Pricing Model and Credit Valuation Adjustment

Tim Xiao

ABSTRACT

This article presents a generic model for pricing derivatives subject to counterparty credit risk. We study various cases: one-way and two-way, single payment and multiple payments, positive and negative payoffs. Our theory shows that the valuation of defaultable derivatives in most situations requires a backward induction procedure. There is an intuitive way of understanding these backward induction behaviours: We can think that any defaultable derivative with two-way credit risk embeds two default options. In other words, when entering a defaultable financial transaction, one party grants the other party an option to default and, at the same time, also receives an option to default itself. In theory, default may occur at any time. Therefore, default options are American style options that normally require a backward induction valuation.

Key Words: credit value adjustment (CVA), credit risk modeling, financial derivative valuation, collateralization, margin and netting.

A broad range of financial instruments bear credit risk. Some derivatives such as, debt instruments by nature contain only one-way credit risk because only the default risk of one party appears to be relevant. Whereas some other derivatives, such as, over the counter (OTC) derivatives and securities financing transactions (SFT), bear two-way credit risk because both parties are susceptible to default risk.

In the market, risk-free values are quoted for most financial derivatives. Historical experience shows that credit risk often leads to significant losses. Therefore, it is obvious to all market participants that credit risk should be taken into account when reporting the fair value of any defaultable derivative.

The adjustment to the risk-free value is known as the credit value adjustment (CVA). CVA offers an opportunity for banks to dynamically price credit risk into new trades and has become a common practice in the financial industry, especially for trading books. By definition, CVA is the difference between the risk-free value and the true (or risky or defaultable) value that takes into account the possibility of default.

There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form (or intensity) models. The structural models, studied by Merton (1974) and Longstaff and Schwartz (1995), regard default as an endogenous event, focusing on the capital structure of a firm. The reduced-form models introduced by Jarrow and Turnbull (1995) and Duffie and Singleton (1999) do not explain the event of default endogenously, but instead characterize it exogenously as a jump process. For pricing and hedging, reduced-form models are the preferred methodology.

Three different recovery models exist in the literature. The default payoff is either 1) a fraction of par (Madan and Unal (1998)), 2) a fraction of an equivalent default-free bond (Jarrow and Turnbull (1995)), or 3) a fraction of market value (Duffie and Singleton (1999)). The last one is most commonly used in the market. In their paper, Duffie and Singleton (1999) do not clearly state whether the market value of a defaultable derivative is a risky value or a risk-free value.

Instead, the authors implicitly treat the market value as a risky value because the market price therein evolves in a defaultable manner. Otherwise, they cannot obtain the desired result. However, most

of the later papers in the literature mistakenly think that the market value of a defaultable derivative is a risk-free value. Consequently, the results are incorrect.

1. Basic Setup

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

The default model is based on the reduced-form approach proposed by Duffie and Singleton (1999) and Jarrow and Turnbull (1994), which does not explain the event of default endogenously, but characterizes it exogenously by a jump process. The stopping (or default) time τ is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default with a stochastic hazard rate or arrival intensity $h(t)$.

It is well-known that the survival probability, conditional on a realization path, from time t to s in this framework is defined by

$$S(t, s) := \Pr(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u) du\right) \quad (1a)$$

where Z is the firm-specific state process that helps forecast that firm's default. The default probability, conditional on the realization path, for the period (t, s) in this framework is defined by

$$Q(t, s) = \Pr(\tau \leq s \mid \tau > t, Z) = 1 - S(t, s) = 1 - \exp\left(-\int_t^s h(u) du\right) \quad (1b)$$

Applying the law of iterated expectations, we express the expected survival probability for the period (t, s) as

$$\bar{S}(t, s) = E\left[\exp\left(-\int_t^s h(u) du\right) \mid \mathcal{F}_t\right] \quad (2a)$$

where $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$. The expected default probability for the period (t, s) is expressed as

$$\bar{Q}(t, s) = 1 - \bar{S}(t, s) = 1 - E\left[\exp\left(-\int_t^s h(u)du\right)|\mathcal{F}_t\right] \quad (2b)$$

There are three different recovery models in the literature. The default payoff is either 1) a fraction of par, 2) a fraction of an equivalent default-free bond, or 3) a fraction of market value. We use the recovery model of market value in our study.

The binomial default rule considers only two possible states: default or survival. For a discrete one-period (t, s) economy, at the end of the period a defaultable derivative either defaults with the default probability $Q(t, s)$ or survives with the survival probability $S(t, s)$. Assume that the market value of the defaultable derivative at time s is $V^D(s)$. The default payoff is a fraction of the market value given by $\varphi(s)V^D(s)$, where $\varphi(s)$ is the default recovery rate at time s , whereas the survival payoff is equal to the market value $V^D(s)$ itself. Therefore, the risky value of the derivative, conditional on the realization path, is the discounted expectation of the payoffs, given by

$$V^D(t) = D(t, s)\left(Q(t, s)\varphi(s)V^D(s) + S(t, s)V^D(s)\right) \quad (3a)$$

where

$$D(t, s) = \exp\left[-\int_t^s r(u)du\right] \quad (3b)$$

If the derivative survives at time s , the survival value is equal to the market value. However, the derivative may default at any time after s . Therefore, the present survival value or the market value at time s is a risky value. In other words, for risk-free valuations, the market value is a risk-free value, but for risky valuations, the market value is actually a risky value.

2. One-way Defaultable Derivatives Valuation and One-way CVA

2.1. Single Payment Cases

Consider a defaultable derivative that promises to pay a X_T from party B to party A at maturity date T , and nothing before date T . The payoff X_T may be positive or negative, i.e. the derivative may be either an asset or a liability to each party. The risk-free value of the derivative at valuation date t is given by

$$V^F(t) = E\{D(t, T)X_T | \mathcal{F}_t\} \quad (4)$$

We divide the time period (t, T) into n very small time intervals (Δt) and assume that a default may occur only at the end of each very small period.

Proposition 1: *The one-way risky value of the single payment derivative is given by*

$$V^D(t) = E\{C_B(t, T)X_T | \mathcal{F}_t\} \quad (5a)$$

where

$$C_B(t, T) = \exp\left[-\sum_{i=0}^{n-1} c_B(t + i\Delta t)\Delta t\right] \quad (5b)$$

$$c_B(t + i\Delta t) = r(t + i\Delta t) + 1_{V^D(t+(i+1)\Delta t) \geq 0} h_B(t + i\Delta t)(1 - \varphi_B(t + i\Delta t)) \quad (5c)$$

Proof: See the Appendix.

We may think of $C_B(t, T)$ as the risk-adjusted discount factor and $c_B(u)$ as the risk-adjusted short rate. Here $\varphi_B(u)$ represents the default recovery rate of party B , and $h_B(u)$ represents the hazard rate of party B . $s_B(u) = h_B(u)(1 - \varphi_B(u))$ is called the credit spread or the short credit spread.

Compared to the risk-free valuation (4), the risky valuation (5) is substantially complex. The intermediate values are vital to determine the final price. For a small time interval, the current risky value has a dependence on the future risky value. Only on the final payment date T , the value of the derivative and the maximum amount of information needed to determine the risk-adjusted discount factor are revealed. This type of problem can be best solved by working backwards in time, with the later risky value feeding into the earlier ones, so that the process builds on itself in a recursive fashion, which is referred to as

backward induction. The most popular backward induction valuation algorithms are lattice/tree and regression-based Monte Carlo.

For an intuitive explanation, we can posit that a defaultable derivative under the one-way credit risk assumption has an embedded default option. In other words, one party entering a defaultable financial transaction actually grants the other party an option to default. If we assume that a default may occur at any time, the default option is an American style option. American options normally have backward recursive natures and require backward induction valuations.

