1 On the Use of Discrete Fourier Transform for
2 Solving Biperiodic Boundary Value Problem of
3 Polyharmonic Equation in the Unit Rectangle

4 Agah D. Garnadi

5 31 December 2017

6 Department of Mathematics,
7 Faculty of Mathematics and Natural Sciences,
8 Bogor Agricultural University
9 Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

10 **Abstract**

11 This note is addressed to solving biperiodic boundary value prob-
12 lem Polyharmonic equation in the unit rectangle. First, we describe
13 the necessary tools, which is discrete Fourier transform for one di-
14 mensional periodic sequence, and then extended the results to 2-
15 dimensional biperiodic sequence. Next, we use the discrete Fourier
16 transform 2-dimensional biperiodic sequence to solve discretization of
17 the biperiodic boundary value problem of Polyharmonic Equation.

18 *MSC:* 15A09; 35K35; 41A15; 41A29; 94A12.

19 *Key words:* Polyharmonic equation, biperiodic boundary value prob-
20 lem, discrete Fourier transform, Partial differential equations, Inter-
21 polation.

22 **1 Introduction**

23 This note is an extension of a work by Henrici [4] on fast solver for Pois-
24 son equation to biperiodic boundary value problem of polyharmonic equa-
25 tion. This note is organized as follows, in the first part we address the

tools needed to solve the problem, i.e. introducing discrete Fourier transform (DFT) follows Henrici's work [4]. The second part, we apply the tools already described in the first part to solve discretization of bi-periodic boundary value problem of polyharmonic equation in the unit rectangle. The need to solve discretization of polyharmonic equation is motivated by image inpainting [5] using biharmonic or harmonic equation. Note that the work of Henrici [4] on fast solver for bi-periodic Poisson equation can be seen as solving harmonic inpainting. For pedagogic reason, results from citeGarnadiff on solving biharmonic equation is reproduced in the second part.

2 Discrete Fourier Transform

2.1 One-dimensional Transform

Let $n \in \mathbb{N}$ and $(x_k)_{k=-\infty}^{\infty}$ is a bilateral infinite sequence of complex number. Let Π_n denote a set of bilateral infinite sequences with period n , i.e.:

$$\Pi_n = \{\mathbf{x} = (x_k)_{k=-\infty}^{\infty}; x_k \in \mathbb{C}, x_{k+n \cdot j} = x_k, j \in \mathbb{Z}\}.$$

Define in Π_n an addition and scalar multiplication by:

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_k + y_k)_{k=-\infty}^{\infty}, \\ c\mathbf{x} &= (c x_k)_{k=-\infty}^{\infty}, \quad \forall c \in \mathbb{C}. \end{aligned}$$

Equipped with these operations, Π_n is a finite dimensional vector space with $\dim(\Pi_n) = n$.

Denote $\omega_n := \exp(2\pi i/n)$, $i^2 = -1$. One dimensional discrete Fourier transform, denoted by $\mathcal{F}_n$, is defined as follows:

$$\begin{aligned} \mathcal{F}_n : \Pi_n &\rightarrow \Pi_n \\ \mathbf{x} &\mapsto \mathbf{y}, \end{aligned}$$

where

$$y_m = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{-mk} x_k.$$

It is clearly that $y_{m+n} = y_m$, holds for all m , as $\omega_n^n = 1$. Hence the map $\mathcal{F}_n$ is well defined. It is easily to show that $\mathcal{F}_n$ is a bijective map, hence invertible. The inverse of $\mathcal{F}_n$ is defined as:

$$\begin{aligned} \mathcal{F}_n^{-1} : \Pi_n &\rightarrow \Pi_n \\ \mathbf{y} &\mapsto \mathbf{x}, \end{aligned}$$

48 where:

$$x_r = \sum_{k=0}^{n-1} \omega_n^{mr} y_r.$$

49 Observe that the inverse has close resemblance with the DFT. Defining the
50 conjugate of DFT $\mathcal{F}_n$ by:

$$(\bar{\mathcal{F}}_n \mathbf{y})_k = \frac{1}{n} \sum_{m=0}^{n-1} \omega_n^{mk} y_m,$$

