What is a Number?

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1 Abstract

While mathematics tends to ensure that topics of interest are well defined, definitions of numbers are often overlooked in number theory. While there may be no single definition that works in all cases, natural numbers act as a foundation for many number systems. It is using the naturals that I seek a definition that works in many cases, and which is easy to explain in a graduate level course on number theory.

2 Introduction

If a reader were to pick up a book on group theory, they would very likely find that the definition of a group is one of the first topics covered. Similarly, if one were to pick up a book on topology, they would probably come across the definition of topology, topological spaces, and so on, very early in their reading. So it is reasonable to assume that if one were to pick up a book on number theory, one of the first topics that would be addressed is the definition of a number. But alas, no definition or discussion on the nature of numbers is generally provided. This absence is not for want of attempts to define numbers. It is simply that, as natural as numbers seem to be, they can be difficult to define mathematically. In this paper, I will attempt to provide a definition of numbers, and argue for validity.

Hume's Principle was utilized by Gottlob Frege, in the late 19th century, as one attempt to at least evidence the existence of numbers[1]. However, showing that an equinumerous property reasonably exists, and describing numbers are two very different tasks. One common method of describing numbers is to use ZF set theory. Once the basic axioms of set theory, including the existence of an empty set, basic set operations, and so on, one can define zero as the empty set, as well as a successor function, and define the naturals[??]. Many other number systems can be defined, starting with the natural numbers, and so one could argue that one of the most fundamental systems of numbers is the natural numbers. It is through the natural numbers that I will define number systems and numbers.

3 Prelims

3.1 Semirings

Aside from a construction of the natural numbers themselves, two main concepts in mathematics are of importance for the definition that I will present. The first is semirings. A semiring is almost a ring, except that it does not require that every element have an additive inverse[3].

3.2 Embeddings

The second necessary concept is that of an embedding. An embedding, for this discussion, is an injective function f , from one semiring R , to another semiring R' , s.t. for $a, b \in R$

1. $f(a + b) = f(a) + f(b)$
2. $f(ab) = f(a)f(b)$
3. $f(e) = e'$, where e is the identity multiplicative identity for R and e' is the multiplicative identity for R' . [2]

4 Definition

We can start by defining a number system as a semiring, which can be embedded into the natural numbers or into which the natural numbers can be embedded. So this would include the integers, the rationals, the reals, complex numbers, and so on. It would also include the p-adics, hyperreals, and other extensions of rational numbers, which are themselves an extension of the natural numbers.

With the definition of a number system out of the way, we can then define a number as any member of a number system. However, There may be an issue with this definition. Polynomials, with real valued coefficients, are extensions of the naturals. But one might ask whether they are actually numbers. They do not look like it, at first glance. But consider complex numbers. It is generally accepted that the complex numbers are indeed numbers. But consider the following class of matrices, under matrix addition and matrix multiplication.

$$X = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

At first glance, this system does not look like a number system either, but it is simply one way of defining complex numbers of the form $a + bi$. Therefore something does not have to immediately look like a number in order for it to be one.

The usual vector space, on the other hand, would not be considered a number

system. While vectors do form a group, they do not form a ring. Multiplication between two vectors produces a scalar, rather than another vector. So at the very least, the definition provided is not so broad as to make it difficult to find a collection of objects that do not constitute a number system.

Similarly, the Octonions are not a number system, under the definition provided in this paper, because octonion multiplication is not associative[?]. An interesting question arises from the Octonions. Because the naturals can easily be embedded into the Octonions, one of the two requirements is met. So there are elements that might be called numbers, within the Octonions. But the Octonions are not a number system, so does that mean that there are elements in the Octonions which are not numbers? And if we can identify numbers and non-numbers, can we investigate other qualities that differentiate the two?

References

- [1] Jeffrey F. Sicha. “Counting and the Natural Numbers”. In: *Philosophy of Science* 37.3 (1970), pp. 405–416. DOI: 10.1086/288316. eprint: <https://doi.org/10.1086/288316>. URL: <https://doi.org/10.1086/288316>.
- [2] Wikipedia contributors. *Ring homomorphism* — *Wikipedia, The Free Encyclopedia*. https://en.wikipedia.org/w/index.php?title=Ring_homomorphism&oldid=865749870. [Online; accessed 28-December-2018]. 2018.
- [3] Wikipedia contributors. *Semiring* — *Wikipedia, The Free Encyclopedia*. <https://en.wikipedia.org/w/index.php?title=Semiring&oldid=871565375>. [Online; accessed 28-December-2018]. 2018.