Proposition 1 can be further expressed in the form of the CTM as:

Corollary 1.1: *The one-way risky value of the single payment derivative under the CTM is given by*

$$V^D(t) = E\{\bar{C}_B(t, T)X_T | \mathcal{F}_t\} \quad (6a)$$

where

$$\bar{C}_B(t, T) = \exp\left[-\int_t^T \bar{c}_B(u) du\right] \quad (6b)$$

$$\bar{c}_B(u) = r(u) + 1_{V^D(u) \geq 0} h_B(u)(1 - \varphi_B(u)) \quad (6c)$$

The proof of this corollary becomes straightforward, according to Proposition 1, by taking the limit as Δt approaches zero.

Proposition 1 provides a general form for pricing a one-way defaultable single payment derivative. Applying it to a particular situation in which we assume that the payoff X_T is non-negative, we derive the following corollary:

Corollary 1.2: *If the payoff X_T is non-negative, the risky value of the single payment derivative under the CTM is given by*

$$V^D(t) = E\{\hat{C}_B(t, T)X_T | \mathcal{F}_t\} \quad (7a)$$

where

$$\hat{C}_B(t, T) = \exp\left[-\int_t^T \hat{c}_B(u) du\right] \quad (7b)$$

$$\hat{c}_B(u) = r(u) + h_B(u)(1 - \varphi_B(u)) \quad (7c)$$

The proof of this corollary is easily obtained according to (6) by setting $V^D(u) \geq 0$. Since the payoff X_T is non-negative, the intermediate value $V^D(u)$ is also non-negative.

The risky valuation becomes quite simple when the payoff is non-negative. This corollary says that a single non-negative payment defaultable derivative can be priced using the present value of the promised payoff X_T discounted by the risk-adjusted discount rate $\hat{c}_B(u)$ instead of the risk-free rate $r(u)$.

By definition, the corresponding CVA of the single non-negative payment derivative can be expressed as

$$CVA^U(t) = V^F(t) - V^D(t) = E\{D(t, T)X_T | \mathcal{F}_t\} - E\{\hat{C}_B(t, T)X_T | \mathcal{F}_t\} \quad (8)$$

Since $\hat{C}_B(t, T)$ is always smaller than $D(t, T)$, the CVA is, in this case, a credit charge. The credit charge covering party A's potential loss comes from the scenario of party B's default when party A is in the money in the portfolio.

Under the CTM, a one-way defaultable derivative has an embedded American style default option, which is continuously defaultable. Since American options are difficult to hedge, a common practice in the market is to use Bermudan options to approximate American ones. Consequently, we assume that a default may only happen at some discrete times. A natural selection is to assume that a default may occur only on the payment dates. The Discrete-Time Risky Valuation Model, or simply the Discrete-Time Model (DTM), assumes that a defaultable derivative may only default on payment dates.

Proposition 2: *The one-way risky value of the single payment derivative under the DTM is given by*

$$V^D(t) = E[G_B(t, T)X_T | \mathcal{F}_t] \quad (9a)$$

where

$$G_B(t, T) = D(t, T)[1 - 1_{X_T \geq 0} Q_B(t, T)(1 - \varphi_B(T))] \quad (9b)$$

Proof: See the Appendix.

Here we may consider $G_B(t, T)$ as the risk-adjusted discount factor. Proposition 2 states that the one-way risky valuation of the single payoff derivative has a dependence on the sign of the payoff. If the payoff is positive, the risky value is equal to the risk-free value minus the discounted potential loss. Otherwise, the risky value is equal to the risk-free value.

The corresponding one-way CVA of the single payment derivative under the DTM can be expressed as

$$\begin{aligned} CVA^U(t) &= V^F(t) - V^D(t) = E\{D(t, T)X_T | \mathcal{F}_t\} - E\{G_B(t, T)X_T | \mathcal{F}_t\} \\ &= E\left[1_{X_T \geq 0} D(t, T)(1 - \varphi_B(T))Q_B(t, T)X_T | \mathcal{F}_t\right] \end{aligned} \quad (10)$$

Proposition 2 has a general form that applies in a particular situation in which we assume that the payoff X_T is non-negative.

Corollary 2.1: *If the payoff X_T is non-negative, the risky value of the single payment derivative under the DTM is given by*

$$V^D(t) = E[\bar{G}_B(t, T)X_T | \mathcal{F}_t] \quad (11a)$$

where

$$\bar{G}_B(t, T) = D(t, T)[1 - Q_B(t, T)(1 - \varphi_B(T))] = D(t, T)(S_B(t, T) + \varphi_B(T)Q_B(t, T)) \quad (11b)$$

The proof of this corollary is straightforward according to Proposition 2 by setting $X_T \geq 0$. Equations (11) are consistent with equations (3).

2.2. Multiple Payments Cases

Suppose that a defaultable derivative has m cash flows. Any of these cash flows may be positive or negative. Let the m cash flows be represented as $X_1, \dots, X_m$ with payment dates $T_1, \dots, T_m$. The risk-free price of the derivative is given by

$$V^F(t) = \sum_{i=1}^m E\{D(t, T_i)X_i | \mathcal{F}_t\} \quad (12)$$

We divide any payment date period (T_{i-1} , T_i) into n_i very small time intervals (Δt) and assume that a default may occur only at the end of each very small period.

Proposition 3: *The one-way risky value of the multiple payments derivative is given by*

$$V^D(t) = \sum_{i=1}^m E[(C_B(t, T_i))X_i | \mathcal{F}_t] \quad (13a)$$

where

$$C_B(t, T_i) = \exp \left[- \sum_{j=0}^{\sum_{k=1}^i n_k - 1} c_B(t + j\Delta t) \Delta t \right] \quad (13b)$$

$$c_B(t + j\Delta t) = r(t + j\Delta t) + 1_{V^D(t+(j+1)\Delta t) \geq 0} h_B(t + j\Delta t) (1 - \varphi_B(t + j\Delta t)) \quad (13c)$$

Proof: See the Appendix.

Proposition 3 says that the pricing process of a multiple payments derivative has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. In other words, the present value takes into account the results of all future decisions. On the final payment date, the value of a derivative and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

Corollary 3.1: *The one-way risky value of the multiple payments derivative under the CTM is given by*

$$V^D(t) = \sum_{i=1}^m E[\bar{C}_B(t, T_i) X_i | \mathcal{F}_t] \quad (14)$$

where

$$\bar{C}_B(t, T_i) = \exp \left[- \int_t^{T_i} \bar{c}_B(u) du \right] \quad (14b)$$

$$\bar{c}_B(u) = r(u) + 1_{V^D(u) \geq 0} h_B(u) (1 - \varphi_B(u)) \quad (14c)$$

The proof of this corollary is easily obtained, according to Proposition 3, by taking the limit as Δt approaches zero.

Proposition 3 has a general form that applies in a particular situation in which we assume that all the payoffs are non-negative.

Corollary 3.2: *If all the payoffs are non-negative, the risky value of the multiple payments derivative under the CTM is given by*

$$V^D(t) = \sum_{i=1}^m E\left[\left(\widehat{C}_B(t, T_i)\right)X_i \middle| \mathcal{F}_t\right] \quad (15a)$$

where

$$\widehat{C}_B(t, T_i) = \exp\left[-\int_t^{T_i} \widehat{c}_B(u) du\right] \quad (15b)$$

$$\widehat{c}_B(u) = r(u) + h_B(u)(1 - \varphi_B(u)) \quad (15c)$$

The proof of this corollary is straightforward, according to (14), by setting $V^D(u) \geq 0$.

