Affine and convex spaces

blending the analytic and geometric viewpoints

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25 April 2011; updated 22 July 2020

This is a short introduction to affine and convex spaces, written especially for physics students. It summarizes different elementary presentations available in the mathematical literature, and blends analytic- and geometric-flavoured presentations. References are also provided, as well as a brief discussion of Grassmann spaces and an example showing the relevance and usefulness of affine spaces in Newtonian physics.

PACS: 02.40.Dr, 02.40.Ft, 45.20.-d MSC: 14R99, 51N10, 52A20

to Louise

1 Spaces that deserve more space

Scientists and science students of different fields are very familiar, in various degrees of sophistication, with vector spaces. Vectors are used to model places, velocities, forces, generators of rotations, electric fields, and even quantum states. Vector spaces are among the building blocks of classical mechanics, electromagnetism, general relativity, quantum theory; they constitute therefore an essential part of physics and mathematics teaching.

Physicists also like to repeat, with Newton (1726 Liber III, regula I; Truesdell et al. 1960 § 293), a maxim variously attributed (Thorburn 1918) to Ockham or Scotus: “*frustra fit per plura quod fieri potest per pauciora*”. Applied to the connexion between mathematics and physics, it says that we should not model a physical phenomenon by means of a mathematical object that has more structure than the phenomenon itself. But this maxim is forsaken in the case of vector spaces, for they are sometimes used where other spaces, having less structure, would suffice. A modern example is given by quantum theory, where “pure states” are usually represented as (complex) vectors; but the vectors ψ and $\lambda\psi$, $\lambda \neq 0$, represent the same state, and the null vector represents none. Clearly the vector-space structure is redundant here. In fact, pure quantum states should more precisely be seen as points in a complex projective space (Haag 1996 § 1.3.1; Bengtsson et al. 2017).

Another example is the notion of reference frame in classical galileian-relativistic mechanics: such a frame is often modelled as a vector space, wherein we describe the *place* occupied by a small body by its “position vector” with respect to some origin. But suppose that I choose two places, for example, on a solar-system scale, those occupied by Pluto and Charon at a given time; and I ask you: what is the *sum* of these places? This question does not make very much sense; and even if you associate two vectors to the two places and then perform a formal sum of those vectors, the resulting place is devoid of any physical meaning. Thus, even if we usually model places as vectors, it is clear that the mathematical structure given by vector addition has no physical counterpart in this case.

On the other hand, I can ask you to determine a place in between the places occupied by Pluto and Charon such that its distances from the two planets are in an inverse ratio as the planets’ masses m_P, m_C ; in other words, their mass centre. You can obtain this place unambiguously, and it also has a physical meaning: it moves as the place occupied by a body with mass $m_P + m_C$ under the total action of the forces acting on Pluto and Charon. It turns out that the operation of assigning a mass-centre does not really need the concept of distance, and can be modelled in a space that has less structure, and is therefore more general, than a vector space: an *affine space*.

Affine spaces have geometrically intuitive properties and are not more difficult to understand than vector spaces. But they are rarely taught to physics students; and when they are, they are presented as by-products of vector spaces. This is reflected in textbooks of mathematical methods in physics. Amongst the old and new, widely and less widely known textbooks that I checked (Courant 1966; Jeffreys et al. 1950; Schouten 1989; Flügge 1955; 1956; Wilf 1978; Arfken et al. 2005; Boas 1983; Reed et al. 1980; Choquet-Bruhat et al. 1996; Marsden et al. 2007; Geroch 1985; Bamberg et al. 1990; Riley et al. 2002; Hassani 2009; Szekeres 2004), only Bamberg & Sternberg (1990), Szekeres (2004), and obviously Schouten (1989) give appropriate space to affine spaces; almost all others do not even mention affine spaces at all, although all of them obviously present the theory of vector spaces.

The students who have heard about affine spaces and would like to know more about them will find heterogeneous material, scattered for the most part in books and textbooks about general geometry. Part of this

material has an analytic flavour, part a geometrical flavour; and to get a more all-round view some patch-work is needed. It is the purpose of these notes to offer such patch-work, emphasizing the dialogue between the analytic and the synthetic-geometric presentations, and to offer some references. An intuitive knowledge of basic geometrical notions is assumed.

Closely related to affine spaces are *convex spaces*. These are also geometrically very intuitive, and are ubiquitous in convex analysis and optimization. Although their range of application in physics is maybe narrower than that of affine and vector spaces, they appear naturally in probability theory and therefore in statistical physics, be it the statistical mechanics of mass-points, fields, or continua; and they are of utmost importance in quantum theory, being behind many of its non-classical properties. Quantum theory is indeed only a particular example of a general plausibilistic physical theory, a particular case of a statistical model; and convex spaces are the most apt spaces to study the latter.

