ON THE DETERMINANT OF TOEPLITZ-HESSSENBERG MATRICES

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ABSTRACT. We derive a formula for calculating determinant of Toeplitz-Hessenberg matrices in terms of the partial Bell polynomials.

1. INTRODUCTION

A lower $n \times n$ Toeplitz-Hessenberg matrix has the following form

$$M_n(a_0; a_1, \dots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least some $k > 0$.

Trudi's formula [3] gives the determinant of these matrices as

$$\det(M_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1, \dots, t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}$$

where

$$\binom{t_1+\cdots+t_n}{t_1, \dots, t_n} = \frac{(t_1+\cdots+t_n)!}{t_1! t_2! \cdots t_n!}$$

is the multinomial coefficient. Our main result (equation (4)) is a much simpler formula than the above which does not involve summation over partitions.

Definition 1 (Partial Bell polynomials). For n and k non-negative integers, the partial (n, k) th partial Bell polynomials in the variables $x_1, x_2, \dots, x_{n-k+1}$ denoted by $B_{n,k} \equiv B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ [1, p. 206] can be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}.$$

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Cvijović [2] gives the following formula for calculating these polynomials

$$B_{n,k+1} = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \overbrace{\binom{n}{\alpha_1} \binom{\alpha_1}{\alpha_2} \cdots \binom{\alpha_{k-1}}{\alpha_k}}^k \cdot x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \cdots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k} \quad (n \geq k+1, k = 1, 2, \dots) \quad (1)$$

We will use the *Faà di Bruno's formula* [1, p. 134]

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=0}^n f^{(k)}(g(x)) \cdot B_{n,k} \left(g'(x), g''(x), \dots, g^{(n-k+1)}(x) \right).$$

2. MAIN RESULT

Theorem 1. *We have*

$$\det(M_n) = \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k k! B_{n,k} (1! a_1, 2! a_2, \dots, (n-k+1)! a_n). \quad (2)$$

Proof. Let $\Psi(x) = \sum_{n \geq 0} a_n x^n$ and $\Phi(x) = \sum_{n \geq 0} b_n x^n$ with $\Psi(x) \Phi(x) = 1$ with $a_0 = b_0 = 1$. Then by [3, Lemma 3] we have

$$b_n = (-1)^n \det(M_n). \quad (3)$$

Letting $f(x) = 1/x$ and $g(x) = \Psi(x)$ in the Faà di Bruno's formula gives us

$$\frac{d^n \Phi(x)}{dx^n} = \sum_{k=0}^n \frac{(-1)^k k!}{\Psi(x)^{k+1}} B_{n,k} \left(\Psi'(x), \Psi''(x), \dots, \Psi^{(n-k+1)}(x) \right).$$

Letting $x \rightarrow 0$ in the above and using Maclaurin series expansion we can conclude our result equation (2). $\square$

Using Cvijović's formula (equation (1)) we get

Corollary 1. *We have*

$$\det(M_n) = -a_n + \sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1} (-1)^{n+k-1} a_{n-\alpha_1} a_{\alpha_1-\alpha_2} \cdots a_{\alpha_{k-1}-\alpha_k} a_{\alpha_k}. \quad (4)$$

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