This is the formula for pricing defaultable bonds in the market. Corollary 3.2 says that if all the payoffs are positive, we can evaluate each payoff separately and sum the corresponding results. In other words, payoffs in this case can be treated as independent.

Proposition 4: *The one-way risky value of the multiple payments derivative under the DTM is given by*

$$V^D(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} G_B(T_j, T_{j+1})\right)X_i \middle| \mathcal{F}_t\right] \quad (16a)$$

where $t = T_0$ and

$$G_B(T_j, T_{j+1}) = D(T_j, T_{j+1}) \left[1 - 1_{(X_{j+1} + V^D(T_{j+1})) \geq 0} Q_B(T_j, T_{j+1})(1 - \varphi_B(T_{j+1}))\right] \quad (16b)$$

Proof: See the Appendix.

Similar to Proposition 3, the individual payoffs under Proposition 4 cannot be evaluated separately. The current risky value depends on the future risky value. This type of problem is usually solved using backward induction algorithms.

Proposition 4 has a general form that applies in a particular situation where we assume that all payoffs are non-negative.

Corollary 4.1: *If all the payoffs are non-negative, the risky value of the multiple payments derivative under the DTM is given by*

$$V^D(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} \bar{G}_B(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] \quad (17a)$$

Where $t = T_0$ and

$$\bar{G}_B(T_j, T_{j+1}) = D(T_j, T_{j+1}) [1 - Q_B(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}))] \quad (17b)$$

The proof of this corollary is easily obtained according to Proposition 4 by setting $(X_{j+1} + V^D(T_{j+1})) \geq 0$.

3. Two-way Defaultable Derivative Valuation and Two-way CVA

A critical ingredient of the pricing of a two-way defaultable derivative is the rules for settlement in default. There are two rules in the market. The *one-way payment rule* was specified by the early International Swap Dealers Association (ISDA) master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the derivative is positive for the defaulting party.

The default indicator $\hat{j}$ for party j ($j=A, B$) is a random variable with a Bernoulli distribution, which takes value 1 with default probability Q_j , and value 0 with survival probability S_j . Consider a pair of random variables $(\hat{A}, \hat{B})$ that has a bivariate Bernoulli distribution as summarized in Table 1.

Table 1. Bivariate Bernoulli Distribution

		Joint Distribution		Marginal Distribution
		$\hat{A} = 1$	$\hat{A} = 0$	
Joint Distribution	$\hat{B} = 1$	$Q_B Q_A + \gamma$	$Q_B S_A - \gamma$	Q_B

	$\hat{B}=0$	$S_B Q_A - \gamma$	$S_B S_A + \gamma$	S_B
Marginal Distribution		Q_A	S_A	

3.1. Single Payment Cases

Consider a defaultable derivative that promises to pay a X_T from party B to party A at maturity date T and nothing before date T . The payoff may be either an asset or a liability to each party.

We divide the time period (t, T) into n very small time intervals (Δt) and assume that a default may occur only at the end of each very small period.

Proposition 5: *The two-way risky value of the single payment derivative is given by*

$$V^D(t) = E\{O(t, T)X_T | \mathcal{F}_t\} \quad (18a)$$

where

$$O(t, T) = \exp\left(-\sum_{i=0}^{n-1} o(t + i\Delta t)\Delta t\right) \quad (18b)$$

$$o(t + i\Delta t) = r(t + i\Delta t) + 1_{V(t+(i+1)\Delta t) \geq 0} p_B(t + i\Delta t) + 1_{V(t+(i+1)\Delta t) < 0} p_A(t + i\Delta t) \quad (18c)$$

$$\begin{aligned} p_B(t + i\Delta t) = & (1 - \varphi_B(t + i\Delta t))h_B(t + i\Delta t) + (1 - \bar{\varphi}_B(t + i\Delta t))h_A(t + i\Delta t) \\ & - (1 - \varphi_B(t + i\Delta t) - \bar{\varphi}_B(t + i\Delta t) + \varphi_{AB}(t + i\Delta t)) \\ & \times \left[\rho \sqrt{h_A(t + i\Delta t)h_B(t + i\Delta t)(1 - h_B(t + i\Delta t)\Delta t)(1 - h_A(t + i\Delta t)\Delta t)} \right. \\ & \left. + h_B(t + i\Delta t)h_A(t + i\Delta t)\Delta t \right] \end{aligned} \quad (18d)$$

$$\begin{aligned} p_A(t + i\Delta t) = & (1 - \varphi_A(t + i\Delta t))h_A(t + i\Delta t) + (1 - \bar{\varphi}_A(t + i\Delta t))h_B(t + i\Delta t) \\ & - (1 - \varphi_A(t + i\Delta t) - \bar{\varphi}_A(t + i\Delta t) + \varphi_{AB}(t + i\Delta t)) \\ & \times \left[\rho \sqrt{h_A(t + i\Delta t)h_B(t + i\Delta t)(1 - h_B(t + i\Delta t)\Delta t)(1 - h_A(t + i\Delta t)\Delta t)} \right. \\ & \left. + h_B(t + i\Delta t)h_A(t + i\Delta t)\Delta t \right] \end{aligned} \quad (18e)$$

Proof: See the Appendix.

We may think of $O(t, T)$ as the two-way risk-adjusted discount factor and $o(u)$ as the two-way risk-adjusted short rate; φ_j ($j=A, B$) represents the default recovery rate of party j , i.e. the fraction of the market value paid by the defaulting party j when the market value is negative for j ; h_j represents the hazard

rate of party j ; $\bar{\varphi}_j$ represents the non-default recovery rate of party j , i.e. the fraction of the market value paid by non-defaulting party j when the market value is negative for j . $\bar{\varphi}_j = 0$ represents the one-way settlement rule, while $\bar{\varphi}_j = 1$ represents the two-way settlement rule. ρ denotes the default correlation coefficient of A and B . φ_{AB} denotes the joint recovery rate when both parties A and B default simultaneously.

Proposition 5 provides a general form for pricing a two-way risky single payment derivative. Applying it to a particular situation in which we assume that parties A and B do not default simultaneously and have independent default risks, i.e. $\rho = 0$ and $\varphi_{AB} = 0$, we derive the following corollary.

Corollary 5.1: *If parties A and B do not default simultaneously and have independent default risks, the two-way risky value of the single payment derivative under the CTM is given by*

$$V^D(t) = E\{\bar{O}(t, T)X_T | \mathcal{F}_t\} \quad (19a)$$

where

$$\bar{O}(t, T) = \exp\left[-\int_t^T \bar{o}(u) du\right] \quad (19b)$$

$$\bar{o}(u) = r(u) + 1_{V^D(u) \geq 0} \bar{p}_B(u) + 1_{V^D(u) < 0} \bar{p}_A(u) \quad (19c)$$

$$\bar{p}_B(u) = (1 - \varphi_B(u))h_B(u) + (1 - \bar{\varphi}_B(u))h_A(u) \quad (19d)$$

$$\bar{p}_A(u) = (1 - \varphi_A(u))h_A(u) + (1 - \bar{\varphi}_A(u))h_B(u) \quad (19e)$$

The proof of this corollary becomes straightforward according to Proposition 5 by setting $\rho = 0$ and $\varphi_{AB} = 0$, taking the limit as Δt approaches zero, and having $h_B(u)h_A(u)\Delta t^2 \approx 0$ for very small Δt .

Corollary 5.1 is the same as equation (2.5') in Duffie and Huang (1996), but their derivation is heuristic rather than rigorous.