Students who have heard about and are interested in the general theory of convex spaces will find even less, and more hidden, material than for affine spaces. These notes offer some references and a general overview of convex spaces, too.

At the end of these notes I shall briefly discuss *Grassmann spaces*, which generalize affine spaces in many remarkable ways, and give an example of application of affine spaces in Newtonian mechanics, related to the previous discussion about Pluto and Charon. Extended application examples for convex spaces are left to a future note.

2 Affine spaces

2.1 Analytic point of view

Affine combinations An *affine space* is a set of *points* that is closed under an operation, *affine combination*, mapping pairs of points (a, b) and pairs of real numbers (λ, μ) summing up to one to another point c of the space:

$$(a, b, \lambda, \mu) \mapsto c = \lambda a \boxplus \mu b, \quad \lambda, \mu \in \mathbf{R}, \quad \lambda + \mu = 1. \quad (1)$$

The intuitive properties of this operation, including the extension to more than two points, I do not list here. It behaves in a way similar to scalar multiplication followed by vector addition in a vector space; but as

the symbol “ $\boxplus$ ” in place of “ $+$ ” reminds us, “multiplication” of a point by a number and “sum” of two points are undefined operations in an affine space: only the combination above makes sense. This operation has a geometric meaning which will be explained in § 2.2. One usually writes simply $\lambda a + \mu b$, a notation that we shall follow. Whereas a vector space has a special vector: the null vector, an affine space has no special points and is therefore more general than a vector space.

Affine basis A set of points is *affinely independent* if none of them can be written as an affine combination of the others. The maximum number of affinely independent points minus one defines the dimension of the affine space. An *affine basis* is a maximal set of affinely independent points. Any point of the space can be uniquely written as an affine combination of basis points, and the coefficients can be called the *weights* of the point with respect to that basis. A choice of basis allows us to baptize each point with a numeric name made of n reals summing up to one, where n is the dimension of the space plus one. This n -tuple can be represented by a column matrix. An affine combination of two or more points corresponds to a sum of their matrices multiplied by their respective coefficients. For particular affine spaces whose points are already numbers, like the real line $\mathbf{R}$, such baptizing ceremonies are usually superfluous.

Affine subspaces Some subsets of an affine space are affine spaces themselves, of lower dimensionality. Given two points a_1, a_2 , the *line* a_1a_2 through them is the locus of all points obtained by their affine combinations for all choices of coefficients (λ_1, λ_2) , $\lambda_1 + \lambda_2 = 1$. Given three affinely independent points a_1, a_2, a_3 , the *plane* $a_1a_2a_3$ through them is the locus of all points obtained by their affine combinations for all choices of coefficients $(\lambda_1, \lambda_2, \lambda_3)$, $\sum_i \lambda_i = 1$. And so on for $n + 1$ points and n -dimensional planes, the latter called *n*-planes for short. All these are affine subspaces: a line, of dimension one; a plane, of dimension two; etc. In general, given a finite set of points $\{a_1, \dots, a_r\}$, not necessarily affinely independent, their *affine span* $\text{aff}\{a_1, \dots, a_r\}$ is the smallest affine subspace containing them. It is simply the locus of all points obtained by affine combinations of the $\{a_i\}$ for all possible choices of coefficients.

Given four points a_1, a_2, b_1, b_2 , the lines a_1a_2 and b_1b_2 are said to be *parallel*, written $a_1a_2 \parallel b_1b_2$, according to the following definition:

$$a_1a_2 \parallel b_1b_2 \iff b_2 = b_1 - \lambda a_1 + \lambda a_2 \text{ for some } \lambda. \quad (2)$$

For two planes $a_1a_2a_3$ and $b_1b_2b_3$ to be parallel we must have

$$a_1a_2a_3 // b_1b_2b_3 \iff \begin{cases} b_2 = b_1 - (\lambda + \mu)a_1 + \lambda a_2 + \mu a_3 & \text{for some } \lambda, \mu, \\ b_3 = b_1 - (\lambda' + \mu')a_1 + \lambda' a_2 + \mu' a_3 & \text{for some } \lambda', \mu'. \end{cases} \quad (3)$$

And so on. We shall see later that these notions coincide with the usual geometric ones. Geometrically, affine dependence means collinearity, coplanarity, etc.

Affine mappings An *affine mapping* or *affinity* from one affine space to another or to itself is a mapping F that preserves affine combinations:

$$F(\sum_i \lambda_i a_i) = \sum_i \lambda_i F(a_i), \quad \sum_i \lambda_i = 1; \quad (4)$$

it therefore maps r -planes into t -planes, where $t \leq r$, and mutually parallel objects into mutually parallel objects. In the following we shall often use the summation convention and omit the normalization condition when clear from the context.

