$

$$f_i : A_i \rightarrow B_i$$

$$\exists! \theta : \beta_1 \circ (f_1 \times f_2) = f_1 \circ \alpha_1 \quad \text{and} \quad \beta_2 \circ (f_1 \times f_2) = f_2 \circ \alpha_2$$