The corresponding two-way CVA of the single payment derivative in this case can be expressed as

$$CVA^B(t) = V^F(t) - V^D(t) = E\{D(t, T)X_T | \mathcal{F}_t\} - E\{\bar{O}(t, T)X_T | \mathcal{F}_t\} \quad (20)$$

Since $\bar{O}(t, T)$ is always smaller than $D(t, T)$, the two-way CVA may be positive or negative depending on the sign of the payoff. In other words, the CVA may be either a credit charge or a credit benefit. A credit charge that is the value of a default loss comes from the scenario of one's counterparty defaulting when one is in the money, while a credit benefit is the value of a default gain in the scenario of one's own default when one is out of money.

Proposition 6: *The two-way risky value of the single payment derivative under the DTM is given by*

$$V^D(t) = E[Y(t, T)X_T | \mathcal{F}_t] \quad (21a)$$

where

$$Y(t, T) = D(t, T) \left(1_{X_T \geq 0} y_B(t, T) + 1_{X_T < 0} y_A(t, T) \right) \quad (21b)$$

$$y_B(t, T) = S_B(t, T)S_A(t, T) + \varphi_B(T)Q_B(t, T)S_A(t, T) + \bar{\varphi}_B(T)S_B(t, T)Q_A(t, T) + \varphi_{AB}(T)Q_B(t, T)Q_A(t, T) + \gamma(t, T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T)) \quad (21c)$$

$$y_A(t, T) = S_B(t, T)S_A(t, T) + \varphi_A(T)Q_A(t, T)S_B(t, T) + \bar{\varphi}_A(T)S_A(t, T)Q_B(t, T) + \varphi_{AB}(T)Q_B(t, T)Q_A(t, T) + \gamma(t, T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T)) \quad (21d)$$

$$\gamma(t, T) = \rho \sqrt{S_B(t, T)Q_B(t, T)S_A(t, T)Q_A(t, T)} \quad (21e)$$

Proof: See the Appendix.

We may think of $Y(t, T)$ as the risk-adjusted discount factor. Proposition 6 tells us that the two-way risky price of a single payment derivative can be expressed as the present value of the payoff discounted by a risk-adjusted discount factor that has a switching-type dependence on the sign of the payoff.

Proposition 6 has a general form that applies in a particular situation where we assume that parties A and B do not default simultaneously and have independent default risks, i.e. $\rho = 0$ and $\varphi_{AB} = 0$.

Corollary 6.1: *If parties A and B do not default simultaneously and have independent default risks, the two-way risky value of the single payment derivative under the DTM is given by*

$$V^D(t) = E\{\bar{Y}(t, T)X_T | \mathcal{F}_t\} \quad (22a)$$

where

$$\bar{Y}(t, T) = D(t, T) \left(1_{X_T \geq 0} \bar{y}_B(t, T) + 1_{X_T < 0} \bar{y}_A(t, T) \right) \quad (22b)$$

$$\bar{y}_B(t, T) = S_B(t, T) S_A(t, T) + \varphi_B(T) Q_B(t, T) S_A(t, T) + \bar{\varphi}_B(T) S_B(t, T) Q_A(t, T) \quad (22c)$$

$$\bar{y}_A(t, T) = S_B(t, T) S_A(t, T) + \varphi_A(T) Q_A(t, T) S_B(t, T) + \bar{\varphi}_A(T) S_A(t, T) Q_B(t, T) \quad (22d)$$

The proof of this corollary is easily obtained according to Proposition 6 by setting $\rho = 0$ and

$$\varphi_{AB} = 0.$$

3.2. Multiple Payments Cases

Suppose that a defaultable derivative has m cash flows. Let the m cash flows be represented as $X_1, \dots, X_m$ with payment dates $T_1, \dots, T_m$. Each cash flow may be positive or negative.

Proposition 7: *The two-way risky value of the multiple payments derivative is given by*

$$V^D(t) = \sum_{i=1}^m E \left[(O(t, T_i)) X_i \middle| \mathcal{F}_t \right] \quad (23a)$$

where

$$O(t, T_i) = \exp \left[- \sum_{j=0}^{\sum_{k=1}^i n_k - 1} o(t + j\Delta t) \Delta t \right] \quad (23b)$$

$$o(t + j\Delta t) = r(t + j\Delta t) + 1_{V(t+(j+1)\Delta t) \geq 0} p_B(t + j\Delta t) + 1_{V(t+(j+1)\Delta t) < 0} p_A(t + j\Delta t) \quad (23c)$$

$$\begin{aligned} p_B(t + j\Delta t) = & (1 - \varphi_B(t + j\Delta t)) h_B(t + j\Delta t) + (1 - \bar{\varphi}_B(t + j\Delta t)) h_A(t + j\Delta t) \\ & - (1 - \varphi_B(t + j\Delta t) - \bar{\varphi}_B(t + j\Delta t) + \varphi_{AB}(t + j\Delta t)) \\ & \times \left[\rho \sqrt{h_A(t + j\Delta t) h_B(t + j\Delta t) (1 - h_B(t + j\Delta t) \Delta t) (1 - h_A(t + j\Delta t) \Delta t)} \right. \\ & \left. + h_B(t + j\Delta t) h_A(t + j\Delta t) \Delta t \right] \end{aligned} \quad (23d)$$

$$\begin{aligned} p_A(t + j\Delta t) = & (1 - \varphi_A(t + j\Delta t)) h_A(t + j\Delta t) + (1 - \bar{\varphi}_A(t + j\Delta t)) h_B(t + j\Delta t) \\ & - (1 - \varphi_A(t + j\Delta t) - \bar{\varphi}_A(t + j\Delta t) + \varphi_{AB}(t + j\Delta t)) \\ & \times \left[\rho \sqrt{h_A(t + j\Delta t) h_B(t + j\Delta t) (1 - h_B(t + j\Delta t) \Delta t) (1 - h_A(t + j\Delta t) \Delta t)} \right. \\ & \left. + h_B(t + j\Delta t) h_A(t + j\Delta t) \Delta t \right] \end{aligned} \quad (23e)$$

Proof: See the Appendix.

Proposition 7 provides a general form for pricing a two-way multiple payment derivative. Applying it to a particular situation in which we assume that parties A and B do not default simultaneously and have independent default risks, i.e. $\rho = 0$ and $\varphi_{AB} = 0$, we derive the following corollary.

Corollary 7.1: *If parties A and B do not default simultaneously and have independent default risks, the two-way risky value of the single payment derivative under the CTM is given by*

$$V^D(t) = \sum_{i=1}^m E[\bar{O}(t, T_i) X_i | \mathcal{F}_t] \quad (24)$$

where $\bar{O}(t, T_i)$ is defined in (19).

The proof of this corollary becomes straightforward according to Proposition 7 by setting $\rho = 0$ and $\varphi_{AB} = 0$, taking the limit as Δt approaches zero, and having $h_B(u)h_A(u)\Delta t^2 \approx 0$ for very small Δt .