Introducing two affine bases in the domain and range of an affine mapping it is easy to see that it can be represented by a stochastic $(m+1, n+1)$ -matrix (i.e., with columns summing up to one), operating on a point through multiplication by the latter's column matrix; n and m are the dimensions of domain and range. This representation is basis-dependent. The rank of the matrix, which is basis-independent (and obviously smaller than $n+1$ and $m+1$), is equal to the dimension of the image of the domain plus one. When domain, range, and the image of the domain have the same dimension the matrix is square and its non-vanishing determinant, also basis-independent, is the *ratio* of the hypervolumes determined by the image of the first space's basis and that formed by the second space's basis; more on ratio of hypervolumes in § 2.2.

We can define affine combinations of the affinities between two affine spaces in a canonical way: $(\lambda F + \mu G)(a) := \lambda F(a) + \mu G(a)$ for any two affinities F, G with same n -dimensional domain and m -dimensional range (note how the expression " $\lambda F(a)$ " by itself has no meaning). The set of these mappings is therefore an affine space itself, of dimension $m(n+1)$.

Affine forms An affinity from an n -dimensional affine space to the real line $\mathbf{R}$ can be represented by a single-row matrix with n entries, instead

of a $(2, n)$ matrix, because as already said the points of the reals can numerically represent themselves without the need of an affine basis. The matrix representation is still dependent on a choice of basis in the domain affine space, though. Such affinities are called *affine forms* or simply *forms*. Their set is an affine space; in fact, it is even a vector space owing to the vector structure of the reals; its dimension in both cases is $n + 1$, thus larger than that of the original affine space (for this reason I find the name “dual space”, used by some, inappropriate). The action of an affine form v on a point a will be denoted by $v \cdot a$.

Choose a basis (e^i) in an affine space. In the space of forms, seen as a *vector* space, we can then choose a vector basis (d_j) such that $d_j \cdot e^i = \delta_j^i$; the d_j are called *dual forms* of the basis (e^i) . The set $\{d_j\}$ is however insufficient as a basis if we see the space of forms as an affine space: it has to be augmented by another form, like the null-form $d_0: a \mapsto 0$ or the unit-form $d_u: a \mapsto 1$. Here we choose the former; (d_0, d_j) is thus an affine basis in the space of forms.

A non-constant affine form can be geometrically seen as a family of parallel hyperplanes in the affine space: the form has a constant value on each hyperplane. This family is usually iconized by drawing only two hyperplanes: one, unmarked, where the form has value zero, and one, marked by e.g. a tick, where it has value one. A constant form has no such hyperplanes of course. The r -multiple of a form has its unity hyperplane at a distance $1/r$ times the original distance from the zero hyperplane. If you wonder what I mean with “distance”, given that no such notion is defined in an affine space, please read the next section. The sum of two forms is a form whose unity hyperplane passes through the intersections

Figure 1 Scalar multiplication and addition of affine forms

of their zero and unity hyperplanes, and whose parallel zero hyperplane passes through the intersection of their zero hyperplanes. See fig. 1 and the nice illustrations in Burke (1987; 1995).

2.2 Geometric point of view

Parallelism and translations From a geometric point of view, an affine space is based on the notions of point, line, plane, space, hyperplane, and so on, and the notion of (Euclidean) parallelism. I shall take these notions, that can be axiomatized in many different ways, for granted. Note that the notions of distance and angle are undefined.

Affine mappings between affine spaces are those that preserve the relation of parallelism: they map pairs of parallel objects, like lines or hyperplanes, into pairs of parallel objects. A special group of affine transformations of an affine space into itself are those that map every object into another parallel to, and of the same dimension as, the original one. They are called *translations*. To specify a translation u we only need to assign a point a and its image $a' := u(a)$. The image $b' := u(b)$ of any other point b outside of the line aa' is determined by requiring that the line bb' be parallel to aa' and the line $a'b'$ to ab , as in fig. 2. The segment bb' can then be used to construct the image of other points on the line aa' ; from this construction it is clear that the case of a one-dimensional affine space requires a different approach. The translations form a commutative group, the identity being the null translation $0: a \mapsto a$, and the inverse of u being the translation $-u$ determined by $u(a)$ and its image $a = -u[u(a)]$.

Figure 2 Construction of the image $b' = u(b)$ of b by the translation determined by a and $a' = u(a)$

The action of this group on the affine space is transitive, faithful, and free.