51 where the index k denoting the k -th component of the sequence, then we
52 have

$$\mathcal{F}_n^{-1} = n \bar{\mathcal{F}}_n.$$

53 Another operator that also useful when working with DFT is the reversion
54 operator denoted by $\mathcal{R}$. This operator is defined as follow, for $\mathbf{x} \in \Pi_n$,

$$(\mathcal{R}\mathbf{x})_m := x_{-m}.$$

55 A left and right composition with $\mathcal{F}_n$, we have the following relations:

$$\begin{aligned} (\mathcal{R} \circ \mathcal{F}_n \mathbf{x})_m &= \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{km} x_k, \\ &= (\bar{\mathcal{F}}_n). \\ (\mathcal{F}_n \circ \mathcal{R}\mathbf{x})_m &= - \sum_{k=0}^{n-1} \omega_n^{-km} x_{-k} = - \sum_{k=0}^{n-1} \omega_n^{km} x_k, \\ &= (\bar{\mathcal{F}}_n). \end{aligned}$$

56 From these composition, it is clear that

$$\mathcal{F}_n^{-1} = n \mathcal{F}_n \mathcal{R} = n \mathcal{R} \mathcal{F}_n.$$

57 We introduce Hadamard product of two sequences $\mathbf{x}$ and $\mathbf{y}$ in Π_n , which
58 is a component-wise product:

$$\mathbf{x} \bullet \mathbf{y} = (x_k y_k)_{k=-\infty}^{\infty}.$$

59 Also we introduce convolution product of two sequences $\mathbf{x}$ and $\mathbf{y}$ in Π_n by:

$$\mathbf{x} * \mathbf{y} = \mathbf{z} = (z_k)_{k=-\infty}^{\infty},$$

60 where:

$$z_k = \sum_{k=0}^{n-1} x_k y_{k-m} = \sum_{k=0}^{n-1} x_{k-m} y_k.$$

Both product relate to each other of as follows. Let $\mathbf{x}$ and $\mathbf{y}$ be two sequences in Π_n , and

$$\mathbf{u} = \mathcal{F}_n \mathbf{x}, \quad \mathbf{v} = \mathcal{F}_n \mathbf{y}.$$

By definition,

$$(\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}))_m = \frac{1}{n} \sum_{k=0}^{n-1} x_k y_k \omega_n^{-km},$$

as $y_k = (\mathcal{F}_n^{-1} \mathbf{y})_k$, then

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} x_k y_k \omega_n^{-km} &= \frac{1}{n} \sum_{k=0}^{n-1} x_k \left(\sum_{l=0}^{n-1} \omega_n^{lk} v_l \right) \omega_n^{-km} \\ &= \sum_{l=0}^{n-1} v_l \left(\frac{1}{n} \sum_{k=0}^{n-1} x_k \omega_n^{-k(m-l)} \right). \end{aligned}$$

The last inner sum is obviously equal to $(\mathcal{F}_n \mathbf{x})_{(m-l)} = u_{m-l}$, hence

$$\begin{aligned} (\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}))_m &= \sum_{l=0}^{n-1} v_l u_{m-l} = (\mathbf{v} * \mathbf{u})_m = (\mathbf{u} * \mathbf{v})_m \\ &= (\mathcal{F}_n(\mathbf{x}) * \mathcal{F}_n(\mathbf{y}))_m \end{aligned}$$

As it holds for all m , then

$$\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}) = \mathcal{F}_n(\mathbf{x}) * \mathcal{F}_n(\mathbf{y}).$$

Working on $\mathbf{u} = \mathcal{F}_n(\mathbf{x})$ and $\mathbf{v} = \mathcal{F}_n(\mathbf{y})$, for convolution form, then

$$\begin{aligned} \mathbf{u} * \mathbf{v} &= n^2 \mathcal{F}_n(\mathcal{F}_n^{-1}(\mathbf{x}) \bullet \mathcal{F}_n^{-1}(\mathbf{y})) \\ &= n^2 \mathcal{F}_n(R \mathcal{F}_n(\mathbf{x}) \bullet R \mathcal{F}_n(\mathbf{y})) \\ &= n^2 \mathcal{F}_n(R(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y}))) \\ &= n(nR \mathcal{F}_n)(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \\ &= n(\mathcal{F}_n^{-1})(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \\ \mathcal{F}_n(\mathbf{u} * \mathbf{v}) &= n(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \end{aligned}$$

Hence for $\mathbf{x}, \mathbf{y} \in \Pi_n$

$$\mathcal{F}_n(\mathbf{u} * \mathbf{v}) = n(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})).$$

69 2.2 Two-dimensional Transform

70 Let $n \in \mathbb{N}$ and $\mathbf{x}^{(2)} = (x_{k_1 k_2})_{k_1, k_2=-\infty}^{\infty}$ is a two dimensional bilateral infi-
 71 nite sequence of complex number. Let $\Pi_n^{(2)}$ denote a set of bilateral infinite
 72 sequences with period n on each direction, i.e.