Proposition 8: *The two-way risky value of the multiple payments derivative under the DTM is given by*

$$V^D(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} Y(T_j, T_{j+1})\right) X_i | \mathcal{F}_t\right] \quad (25a)$$

where $t = T_0$ and

$$Y(T_j, T_{j+1}) = D(T_j, T_{j+1}) \left(1_{(X_{j+1} + V^D(T_{j+1})) \geq 0} y_B(T_j, T_{j+1}) + 1_{(X_{j+1} + V^D(T_{j+1})) < 0} y_A(T_j, T_{j+1}) \right) \quad (25b)$$

$$\begin{aligned} y_B(T_j, T_{j+1}) = & S_B(T_j, T_{j+1}) S_A(T_j, T_{j+1}) + \varphi_B(T_{j+1}) Q_B(T_j, T_{j+1}) S_A(T_j, T_{j+1}) + \bar{\varphi}_B(T_{j+1}) S_B(T_j, T_{j+1}) Q_A(T_j, T_{j+1}) \\ & + \varphi_{AB}(T_{j+1}) Q_B(T_j, T_{j+1}) Q_A(T_j, T_{j+1}) + \gamma(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}) - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})) \end{aligned} \quad (25c)$$

$$\begin{aligned} y_A(T_j, T_{j+1}) = & S_B(T_j, T_{j+1}) S_A(T_j, T_{j+1}) + \varphi_A(T_{j+1}) Q_A(T_j, T_{j+1}) S_B(T_j, T_{j+1}) + \bar{\varphi}_A(T_{j+1}) S_A(T_j, T_{j+1}) Q_B(T_j, T_{j+1}) \\ & + \varphi_{AB}(T_{j+1}) Q_B(T_j, T_{j+1}) Q_A(T_j, T_{j+1}) + \gamma(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}) - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})) \end{aligned} \quad (25d)$$

$$\gamma(T_j, T_{j+1}) = \rho \sqrt{S_B(T_j, T_{j+1}) Q_B(T_j, T_{j+1}) S_A(T_j, T_{j+1}) Q_A(T_j, T_{j+1})} \quad (25e)$$

Proof: See the Appendix.

Proposition 8 has a general form that applies in a particular situation where we assume that parties A and B do not default simultaneously and have independent default risks, i.e. $\rho = 0$ and $\varphi_{AB} = 0$.

Corollary 8.1: *If parties A and B do not default simultaneously and have independent default risks, the two-way risky value of the single payment derivative under the DTM is given by*

$$V^D(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} \bar{Y}(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] \quad (26)$$

where $\bar{Y}(T_j, T_{j+1})$ is defined in (22).

The proof of this corollary is easily obtained according to Proposition 8 by setting $\rho = 0$ and

$$\varphi_{AB} = 0.$$

4. Numerical Results

5.1. Comparison Between the CTM and the DTM

Here we use one-way credit risk as an example. Assume that party B has an ‘A’ rating and a constant default recovery rate of 70%. Party A is default-free. All calculations are from the perspective of party A .

Let us consider single payment derivatives first. We use a 6-month zero-coupon bond and a 1-year zero-coupon bond to analog single payment instruments. Under the CTM, the derivatives are continuously defaultable, whereas under the DTM, the derivatives may default only at 6 months or 1 year. The principals are a unit, and the hazard rates are bootstrapped from CDS spreads. The risk-free and risky values and the CVAs are calculated according to Corollary 1.2 and Corollary 2.1, shown in Table 2 and Table 3. The results demonstrate that the CVAs under both the CTM and the DTM are very close (relative difference < 0.3%).

Then, we use a 10-year semi-annual fixed rate coupon bond to analog a multiple payments instrument. Assume that the principal is a unit and that the annual coupon rate of the fixed-rate coupon bond is 4%. The risk-free and risky values and the CVAs are calculated according to Corollary 3.2 and Corollary 4.1, shown in Table 4. The results confirm that the DTM and the CTM produce very close results.

Table 2. The Numerical Results of a 6-month Maturity Single Payment Derivative

Risk-Free Value	CTM	DTM	
-----------------	-----	-----	--

	Risky Value	CVA	Risky Value	CVA	Difference of CVA
0.998168	0.997026	0.001142	0.997028	0.001140	0.1334%

Table 3. The Numerical Results of a 1-year Maturity Single Payment Derivative

Risk-Free Value	CTM		DTM		Difference of CVA
	Risky Value	CVA	Risky Value	CVA	
0.995693	0.993422	0.002271	0.993428	0.002265	0.2658%

Table 4 The Numerical Results of a 10-Year Maturity Multiple Payments Derivative

Risk-Free Value	CTM		DTM		Difference of CVA
	Risky Value	CVA	Risky Value	CVA	
1.122892	1.032344	0.090547	1.032901	0.089990	0.6153%

5.2. Impact of Margin Agreements

In this section, we use a portfolio to study the impact of margin agreements. The portfolio consists of a number of derivatives on interest rate, equity, and foreign exchange. In the real world, it is very rare that a portfolio is positive all the time and in all scenarios. The number of simulation scenarios (or paths) is 20,000. The time buckets are set weekly. If the computational requirements exceed the system limit, one can reduce both the number of scenarios and the number of time buckets. The time buckets can be designed as fine-granularity at the short end, and coarse-granularity at the far end, e.g. daily, weekly, monthly and yearly, etc. The rationale is that calculations become less accurate due to accumulated errors from simulation discretization and inherited errors from the calibration of the underlying models, such as those

due to a change in macro-economic climate. The collateral margin period of risk is assumed to be 14 days (2 weeks).

Table 5 illustrates that if party A has an infinite collateral threshold $H_A = \infty$ i.e. no collateral requirement on A , the two-way CVA value increases, while the threshold H_B increases. Table 6 shows that if party B has an infinite collateral threshold $H_B = \infty$, the two-way CVA value actually decreases, while the threshold H_A increases. This reflects the two-way impact of the collaterals on the two-way CVA. The impact is mixed in Table 7 when both parties have finite collateral thresholds.

Table 5. The Impact of Collateral Threshold H_B on the Two-way CVA

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	19,550.91	20,528.65	21,368.44	22,059.30

Table 6. The Impact of Collateral Threshold H_A on the Two-way CVA

Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	28,283.64	25,608.92	23,979.11	22,059.30

Table 7. The Impact of Both Collateral Thresholds on the Two-way CVA

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	25,752.98	22,448.45	23,288.24	22,059.30

5.3. Impact of Wrong or Right Way Risk

We use an equity swap as an example. Assume the correlation between the underlying equity price and the credit quality (hazard rate) of party B is ρ . The impact of the correlation on the CVA is shown in Table 8. The results state that the CVA increases when the absolute value of the negative correlation increases.

Table 8. Impact of Wrong Way Risk on the Two-way CVA

Correlation ρ	0	-50%	-100%
CVA	165.15	205.95	236.99

5. Conclusion

This article presents a theory for pricing defaultable financial derivatives and their CVAs. First, we want to indicate that the market value of a defaultable derivative is actually a risky value rather than a risk-free value. In fact, in applying the upfront CVA, we already converted the market value of a defaultable derivative from the risk-free value to the risky value.

For completeness, our theoretical study covers various cases. We find that the valuation of defaultable derivatives and their CVAs, in most situations, has a backward recursive nature and requires a backward induction valuation.

Based on our theory, we propose a novel cash-flow-based practical framework for calculating the two-way risky value and two-way CVA at the counterparty portfolio level. This framework can easily incorporate various credit mitigation techniques, such as netting agreements and margin agreements, and can capture wrong/right way risk. Numerical results show that these credit mitigation techniques and wrong/right way risk have significant impacts on CVA.