Ratio of n -areas If the point a_1 is the image of a under u , and a_2 the image of a under a double application, $u' := 2u := u + u := u \circ u$, of the same translation, we can say that the oriented segment $\overrightarrow{aa_2}$ is twice $\overrightarrow{aa_1}$, or that the latter is half the former, and we can write $u = u'/2$. Generalizing this construction we can define rational multiples of a translation, and thence generic real multiples λu , $\lambda \in \mathbf{R}$, through a Dedekind-section-like construction (Coxeter 1969 § 13.3). Negative values indicate a change in orientation. Translations form therefore a vector space over the reals, sometimes called the *translation space* of the original affine space, and they allow us to speak of the *ratio* of two lengths along parallel lines (but not along non-parallel ones), of two areas on parallel planes, and so on with n -areas, up to the ratio of any two hypervolumes. The procedure is to divide the first n -area into smaller and smaller equal n -rectangles, and to see how many of them are in the limit needed to fill, by translation, the second n -area; see the example in fig. 3. The ratio between two hypervolumes provides a geometric definition of the determinant of an affinity, defined in § 2.1 in terms of the matrix representing the affinity. It should now be clear what we meant, in the section about affine forms, when we said that the distance between two parallel hyperplanes is r times the distance between two other parallel hyperplanes: draw

Figure 3 The orange and blue areas on the two parallel planes are in the ratio 21/24, as the decomposition into smaller equal rectangles shows underneath. With a limit construction we can compare parallel areas with curvilinear boundaries

an arbitrary line intersecting all these hyperplanes; then the segment intercepted on the line by the first two hyperplanes and that intercepted by the last two hyperplanes are in the ratio r .

2.3 Relation between analytic and geometric points of view

The action of a translation u on the point b is usually denoted by $b + u := u(b) = c$. We also write $u = c - b$ to denote the fact that u is uniquely determined by some point b and its image c . Then, by what we said in § 2.2, the translation $\lambda u = \lambda(c - b)$ maps b to a point c' such that $\overrightarrow{bc'}$ is λ times $\overrightarrow{bc}$ (negative λ indicating a change in orientation). The action of the same translation λu on the point a can then be written $a + \lambda(c - b)$. Given another translation $\mu v = \mu(d - b)$, the action of the composite translation $\lambda u + \mu v$ on a can be written as $a + \lambda(c - b) + \mu(d - b)$. Generalizing this we obtain expressions which are formal sums of affine points with coefficients summing up to unity. This provides a link between the geometric and analytic presentations of an affine space: any affine combination $\lambda_i a^i$ can be written and interpreted as the image $a + \sum_i \lambda_i (a^i - a)$ of some point a under the composition of the translations $\lambda_j (a^j - a)$, and vice versa. Note again that the expression “ $a - b$ ” does not denote a point of the affine space but a particular mapping (translation) onto the space.

An expression like “ $a - b - c$ ” has no meaning in an affine space, not even in terms of translations. In § 4, however, we will briefly discuss spaces for which such an expression makes sense and moreover the difference between points and translations disappears.

A geometric interpretation of the affine combination $b = \lambda_1 a_1 + \lambda_2 a_2$ is that b is a point on the line determined by a_1 and a_2 and such that the unoriented segment $\overline{a_2 b}$, i.e. the one a_1 is not generally an endpoint of, is λ_1 times the segment $\overline{a_1 a_2}$, a negative ratio indicating that b and a_1 lie on opposite sides of a_2 ; and analogously for $\overline{a_1 b}$ and λ_2 . You can prove for yourself that the geometric interpretation of the combination $b = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$, with the a_i affinely independent, is that b is a point in the plane determined by the a_i and such that the triangle $a_2 a_3 b$, i.e. the one a_1 is not generally a vertex of, is λ_1 times the triangle $a_1 a_2 a_3$, the ratio being negative if b and a_1 lie on opposite sides of the line $a_2 a_3$; and analogously for the other triangles with b as a vertex and the other coefficients; see fig. 4.

Figure 4 Geometric meaning of the affine combination $b = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$: the ratios of the triangles $a_2 a_3 b$, $a_1 a_3 b$, and $a_1 a_2 b$ to $a_1 a_2 a_3$ are $|\lambda_1|$, $|\lambda_2|$, and $|\lambda_3|$. The coefficient of a_1 is negative: $\lambda_1 < 0$, because b and a_1 lie on opposite sides of the line $a_2 a_3$

Note again that lengths, areas, etc. belonging to non-parallel subspaces cannot be directly compared. For that purpose one can use affine forms, two-forms, twisted forms, etc., which however will not be discussed in this note. For those I refer you to the works of Burke (1987; 1995), Bossavit (1991; 2002), and also Schouten (1989).

2.4 References

Excellent analytic and geometric introductions to affine spaces and mappings can be found in ch. 13 of Coxeter (1969), ch. II of Artin (1955), § I.1 of Burke (1987), and also in chs I–III of Schouten (1989) and in Boehm & Prautzsch (2000).