$$\begin{aligned} \Pi_n^{(2)} = \{ \mathbf{x} = (x_{k_1 k_2})_{k_1, k_2=-\infty}^{\infty}; x_{k_1 k_2} \in \mathbb{C}, \\ x_{k_1+n, j, k_2} = x_{k_1 k_2}, x_{k_1, k_2+n, j} = x_{k_1 k_2}, j \in \mathbb{Z} \}. \end{aligned}$$

73 Observe that this set of two-dimensional array is isomorph with a set of bilat-
 74 eral infinite sequence of element in Π_n , i.e. $\{(\mathbf{x}_{k_2})_{k_2=-\infty}^{\infty}; \mathbf{x}_{k_2} \in \Pi_n, \mathbf{x}_{k_2+n, j} =$
 75 $\mathbf{x}_{k_2}, j \in \mathbb{Z}\}$.

76 Addition, scalar multiplication, Hadamard product, and convolution in
 77 Π_n easily extended to $\Pi_n^{(2)}$ by using the fact that $\Pi_n^{(2)}$ is a set of bilateral
 78 infinite array of element in Π_n .

79 The 2-dimensional DFT is defined by

$$\mathbf{y}^{(2)} = \mathcal{F}_n^{(2)} \mathbf{x}^{(2)},$$

80 where

$$y_{m_1 m_2} = \frac{1}{n^2} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \omega_n^{-(m_1 k_1 + m_2 k_2)} x_{k_1 k_2}.$$

81 Rearrangement of the right hand side, we have the following form:

$$y_{m_1 m_2} = \frac{1}{n} \sum_{k_1=0}^{n-1} \omega_n^{-m_1 k_1} \left(\frac{1}{n} \sum_{k_2=0}^{n-1} \omega_n^{-m_2 k_2} x_{k_1 k_2} \right),$$

82 Clearly that the 2-dimensional DFT can be seen as a successive of two one-
 83 dimensional DFT. Hence, the 2-dimensional DFT is a bijective map, so it is
 84 invertible. The inverse is given by:

$$(\mathcal{F}_n^{(2)})^{-1} = n^2 \mathcal{F}_n^{(2)*}.$$

85 As 2-dimensional DFT on Hadamard product and convolutions of two ele-
 86 ments in $\Pi_n^{(2)}$, the following are satisfied:

$$\begin{aligned} \mathcal{F}_n^{(2)}(\mathbf{x} \bullet \mathbf{y}) &= \mathcal{F}_n^{(2)}(\mathbf{x}) * \mathcal{F}_n^{(2)}(\mathbf{y}), \\ \mathcal{F}_n^{(2)}(\mathbf{u} * \mathbf{v}) &= n^2 (\mathcal{F}_n^{(2)}(\mathbf{x}) \bullet \mathcal{F}_n^{(2)}(\mathbf{y})). \end{aligned}$$

87 2.3 Applying DFT for solving Polyharmonic equations

88 Suppose that S is a unit square,

$$S = \{(\xi, \eta); 0 \leq \xi, \eta \leq 1\}.$$

89 We are going to solve polyharmonic equation:

$$\Delta^{(2k)}u(x, y) = f(x, y), k = 1, 2, \dots$$

90 in the rectangle S , where f is bi-periodic on S . Furthermore, we restricts u
91 to be bi-periodic, i.e. bi-periodic boundary value problem.

92 To solve the problem, rewrite the polyharmonic problem in mixed form
93 as follows :

$$\Delta^2(\dots(\Delta^2u(x, y))\dots) = f(x, y),$$

94 so to solve the problem of polyharmonic equations of even power, is equiv-
95 alent to solve a sequence of biharmonic equation recursively.

96 Let we discretize the equation over a discrete grid on S . Suppose that on
97 S we construct a uniform grid with constant spacing $h = 1/(2n)$ on each
98 direction. Let

$$\xi_i = ih ; \quad \eta_j = jh \quad i, j = 0, 1, \dots, 2n,$$

99 and obtains approximate values $u_{i,j}$ of $u(\xi_i, \eta_j)$ by solving linear equation sys-
100 tem from thirteen-points finite difference discretization of biharmonic equa-
101 tions, we have the following discrete equation for biharmonic equation ([7])

$$\begin{aligned} [20u_{i,j} &- 8(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) + \\ &+ 2(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}) + \\ &+ (u_{i+2,j} + u_{i-2,j} + u_{i,j+1} + u_{i,j-1})] \\ &= h^4 f_{i,j}, \end{aligned}$$