Appendix

Proof of Proposition 1: Under the one-way credit risk assumption, only the default risk of one party appears to be relevant, i.e., we only consider the default risk when the derivative is in the money. We divide the time period (t, T) into n very small intervals (Δt) and use the approximation $\exp(y) \approx 1 + y$ for very small y . Assume that a default may occur only at the end of each small period. The survival and the default probabilities for the period $(t, t + \Delta t)$ are given by

$$\widehat{S}_B(t) := S_B(t, t + \Delta t) = \exp(-h_B(t)\Delta t) \approx 1 - h_B(t)\Delta t \quad (\text{A1a})$$

$$\widehat{Q}_B(t) := Q_B(t, t + \Delta t) = 1 - \exp(-h_B(t)\Delta t) \approx h_B(t)\Delta t \quad (\text{A1b})$$

At time $t + \Delta t$ the derivative either defaults or survives. The risky value of the derivative at t is given by

$$\begin{aligned} V^D(t) &= E \left\{ \exp(-r(t)\Delta t) \left[1_{V^D(t+\Delta t) \geq 0} \left(\widehat{S}_B(t) + \varphi_B(t)\widehat{Q}_B(t) \right) + 1_{V^D(t+\Delta t) < 0} \right] V^D(t + \Delta t) \middle| \mathcal{F}_t \right\} \\ &\approx E \left\{ \exp(-r(t)\Delta t) \left[1_{V^D(t+\Delta t) \geq 0} (1 - h_B(t)\Delta t + \varphi_B(t)h_B(t)\Delta t) + 1_{V^D(t+\Delta t) < 0} \right] V^D(t + \Delta t) \middle| \mathcal{F}_t \right\} \\ &\approx E \left\{ \exp(-c_B(t)\Delta t) V^D(t + \Delta t) \middle| \mathcal{F}_t \right\} \end{aligned} \quad (\text{A2a})$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise; and

$$1_{V^D(t+\Delta t) \geq 0} + 1_{V^D(t+\Delta t) < 0} = 1 \quad (\text{A2b})$$

$$c_B(t) = r(t) + 1_{V^D(t+\Delta t) \geq 0} h_B(t)(1 - \varphi_B(t)) \quad (\text{A2c})$$

Similarly, we have

$$V^D(t + \Delta t) = E \left\{ \exp(-c_B(t + \Delta t)\Delta t) V^D(t + 2\Delta t) \middle| \mathcal{F}_{t+\Delta t} \right\} \quad (\text{A3})$$

Note that $\exp(-c_B(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t + \Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned}
V^D(t) &= E\left\{\exp(-c_B(t)\Delta t)V^D(t+\Delta t)|\mathcal{F}_t\right\} \\
&= E\left\{\exp(-c_B(t)\Delta t)E\left[\exp(-c_B(t+\Delta t)\Delta t)V^D(t+2\Delta t)|\mathcal{F}_{t+\Delta t}\right]|\mathcal{F}_t\right\} \\
&= E\left\{E\left[\exp\left(-\sum_{i=0}^1 c_B(t+i\Delta t)\Delta t\right)V^D(t+2\Delta t)|\mathcal{F}_{t+\Delta t}\right]|\mathcal{F}_t\right\} \\
&= E\left\{\exp\left(-\sum_{i=0}^1 c_B(t+i\Delta t)\Delta t\right)V^D(t+2\Delta t)|\mathcal{F}_t\right\}
\end{aligned} \tag{A4}$$

By recursively deriving from t forward over T , where $V^D(T) = X_T$, the price can be expressed as

$$V^D(t) = E\left\{\exp\left[-\sum_{i=0}^{n-1} c_B(t+i\Delta t)\Delta t\right]X_T|\mathcal{F}_t\right\} = E\left\{C_B(t, T)X_T|\mathcal{F}_t\right\} \tag{A5a}$$

where

$$c_B(t+i\Delta t) = r(t+i\Delta t) + 1_{V^D(t+(i+1)\Delta t) \geq 0} h_B(t+i\Delta t)(1 - \varphi_B(t+i\Delta t)) \tag{A5b}$$

Proof of Proposition 2: Under the one-way credit risk assumption, we only consider the default risk when the derivative is in the money. Assume that a default may only occur on the payment date.

Therefore, the risky value of the derivative at t is given by

$$\begin{aligned}
V^D(t) &= E\left\{D(t, T)\left[1_{X_T \geq 0}(S_B(t, T) + \varphi_B(T)Q_B(t, T)) + 1_{X_T < 0}\right]X_T|\mathcal{F}_t\right\} \\
&= E\left\{D(t, T)\left[1 - 1_{X_T \geq 0}(1 - \varphi_B(T))Q_B(t, T)\right]X_T|\mathcal{F}_t\right\} = E\left[G_B(t, T)X_T|\mathcal{F}_t\right]
\end{aligned} \tag{A6a}$$

where

$$1_{X_T \geq 0} + 1_{X_T < 0} = 1 \tag{A6b}$$

$$S_B(t, T) = 1 - Q_B(t, T) \tag{A6c}$$

$$G_B(t, T) = D(t, T)\left[1 - 1_{X_T \geq 0}Q_B(t, T)(1 - \varphi_B(T))\right] \tag{A6d}$$

Proof of Proposition 3: We divide any payment date period (T_{i-1}, T_i) into n_i very small time intervals (Δt) . On the first cash flow payment date T_1 , let $V^D(T_1)$ denote the value of the risky derivative excluding the current cash flow X_1 . Both $V^D(T_1)$ and X_1 could be positive or negative. According to Proposition 1, the risky value of the derivative at t is given by

$$V^D(t) = E\left[C_B(t, T_1)(X_1 + V^D(T_1))|\mathcal{F}_t\right] \tag{A7}$$

where $C_B(t, T_1)$ is defined in (A5)

Similarly, we have

$$V^D(T_1) = E\left[C_B(T_1, T_2)(X_2 + V^D(T_2))\middle|\mathcal{F}_{T_1}\right] \quad (\text{A8})$$

Note that $C_B(t, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V^D(t) &= E\left[C_B(t, T_1)(X_1 + V^D(T_1))\middle|\mathcal{F}_t\right] = E\left[C_B(t, T_1)X_1\middle|\mathcal{F}_t\right] \\ &\quad + E\left\{C_B(t, T_1)\left[E\left(C_B(T_1, T_2)X_2\middle|\mathcal{F}_{T_1}\right) + E\left(C_B(T_1, T_2)V^D(T_2)\middle|\mathcal{F}_{T_1}\right)\right]\middle|\mathcal{F}_t\right\} \\ &= \sum_{i=1}^2 E\left\{C_B(t, T_i)X_i\middle|\mathcal{F}_t\right\} + E\left[C_B(t, T_2)V^D(T_2)\middle|\mathcal{F}_t\right] \end{aligned} \quad (\text{A9a})$$

where

$$C_B(t, T_2) = C_B(t, T_1)C_B(T_1, T_2) = \exp\left[-\sum_{j=0}^{n_1+n_2-1} c_B(t+j\Delta t)\Delta t\right] \quad (\text{A9b})$$

By recursively deriving from T_2 forward over T_m , where $V^D(T_m) = X_m$, we have

$$V^D(t) = \sum_{i=1}^m E\left[(C_B(t, T_i))X_i\middle|\mathcal{F}_t\right] \quad (\text{A10})$$

Proof of Proposition 4: Let $t = T_0$. Assume that a default may only occur on the payment dates.