3 Convex spaces

3.1 Convex combinations and mixture spaces

Mixture spaces and convex spaces A convex space is analytically defined as a set of points which is closed under the operation of *convex*

combination, mapping pairs of points and pairs of non-negative real numbers summing up to one to another point of the space:

$$(a, b, \lambda, \mu) \mapsto c = \lambda a \dot{+} \mu b, \quad \lambda, \mu \in [0, 1], \quad \lambda + \mu = 1. \quad (5)$$

This operation satisfies additional properties, and their analysis is interesting: Three of them,

$$1 a \dot{+} 0 b = a, \quad (6a)$$

$$\mu b \dot{+} \lambda a = \lambda a \dot{+} \mu b, \quad (6b)$$

$$\mu [\lambda a \dot{+} (1 - \lambda) b] \dot{+} (1 - \mu) b = \lambda \mu a \dot{+} (1 - \lambda \mu) b, \quad (6c)$$

define a *mixture space*. To define a convex space, which is less general than a mixture space, we need two additional properties:

$$b \mapsto \lambda a \dot{+} (1 - \lambda) b \quad \text{is injective for all } \lambda \in]0, 1[\text{ and all } a, \quad (6d)$$

and

$$\mu [\lambda a \dot{+} (1 - \lambda) b] \dot{+} (1 - \mu) c =$$

$$\lambda \mu a \dot{+} (1 - \lambda \mu) \left[\frac{(1 - \lambda)\mu}{1 - \lambda \mu} b \dot{+} \frac{1 - \mu}{1 - \lambda \mu} c \right]$$

$$\text{for all } \lambda, \mu \in [0, 1] \text{ with } \lambda \mu \neq 1. \quad (6e)$$

Convex spaces are special amongst mixture spaces because they can always be represented as convex subsets of some affine space; this property does not need to hold for a generic mixture space (Mongin 2001; Wakker 1988 § VII.2). All such representations of a convex space are isomorphic to one another, and their affine spans are also isomorphic. This allows us to rewrite expressions like (5) as $\lambda a + \mu b$ and to interpret them in the affine sense (1); it also allows us to speak of the dimension of a convex space, defined as the dimension of the affine span of any of its representations, and to speak of other notions like parallelism and compactness. From now on we shall only consider convex rather than mixture spaces, and finite-dimensional, compact convex spaces in particular. See fig. 5 for some examples of equivalent and inequivalent convex spaces.

Extreme points and bases A set of points is *convexly independent* if none of them can be written as a convex combination of the others. The *extreme points* of a convex space are the convexly independent points that convexly span the whole convex space (their set can be empty for

Figure 5 The two upper quadrilateral figures are the same convex space, whereas the lower, dashed, darker quadrilateral one is a different convex space; analogously for the three rounded figures

non-compact convex spaces). Equivalent characterizations are possible, e.g. a point is extreme if its exclusion from the original convex set leaves a set that is still convex (Klee 1957). A point can generally be written as a convex combination of extreme points in more than one way, so we cannot use them as a “convex basis” to assign unambiguous numeric names to the other points (one can select a unique convex combination through additional requirements, e.g. that its weights have maximum Shannon entropy). See fig. 6. But through the representation of the convex space in an affine space we can introduce an *affine basis*, whose elements can lie outside the convex space, and write every point of the convex space uniquely as an *affine* combination of these basis elements; this affine combination will not in general be a convex combination, i.e. its weights can be strictly negative or greater than unity. A weight lying in $[0, 1]$ will be called *proper*, otherwise *improper*.

A *face* of a convex space is a subset which is also convex and which contains all points that can be convexly combined into each of its points

Figure 6 Example of a convex set. The points a_1, a_2, a_3, a_4 are extreme points of the set, as well as all points on the (thicker) curved part of the boundary. The point b can be written as a convex combination of extreme points in at least two different ways: as $a_1/2 + a_3/2$ or as $a_2/3 + 2a_4/3$. The points a_4 and a_3 are faces, as is each point on the (thicker) curved part of the boundary

(e.g. Valentine 1964 § XI.B; Rockafellar 1972 § 18); in formulae, F is a face if and only if

$$\begin{cases} a_1, a_2 \in F \text{ and } a = \lambda a_1 + (1 - \lambda) a_2 \implies a \in F, \\ a \in F \text{ and } a = \lambda a_1 + (1 - \lambda) a_2 \implies a_1, a_2 \in F. \end{cases} \quad (7)$$

Quantum physicists call the first property (the convexity of the set) “invariance under mixing” and the second “invariance under purification” (Bengtsson et al. 2017 § 1.1). See fig. 6. A *facet* is a face of one less dimension than the convex space.