102 where $f_{i,j} =: \mathbf{v} \in \Pi_{2n}^{(2)}$. Observe that the left hand side is a convolution of
103 $\mathbf{u} \in \Pi_{2n}^{(2)}$ with a bilateral sequence $\mathbf{d} = (d_{km})$ with vanishes element except
104 that

$$\begin{aligned} d_{0,0} &= 20, \\ d_{1,0} = d_{-1,0} = d_{0,1} = d_{0,-1} &= -8, \\ d_{1,1} = d_{1,-1} = d_{-1,1} = d_{-1,-1} &= 2, \\ d_{2,0} = d_{-2,0} = d_{0,2} = d_{0,-2} &= 1. \end{aligned}$$

105 Then the discrete equation can be written as:

$$\mathbf{d} * \mathbf{u} = h^4 \mathbf{v}.$$

106 Applying $\mathcal{F}_{2n}^{(2)}$ to both side, we obtain

$$(2n)^2 \hat{\mathbf{d}} \bullet \hat{\mathbf{u}} = h^4 \hat{\mathbf{v}},$$

107 where $\hat{\mathbf{d}}$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are images under $\mathcal{F}_{2n}^{(2)}$ of $\mathbf{d}$, $\mathbf{u}$, and, $\mathbf{v}$ respectively, and the
108 images of $\mathbf{d}$ is

$$\begin{aligned} \hat{\mathbf{d}} = & [20 - (\cos(\frac{k_1 + k_2}{2n}\pi) \cos(\frac{k_1 - k_2}{2n}\pi)) + \\ & + 4(\cos(\frac{k_1 + k_2}{n}\pi) \cos(\frac{k_1 - k_2}{n}\pi)) + \\ & + 8(\cos(\frac{k_1}{n}\pi) \cos(\frac{k_2}{n}\pi))]_{k_1, k_2 = -\infty}^{\infty}. \end{aligned}$$

109 Multiplying using Hadamard product of $\hat{\mathbf{d}}^{-1}$, we have

$$\hat{\mathbf{u}} = \frac{h^4}{(2n)^2} \hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}},$$

110 inverting using 2-dimensional inverse DFT, we get

$$\mathbf{u} = \frac{h^4}{(2n)^2} \mathcal{F}_{2n}^{(2)-1}(\hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}}),$$

111 and we get the solutions of discrete biharmonic problem

$$\mathbf{u} = h^4 \bar{\mathcal{F}}_{2n}^{(2)}(\hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}}),$$

112 using the fact that

$$\frac{1}{(2n)^2} \mathcal{F}_{2n}^{(2)-1} = \bar{\mathcal{F}}_{2n}^{(2)}.$$

113 Remark: For the case polyharmonic of odd-power, please refer to the
114 problem of triharmonic equation addressed in [1].

115 3 Epilogue

116 We have demonstrated an algorithmic development of solving discretization
117 of biperiodic polyharmonic equations in unit rectangle. The availability of
118 DFT in several computational platforms such as MATLAB or Scilab, means
119 that the algorithms is easily benchmarked with several necessary additional
120 tools such as Hadamard product, which is unfortunately is not widely avail-
121 able in most computational platforms, hence opening for further work for
122 implementation to be pursued.

123 References

- 124 [1] A.D. Garnadi, On the Use of Discrete Fourier Transform for Solving
125 Biperiodic Boundary Value Problem of Triharmonic Equation in the
126 Unit Rectangle, INARXiv, DOI:10.17605/OSF.IO/G9D4W
- 127 [2] A.D. Garnadi, On the Use of Discrete Fourier Transform for Solving
128 Biperiodic Boundary Value Problem of Biharmonic Equation in the Unit
129 Rectangle, INARXiv, DOI: 10.17605/OSF.IO/TCFB4
- 130 [3] M.M. Gupta, Numerical methods for viscous flow problems. In:
131 A.S. Mujumdar(eds.), *Advances in Transport Processes*, Wiley Eastern,
132 New Delhi, 1980.
- 133 [4] P. Henrici, Fast Fourier methods in computational complex analysis,
134 *SIAM Rev.* **21** (1979) 481–527.
- 135 [5] Hoang, NS and Damelin, SB, 2017, On surface completion and image in-
136 painting by biharmonic functions: Numerical aspects, **arXiv preprint**
137 **arXiv:1707.06567**
- 138 [6] I. Sneddon, *Elements in Partial Differential Equations*, McGraw-Hill,
139 Tokyo, 1984.
- 140 [7] G.D. Smith, *Numerical Solution of Partial Differential Equations*, Ox-
141 ford Univ. Press, London, 1975.