On the first payment date T_1 , let $V^D(T_1)$ denote the risky value of the derivative excluding the current cash flow X_1 . According to Proposition 2, the risky value of the derivative at t is given by

$$V^D(t) = E\left[G_B(T_0, T_1)(X_1 + V^D(T_1))\middle|\mathcal{F}_t\right] \quad (\text{A11a})$$

where

$$G_B(T_0, T_1) = D(T_0, T) \left[1 - 1_{(X_1 + V^D(T_1)) \geq 0} Q_B(T_0, T_1)(1 - \varphi_B(T_1))\right] \quad (\text{A11b})$$

Similarly, we have

$$V^D(T_1) = E\left[G_B(T_1, T_2)(X_2 + V^D(T_2))\middle|\mathcal{F}_{T_1}\right] \quad (\text{A12})$$

Note that $G_B(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned}
V^D(t) &= E[G_B(T_0, T_1)(X_1 + V^D(T_1)) | \mathcal{F}_t] = E[G_B(T_0, T_1)X_1 | \mathcal{F}_t] \\
&\quad + E\left\{G_B(T_0, T_1)\left[E(G_B(T_1, T_2)X_2 | \mathcal{F}_{T_1}) + E(G_B(T_1, T_2)V^D(T_2) | \mathcal{F}_{T_1})\right] | \mathcal{F}_t\right\} \\
&= \sum_{i=1}^2 E\left[\left(\prod_{j=0}^{i-1} G_B(T_j, T_{j+1})\right)X_i | \mathcal{F}_t\right] + E\left[\left(\prod_{j=0}^1 G_B(T_j, T_{j+1})\right)V^D(T_2) | \mathcal{F}_t\right]
\end{aligned} \tag{A13}$$

By recursively deriving from T_2 forward over T_m , where $V^D(T_m) = X_m$, we have

$$V^D(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} G_B(T_j, T_{j+1})\right)X_i | \mathcal{F}_t\right] \tag{A14}$$

Proof of Proposition 5: We divide the time period (t, T) into n very small intervals (Δt) and use the approximation $\exp(y) \approx 1 + y$ for very small y . The survival and the default probabilities of party j ($j=A, B$) for the period $(t, t + \Delta t)$ are given by

$$\hat{S}_j(t) := S_j(t, t + \Delta t) = \exp(-h_j(t)\Delta t) \approx 1 - h_j(t)\Delta t$$

$$\hat{Q}_j(t) := Q_j(t, t + \Delta t) = 1 - \exp(-h_j(t)\Delta t) \approx h_j(t)\Delta t$$

At time $t + \Delta t$, there are four possible states: 1) both A and B survive, 2) A defaults but B survives, 3) A survives but B defaults, and 4) both A and B default. The joint distribution of A and B is shown in Table 1. Depending on whether the market value of the derivative is an asset or a liability at $t + \Delta t$, we have

$$\begin{aligned}
V^D(t) &= E\left\{\exp(-r(t)\Delta t)\left[1_{V^D(t+\Delta t) \geq 0}\left(\left(\hat{S}_B(t)\hat{S}_A(t) + \gamma(t)\right) + \varphi_B(t)\left(\hat{Q}_B(t)\hat{S}_A(t) - \gamma(t)\right)\right.\right.\right. \\
&\quad \left.\left. + \bar{\varphi}_B(t)\left(\hat{S}_B(t)\hat{Q}_A(t) - \gamma(t)\right) + \varphi_{AB}(t)\left(\hat{Q}_B(t)\hat{Q}_A(t) + \gamma(t)\right)\right)V^D(t + \Delta t) | \mathcal{F}_t\right. \\
&\quad \left. + 1_{V^D(t+\Delta t) < 0}\left(\left(\hat{S}_B(t)\hat{S}_A(t) + \gamma(t)\right) + \bar{\varphi}_A(t)\left(\hat{Q}_B(t)\hat{S}_A(t) - \gamma(t)\right)\right.\right. \\
&\quad \left.\left. + \varphi_A(t)\left(\hat{S}_B(t)\hat{Q}_A(t) - \gamma(t)\right) + \varphi_{AB}(t)\left(\hat{Q}_B(t)\hat{Q}_A(t) + \gamma(t)\right)\right)V^D(t + \Delta t) | \mathcal{F}_t\right\} \\
&\approx E\left\{\exp(-r(t)\Delta t)\left[1_{V^D(t+\Delta t) \geq 0}\left(1 - h_A(t)(1 - \bar{\varphi}_B(t))\Delta t - h_B(t)(1 - \varphi_B(t))\Delta t\right.\right.\right. \\
&\quad \left.\left. + (1 - \varphi_B(t) - \bar{\varphi}_B(t) + \varphi_{AB}(t))\left(\gamma(t) + h_B(t)h_A(t)\Delta t^2\right)\right)V^D(t + \Delta t) | \mathcal{F}_t\right. \\
&\quad \left. + 1_{V^D(t+\Delta t) < 0}\left(1 - h_A(t)(1 - \varphi_A(t))\Delta t - h_B(t)(1 - \bar{\varphi}_A(t))\Delta t\right.\right. \\
&\quad \left.\left. + (1 - \varphi_A(t) - \bar{\varphi}_A(t) + \varphi_{AB}(t))\left(\gamma(t) + h_B(t)h_A(t)\Delta t^2\right)\right)V^D(t + \Delta t) | \mathcal{F}_t\right\} \\
&\approx E\left\{\exp(-o(t)\Delta t)V^D(t + \Delta t) | \mathcal{F}_t\right\}
\end{aligned} \tag{A15a}$$

where

$$o(t) = r(t) + 1_{V^D(t+\Delta t) \geq 0} p_B(t) + 1_{V^D(t+\Delta t) < 0} p_A(t) \quad (\text{A15b})$$

$$p_B(t) = (1 - \varphi_B(t))h_B(t) + (1 - \bar{\varphi}_B(t))h_A(t) - (1 - \varphi_B(t) - \bar{\varphi}_B(t) + \varphi_{AB}(t)) \times (\rho \sqrt{h_A(t)h_B(t)(1-h_B(t)\Delta t)(1-h_A(t)\Delta t)} + h_B(t)h_A(t)\Delta t) \quad (\text{A15c})$$

$$p_A(t) = (1 - \varphi_A(t))h_A(t) + (1 - \bar{\varphi}_A(t))h_B(t) - (1 - \varphi_B(t) - \bar{\varphi}_B(t) + \varphi_{AB}(t)) \times (\rho \sqrt{h_A(t)h_B(t)(1-h_B(t)\Delta t)(1-h_A(t)\Delta t)} + h_B(t)h_A(t)\Delta t) \quad (\text{A15d})$$

$$\gamma(t) = \rho \sqrt{h_A(t)h_B(t)(1-h_B(t)\Delta t)(1-h_A(t)\Delta t)} \Delta t^2 \quad (\text{A15e})$$

where φ_j ($j=A, B$) represents the default recovery rate of party j ; h_j represents the hazard rate of party j ; $\bar{\varphi}_j$ represents the non-default recovery rate of party j . $\bar{\varphi}_j=0$ represents the one-way settlement rule, while $\bar{\varphi}_j=1$ represents two-way settlement rule. ρ denotes the default correlation of A and B . φ_{AB} represents the joint recovery rate when A and B default simultaneously.

Similarly, we have

$$V^D(t + \Delta t) = E\{\exp(-o(t + \Delta t)\Delta t)V^D(t + 2\Delta t)|\mathcal{F}_{t+\Delta t}\} \quad (\text{A16})$$

Note that $\exp(-o(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V^D(t) &= E\{\exp(-o(t)\Delta t)V^D(t + \Delta t)|\mathcal{F}_t\} \\ &= E\{\exp(-o(t)\Delta t)E[\exp(-o(t + \Delta t)\Delta t)V^D(t + 2\Delta t)|\mathcal{F}_{t+\Delta t}]\big|\mathcal{F}_t\} \\ &= E\{\exp(-\sum_{i=0}^1 o(t + i\Delta t)\Delta t)V^D(t + 2\Delta t)|\mathcal{F}_t\} \end{aligned} \quad (\text{A17})$$

By recursively deriving from t forward over T , where $V^D(T) = X_T$, the price can be expressed as:

$$V^D(t) = E\{\exp(-\sum_{i=0}^{n-1} o(t + i\Delta t)\Delta t)X_T|\mathcal{F}_t\} = E\{O(t, T)X_T|\mathcal{F}_t\} \quad (\text{A18})$$