The boundary points of a convex space can be classified according to several other properties (Valentine 1964 § XI.B; Rockafellar 1972 § 18; Valentine 1964 parts IV, XI; Alfsen 1971 § II.5; Brøndsted 1983 § 5), for example *exposed points*, which we briefly mention again in the next section. Such properties often correspond to important physical properties in the physical theories where convex sets find application. Examples are thermostatics (Wightman 1979), where they for example indicate mixed phases (exposed faces) and critical points (non-exposed faces), and quantum theory (Bengtsson et al. 2017; Kimura 2003; Kimura et al. 2005; Peres et al. 1998).

A *simplex* is a convex space with a number of extreme points exceeding its dimension by one. The extreme-point decomposition of a point of a simplex is always unique, hence a simplex' extreme points constitute a canonical affine basis. A *parallelotope* is a convex space whose facets are pairwise parallel; it can be represented as a hypercube. See the right side of fig. 8 for the two-dimensional case.

Convex forms We can consider mappings from a convex space to another, mappings that preserve convex combinations. When the range is the real numbers, we can speak of an *affine form*, since such a convex mapping can be uniquely extended to an affine form on the affine span of the convex space. This kind of mappings are also defined for a mixture space, and properties (6d) and (6e) are equivalent to say that the mixture space is *separated* or *non-degenerate*, viz, for any pair of points there is a form having distinct values on them. In other words, a convex space is a mixture space in which each pair of points can be distinguished by a form (Mongin 2001 § 3).

A surjective affine mapping from a convex space onto one of equal or lower dimensionality can be called a (parallel) *projection*. An injective affine mapping from a convex space into one of higher dimensionality can be called an (affine) *embedding*.

Affine forms from a convex space $\mathcal{S}$ to the interval $[0, 1]$ are especially important. We call them *convex forms*. Convex combinations of these can be naturally defined; they therefore constitute a convex space, which can be given the name of *convex-form space* (or simply form space) of $\mathcal{S}$, denoted by $\mathcal{P}(\mathcal{S})$:

$$\mathcal{P}(\mathcal{S}) := \{v : \mathcal{S} \rightarrow [0, 1] \mid v \text{ is affine (or convex)}\}. \quad (8)$$

I avoid the name “dual space” because it risks to become overloaded and easily confused with other notions of duality (see e.g. Grünbaum 2003 § 3.4). The action of a convex form v on a point a will be denoted by $v \cdot a$ (confusion with affine forms on affine spaces is not likely to arise); once an affine basis is chosen in the convex space, this action can be written as matrix multiplication, as for affine forms. The forms $v_0 : a \mapsto 0$ and $v_1 : a \mapsto 1$ are called *null-form* and *unit-form*.

A non-constant convex form on a convex space can be geometrically seen as a family of parallel hyperplanes (in the embedding affine space) between two given ones that do not intersect the space's interior. On each hyperplane the form has a constant value, with values zero and

Figure 7 On the left, v is a convex form for the five-sided convex space; two lines are indicated where the form has values $1/2$ and $2/3$ on the space. On the right, w cannot be a convex form (although it is an affine form) because it assigns strictly negative values to some points of the convex space

unity on the utmost ones. These hyperplanes are also used to iconize the convex form, a mark being put on the unit one; see fig. 7.

Convex forms allow us to give this definition: an *exposed face* of a convex space is a face on which a convex form has value 0. Not all faces are exposed faces; for example, the point a_1 and the similar point at the other end of the curved boundary are non-exposed, zero-dimensional faces. The presence of non-exposed faces has important consequences for convex optimization, i.e. the search for the extremum of a function over a convex space (see references in the next section).

The convex structure of a convex-form space is determined by that of its respective convex space. Its affine span is the space of affine forms on the affine span of the original convex space. This means, from what we said about dual forms and bases in § 2.1, that a form space has one more dimension than the original convex space. A form space is always a bi-cone with the null-form v_0 and the unit-form v_1 as vertices; indeed, it is centro-symmetric with centre of symmetry $(v_0 + v_1)/2$. Its number of extreme points besides v_0 and v_1 is determined by the structure of the faces of the original convex space (for example, if the convex space is two-dimensional, the number of extreme points of its form space is equal to $2m + 2$, where m is the number of bounding directions of the convex space). See fig. 8 for two two-dimensional examples. The convex-form

Figure 8 Two two-dimensional convex spaces, on the left, with their three-dimensional convex-form spaces, on the right. The convex forms w, u are represented as pairs of parallel lines on the convex spaces and as points on the convex-form spaces. v_0 and v_1 are the null- and unit-forms.

space of an n -simplex is an $(n + 1)$ -parallelotope (which has 2^{n+1} extreme points).

