ANOTHER PROOF OF ZHANG'S CONGRUENCE FOR THE EULER NUMBERS

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ABSTRACT. We give a proof of Zhang's congruence for the Euler numbers. The proof uses an explicit formula for the Euler numbers in terms of the Stirling numbers of the second kind.

Definition 1. The *Euler numbers* are a sequence of integers, denoted by E_n , which can be defined by the following Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine.

Zhang [1] proved the following

Theorem 1. *We have, for all odd prime p ,*

$$(-1)^{\frac{p-1}{2}} E_{p-1} \equiv -1 + (-1)^{\frac{p-1}{2}} \pmod{p}. \quad (1)$$

Above result was proved in [1] by studying the calculating problem of the summation involving the Euler numbers, that is,

$$\sum_{a_1+a_2+\dots+a_n=k} \frac{E_{2a_1} E_{2a_2} \cdot E_{2a_k}}{(2a_1)! (2a_2)! \cdots (2a_k)!}$$

where the summation is over all k -dimension nonnegative integer coefficients $(a_1, a_2, \dots, a_k)$ such that $a_1 + a_2 + \dots + a_k = n$, and k is any odd number with $k > 1$.

We prove Theorem 1 using the following explicit formula for E_r , where r is even.

Lemma 1. *We have, for every even r ,*

$$E_r = 2 \sum_{k=1}^r (-1)^k \cdot k! \cdot S(r, k) \left(\frac{1+i}{2} \right)^{k+1}, \quad (2)$$

where $i = \sqrt{-1}$, and $S(r, k)$ represents Stirling numbers of the second kind.

2010 *Mathematics Subject Classification.* 11B68.

Key words and phrases. Euler numbers, Zhang's congruence.

Proof. Let A_r be the number of alternating permutations of the set $\{1, 2, \dots, r\}$. (Let S_n be the symmetric group of all permutations of $[n] := \{1, 2, \dots, n\}$. A permutation $w = a_1 a_2 \dots a_n \in S_n$ is called *alternating* if $a_1 > a_2 < a_3 > a_4 < \dots$. In other words, $a_i < a_{i+1}$ for i even, and $a_i > a_{i+1}$ for i odd). The formula (2) is a consequence of the following explicit formula for A_r

$$A_r = 2i^{r+1} \mathbf{Li}_{-r}(-i) = i^r \sum_{k=1}^r \frac{(-1)^k \cdot k! \cdot S(r, k)}{2^k} (i+1)^{k+1}, \quad (3)$$

where $\mathbf{Li}_{-r}(\cdot)$ is negative polylogarithm function, and $S(r, k)$ denotes the Stirling number of the second kind.

When r is even, we have, $A_r = (-1)^{\frac{r}{2}} E_r$. We, now, proceed to prove the above result.

The *polylogarithm function* $\mathbf{Li}_s(z)$ is defined by the following power series in z

$$\mathbf{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad (4)$$

where $|z| < 1$. We know that, from [2],

$$\mathbf{Li}_s(-i) = -2^{-s} \eta(s) - i\beta(s)$$

where $\eta(s)$ and $\beta(s)$ are the Dirichlet eta and beta functions, respectively. Letting, $s = -r$, a negative integer in the above relation, we get

$$\mathbf{Li}_{-r}(-i) = -2^{-s} \eta(-r) - i\beta(-r) = - \left(2^r \frac{2^{r+1} - 1}{r+1} B_{r+1} + i \frac{E_r}{2} \right),$$

where B_{r+1} and E_r represent Bernoulli and Euler numbers, respectively.

We conclude our result (3) after the following fact, from [3],

$$A_r = \begin{cases} (-1)^{\frac{r-1}{2}} \frac{2^{r+1}(2^{r+1}-1)B_{r+1}}{r+1} & \text{when } r \text{ is odd} \\ (-1)^{\frac{r}{2}} E_r & \text{when } r \text{ is even} \end{cases},$$

and the following representation from the note [4]

$$\mathbf{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-x)^k.$$

□

We would also require the following congruence involving Stirling numbers of the second kind

Lemma 2. *We have, for all odd prime p ,*

$$(-1)^k \cdot k! \cdot S(p-1, k) \equiv -1 \pmod{p}, \quad (5)$$

for $1 \leq k \leq p-1$.

Proof. We have the following explicit formula for Stirling numbers of the second kind [5]

$$S(p-1, k) = \frac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} j^{p-1}.$$

Using the above, and Fermat's little theorem we can conclude our result. $\square$

Proof of Theorem 1. Letting $r = p-1$, where p is an odd prime, in equation (2) gives us

$$E_{p-1} = 2 \sum_{k=1}^{p-1} (-1)^k \cdot k! \cdot S(p-1, k) \left(\frac{1+i}{2} \right)^{k+1}.$$

Considering the fact that $2^{p-1} \equiv 1 \pmod{p}$, and Lemma 2, we have,

$$\begin{aligned} E_{p-1} &\equiv 2^{p-1} E_{p-1} \equiv - \sum_{k=1}^{p-1} 2^p \cdot \left(\frac{1+i}{2} \right)^{k+1} \pmod{p} \\ &\equiv -2^p \cdot \left(\frac{-1}{2} + 2^{-\frac{p}{2}} \sin \left(\frac{\pi p}{4} \right) + i \left(\frac{1}{2} - 2^{-p/2} \cos \left(\frac{\pi p}{4} \right) \right) \right) \pmod{p}. \end{aligned}$$

Now, discarding the imaginary part in the above, since, E_{p-1} are integers, we get

$$\begin{aligned} E_{p-1} &\equiv -2^p \cdot \left(\frac{-1}{2} + 2^{-\frac{p}{2}} \sin \left(\frac{\pi p}{4} \right) \right) \pmod{p} \\ &\equiv 2^{p-1} - 2^{\frac{p-1}{2}} \cdot \sqrt{2} \cdot \sin \left(\frac{\pi p}{4} \right) \pmod{p}. \end{aligned}$$

But, we have

$$2^{\frac{p-1}{2}} \equiv \left(\frac{2}{p} \right) \equiv (-1)^{\frac{p^2-1}{8}} \pmod{p},$$

where $\left(\frac{a}{b} \right)$ represents the Legendre's symbol. As a result, we can conclude that

$$E_{p-1} \equiv 1 - (-1)^{\frac{p^2-1}{8}} \cdot \sqrt{2} \sin \left(\frac{\pi p}{4} \right) \pmod{p}.$$

Now, since $\sqrt{2} \sin \left(\frac{\pi p}{4} \right) = (-1)^{\frac{-(p-1)(p-3)}{8}}$, we have

$$E_{p-1} \equiv 1 + (-1)^{\frac{p+1}{2}} \pmod{p}$$

which is our result Theorem 1. $\square$

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