Proof of Proposition 6: We assume that a default may only occur on the payment date. At time T , there are four possible states: 1) both A and B survive, 2) A defaults but B survives, 3) A survives but B defaults, and 4) both A and B default. The joint distribution of A and B is shown in Table 1. Depending on whether the payoff is in the money or out of the money at T , we have

$$\begin{aligned}
V^D(t) &= E\left\{D(t,T)\left[1_{X_T \geq 0}\left(\langle S_B(t,T)S_A(t,T) + \gamma(T) \rangle + \varphi_B(T)(Q_B(t,T)S_A(t,T) - \gamma(T))\right.\right.\right. \\
&\quad \left.\left. + \bar{\varphi}_B(T)(S_B(t,T)Q_A(t,T) - \gamma(T)) + \varphi_{AB}(T)(Q_B(t,T)Q_A(t,T) + \gamma(T))\right)\right]X_T \Big| \mathcal{F}_t \\
&\quad + 1_{X_T < 0}\left(\langle S_B(t,T)S_A(t,T) + \gamma(T) \rangle + \bar{\varphi}_A(T)(Q_B(t,T)S_A(t,T) - \gamma(T))\right. \\
&\quad \left. + \varphi_A(T)(S_B(t,T)Q_A(t,T) - \gamma(T)) + \varphi_{AB}(T)(Q_B(t,T)Q_A(t,T) + \gamma(T))\right)\right]X_T \Big| \mathcal{F}_t\Big\} \\
&= E(Y(t,T)X_T \Big| \mathcal{F}_t)
\end{aligned} \tag{A19a}$$

where

$$Y(t,T) = D(t,T)\left(1_{X_T \geq 0}y_B(t,T) + 1_{X_T < 0}y_A(t,T)\right) \tag{A19b}$$

$$\begin{aligned}
y_B(t,T) &= S_B(t,T)S_A(t,T) + \varphi_B(T)Q_B(t,T)S_A(t,T) + \bar{\varphi}_B(T)S_B(t,T)Q_A(t,T) \\
&\quad + \varphi_{AB}(T)Q_B(t,T)Q_A(t,T) + \gamma(t,T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T))
\end{aligned} \tag{A19c}$$

$$\begin{aligned}
y_A(t,T) &= S_B(t,T)S_A(t,T) + \varphi_A(T)Q_A(t,T)S_B(t,T) + \bar{\varphi}_A(T)S_A(t,T)Q_B(t,T) \\
&\quad + \varphi_{AB}(T)Q_B(t,T)Q_A(t,T) + \gamma(t,T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T))
\end{aligned} \tag{A19d}$$

$$\gamma(t,T) = \rho \sqrt{S_B(t,T)Q_B(t,T)S_A(t,T)Q_A(t,T)} \tag{A19e}$$

Proof of Proposition 7. On the first cash flow payment date T_1 , let $V^D(T_1)$ denote the risky value of the derivative excluding the current cash flow X_1 . According to Proposition 5, we have

$$V^D(t) = E\left[O(t,T_1)(X_1 + V^D(T_1)) \Big| \mathcal{F}_t\right] \tag{A20a}$$

Where $O(t,T_1)$ is defined in (A18) and (A15).

Similarly, we have

$$V^D(T_1) = E\left[O(T_1,T_2)(X_2 + V^D(T_2)) \Big| \mathcal{F}_{T_1}\right] \tag{A21}$$

Note that $O(t,T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned}
V^D(t) &= E\left[O(t,T_1)(X_1 + V^D(T_1)) \Big| \mathcal{F}_t\right] = E\left[O(t,T_1)X_1 \Big| \mathcal{F}_t\right] \\
&\quad + E\left\{O(t,T_1)\left[E(O(T_1,T_2)X_2 \Big| \mathcal{F}_{T_1}) + E(O(T_1,T_2)V^D(T_2) \Big| \mathcal{F}_{T_1})\right] \Big| \mathcal{F}_t\right\} \\
&= \sum_{i=1}^2 E\left[O(t,T_i)X_i \Big| \mathcal{F}_t\right] + E\left[O(t,T_2)V^D(T_2) \Big| \mathcal{F}_t\right]
\end{aligned} \tag{A22a}$$

where

$$O(t, T_2) = O(t, T_1)O(T_1, T_2) \quad (\text{A22b})$$

By recursively deriving from T_2 forward over T_m , where $V^D(T_m) = X_m$, we have

$$V^D(t) = \sum_{i=1}^m E[O(t, T_i)X_i | \mathcal{F}_t] \quad (\text{A23})$$

Proof of Proposition 8. We assume that a default may only occur on the payment dates. Let $t = T_0$

. On the first cash flow payment date T_1 , let $V^D(T_1)$ denote the risky value of the derivative excluding the current cash flow X_1 . According to Proposition 6, we have

$$V^D(t) = E[Y(T_0, T_1)(X_1 + V^D(T_1)) | \mathcal{F}_t] \quad (\text{A24a})$$

where

$$Y(T_0, T_1) = D(T_0, T_1) \left(1_{(X_1 + V^D(T_1)) \geq 0} y_B(T_0, T_1) + 1_{(X_1 + V^D(T_1)) < 0} y_A(T_0, T_1) \right) \quad (\text{A24b})$$

where $y_B(T_0, T_1)$ and $y_A(T_0, T_1)$ are defined in (A19).

Similarly, we have

$$V^D(T_1) = E[Y(T_1, T_2)(X_2 + V^D(T_2)) | \mathcal{F}_{T_1}] \quad (\text{A25})$$

Note that $Y(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V^D(t) &= E[Y(T_0, T_1)(X_1 + V^D(T_1)) | \mathcal{F}_t] = E[Y(T_0, T_1)X_1 | \mathcal{F}_t] \\ &\quad + E\left\{Y(T_0, T_1) \left[E(Y(T_1, T_2)X_2 | \mathcal{F}_{T_1}) + E(Y(T_1, T_2)V^D(T_2) | \mathcal{F}_{T_1}) \right] | \mathcal{F}_t\right\} \\ &= E[Y(T_0, T_1)X_1 | \mathcal{F}_t] + E\left[\left(\prod_{j=0}^1 Y(T_j, T_{j+1})\right)X_2 | \mathcal{F}_t\right] + E\left[\left(\prod_{j=0}^1 Y(T_j, T_{j+1})\right)V^D(T_2) | \mathcal{F}_t\right] \end{aligned} \quad (\text{A26})$$

By recursively deriving from T_2 forward over T_m , where $V^D(T_m) = X_m$, we have

$$V^D(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} Y(T_j, T_{j+1})\right)X_i | \mathcal{F}_t\right] \quad (\text{A27})$$

Reference

Duffie, Darrell, and Kenneth J. Singleton, 1999, Modeling term structure of defaultable bonds, Review of Financial Studies, 12, 687-720.

FinPricing, 2019, Valuation service, <https://finpricing.com/lib/EqCorrelationSwap.html>

Jarrow, Robert A., and Stuart M. Turnbull, 1995, Pricing derivatives on financial securities subject to credit risk, Journal of Finance, 50, 53-85.

Longstaff, Francis A., and Eduardo S. Schwartz, 1995, A simple approach to valuing risky fixed and floating debt, Journal of Finance, 50, 789-819.

Madian, Dilip.B., and Unal, Haluk, 1998, Pricing the risks of default, Review of Derivatives Research, 2, 121-160.

Merton, Robert C., 1974, On the pricing of corporate debt: the risk structure of interest rates, Journal of Finance, 29, 449-470.

Pykhtin, Michael, and Steven Zhu, 2007, A guide to modeling counterparty credit risk, GARP Risk Review, July/August, 16-22.