3.2 References

Books on or touching convex spaces are Grünbaum's (2003), Valentine's (1964), Alfsen's (1971), Brøndsted's (1983), Eggleston (1958). Studies and examples of the difference between mixture and convex spaces are presented by Mongin (2001) and Wakker (1988 § VII.2). Other examples, axiomatizations, and applications can be found in Stone (1949), Herstein et al. (1953), Hausner (1954), Luce et al. (1971), Luce (1973), Krantz (1975a,b), Vincke (1980), and Holevo (1982). See also Gale (1953), Gale et al. (1968), Rockafellar (1972), McMullen et al. (1971), Gruber et al. (1993a,b), Schneider (1993), Webster (1994), Ewald (1996), Ball (1997), and Bengtsson et al. (2017) for related topics.

Convex optimization is a topic full of intriguing subtleties and is openly or silently present in every branch of science. It has a vast literature, sadly scattered between disciplines that do not talk with one another very much. See Fang et al. (1997), Boyd et al. (2009), Borwein et al. (2000), and Berkovitz (2002) as possible starting points.

Infinite-dimensional convex spaces are less intuitive and require care in their study. The studies of Klee and others (Klee 1948; 1949a,b; 1950; 1951a,b,c; 1953; 1954; 1955a,b; 1956; 1957; 1958; 1961; 1963; 1969a,b,c,d,e; 1971; 1977; 1980; Burger et al. 1996) are very interesting and provide appropriate references.

4 Generalizations: Grassmann spaces

The operation of affine combination suggests several generalizations.

A question comes quite naturally to mind, for example: what if the coefficients (λ_i) of an affine combination $\sum_i \lambda_i a^i$ do not sum up to one? In fact, in numeric applications it can be a nuisance to make sure that the coefficients satisfy this requirement. It turns out that an affine space can be seen as a special case of a more general space which has various names in the literature; we call it *Grassmann space*. A Grassmann space is closed under an operation that looks like (1), with the exception that the coefficients λ and μ can assume arbitrary real values. Points in a Grassmann space and in an affine space, however, differ: the former are equipped with a *weight*, which can be positive or negative. The operation

$\lambda a + \mu b$ in a Grassmann space yields a point with weight $\lambda + \mu$. It is easy to guess that an affine space is like a Grassmann space where we only consider points of unit weight. A remarkable consequence of this generalization is that vectors and translations (§§ 2.2–2.3) turn out to be *points* with zero weight. Goldman (2002; 2000) gives a brilliant introduction to these spaces.

A second question can naturally come to mind: could we consider affine combinations not just of points, but also of straight lines, planes, and analogous objects of higher dimensions? Also in this case the answer is positive; in fact, we can define combinations with arbitrary coefficients. The spaces where this is possible are again Grassmann spaces; Peano (1888 ch. I) gives an introduction to these generalized combinations.

In fact, in a Grassmann space we can also define multiplicative operations that combine points and lines, planes, and so on. This kind of spaces was first consistently introduced by Grassmann (1878; 1862); Peano (1888) also gave a very accessible introduction to them. Unfortunately their subsequent history – which includes figures like Clifford (1878) and Cartan (1923) – has been very convoluted. Their multiplicative operations have been developed by different groups of mathematicians in ways that are inequivalent and, worst of all, overly complicated. Interested readers can explore the approach by Barnabei, Brini, Rota, and others (Barnabei et al. 1985; Crapo 2009; Brini et al. 2011); the approach by Hestenes, Doran, Lasenby, Dorst, and others (Hestenes 1968; Hestenes et al. 1987; Dorst et al. 2002; Li 2008; Dorst et al. 2011); the approaches by Gunn (2011), Browne (2012), González Calvet (2016) – and there are many others out there (see Vargas 2016 remarks, § 1.4). Bengtsson and I (Porta Mana et al. 2017) hope to soon present an approach that makes Grassmann spaces accessible to high-school students.

5 An application: reference frames in classical mechanics

In the introduction I hinted at the fact that in classical galileian-relativistic mechanics *places* are often represented by “position vectors” though they need not be modelled by vectors at all; in fact some operations that we can do with vectors, e.g. sums, do not have any physically meaningful counterpart for places. Places can instead be modelled by a Euclidean space, which is a particular example of affine space, one in which the additional notions of distance and angle are defined. Velocities,

accelerations, forces maintain their vectorial character nevertheless. This is done as follows in the special case of point-mass mechanics:

We assume as primitives the notions of point-mass, time, time lapse (i.e. a metric on the time manifold), and distance between any pair of point masses at each time instant. We postulate that, at each time instant, the net of distances among all point masses has a three-dimensional Euclidean character (e.g., theorems concerning triangles equalities and triangle inequalities are satisfied). This net of distances determines precise affine relations among the point masses; these relations are of course variable with time like the distances themselves. The point masses can therefore be made to span a three-dimensional *affine* space at each time instant. The points of this affine space are what we call *places*, and each place is determined, in many equivalent ways, by an affine combination of the point masses. For example, at an instant t the affine combination $a_1(t)/4 - 3a_2(t)/4 + 6a_3(t)/4$ determines a unique place b in terms of the point masses $a_1(t), a_2(t), a_3(t)$. Different affine combinations can determine the same point: e.g., if $a_4(t) = 2a_3(t) - a_2(t)$ at t , then b is equivalently given by $a_1(t)/4 + 3a_4(t)/4$. Note that, once the affine relations among the point masses are given, we do not need a notion of absolute distance to determine b , nor the ability to compare distances along unparallel directions; i.e. we do not need the Euclidean structure.

At another time instant t' the mutual distances and affine relations between the point masses will be different; we may have e.g. $a_4(t') \neq 2a_3(t') - a_2(t')$. So it does not make sense to try to identify at t' the place b that we defined at t : should it be given by $a_1(t')/4 - 3a_2(t')/4 + 6a_3(t')/4$? or by $a_1(t')/4 + 3a_4(t')/4$? — the two combinations are inequivalent now. In other words, there is no *canonical* identification between the whole affine (and Euclidean) spaces at two different instants of time. This also means that there is no “absolute space”. See fig. 9.

But absence of a canonical identification does not mean that no identification at all is possible. A *frame of reference* is a particular, *arbitrary* identification of the places of the affine spaces at any two instants of time, respecting the affine and Euclidean structure; i.e., it is a mapping, defined for any two instants t and t' ,

$$F_{t',t}: A_{t'} \rightarrow A_t, \quad (9)$$

between the Euclidean-affine spaces $A'_{t'}, A_t$ spanned by the point masses at those two instants, that preserves distances. It therefore preserves

Figure 9 The Euclidean net of distances among the point masses a_1, a_2, a_3, a_4 determines an affine space at each time instant, e.g. t_0 and t_1 . The point masses can be identified at each instant, but a generic place b_1 determined at t_0 by a particular affine combination of the point masses has no counterpart at t_1 because the affine relations among the point masses have changed.

affine combinations:

$$F_{t',t}(\lambda a' + \mu b') = \lambda F_{t',t}(a') + \mu F_{t',t}(b'). \quad (10)$$

A frame of reference allows us to say that a particular place at time t is the “same” as some place at time t' , so that we can use only one affine space for all times and we can say that a particular point mass “moved” from a place at t to another place at t' . See fig. 10.

The velocity of a point mass’ motion at the instant t_0 in a particular frame F is defined as

$$v(t_0) := \lim_{t \rightarrow t_0} \frac{F_{t,t_0}[p(t)] - p(t_0)}{t - t_0}, \quad (11)$$

$p(t)$ being the place occupied by the point mass at time t . The argument of the limit is, for each t , the “difference” between two points in the affine space associated to the instant t_0 : it is namely a translation, as discussed in § 2.2, and therefore a vector. The limit is hence a vector, too. In this way we obtain the vectorial character of velocities, accelerations, and in a similar way of forces, without the need to model places as vectors. Note again that only the affine structure of space enters in the

Figure 10 A *frame of reference* is an arbitrary isomorphism between the places of the Euclidean-affine spaces at any two times. With respect to the mapping above we can say, e.g., that the point mass a_2 occupies the same place at t_0 and t_1 , while the other point masses change place. Note, however, that the physical situation at t_1 (and t_0) in this figure and fig. 9 is exactly the same.

expression above, not the Euclidean one (but we have a metric on the one-dimensional manifold that models time, as implied by the denominator of the fraction).

This way of modelling space in classical mechanics is based upon and combines the works of Noll (1959; 1967), Truesdell (1991), and Zanstra (1922; 1923; 1924; 1946). Apart from mathematical economy, it has the pedagogic advantage of presenting galileian-relativistic mechanics in a fashion closer to that of general relativity: in general relativity the set of events is a manifold that cannot be modelled as a (four-dimensional) vector space. Only (four-)velocities, accelerations, momenta have a vectorial character.

Thanks

... to Ingemar Bengtsson for support, for many, always insightful convex and non-convex discussions, for references, and for pointing out deficiencies in previous versions of this note; any deficiencies that ~~may~~ remain are my fault. To Mari & Miri for continuous encouragement and affection. To Buster Keaton and Saitama for filling life with awe and inspiration. To

the developers and maintainers of L^AT_EX, Emacs, AUC_TE_X, Open Science Framework, biorXiv, Hal archives, Python, Inkscape, Sci-Hub for making a free, unfiltered, and unmoderated scientific exchange possible. ☒

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