Long trail in a graph

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Abstract

We show that searching for a long trail in a given graph is a hard task.

I. PROPOSITION

Suppose $G = (V, E)$ is a given graph, a trail in G is an edge-disjoint path in it. In particular, a trail can visit a vertex more than once but never includes the same edge twice, even in different directions. The problem, that we are interested in this note, denoted by LONG TRAIL, is asking whether there exists a trail of length k in G . We will restrict our problem to graphs of maximum vertex degree 3. The following proposition turns out to be very helpful.

Proposition: In a graph of maximum degree 3, a trail can visit at most 2 vertices twice, cannot visit any vertex for a third time.

Proof: In a given graph, if a trail visit a vertex v for three times, then v 's degree is at least 5. In fact, if v is not the starting vertex of the trail, its degree must be at least 6. But, we are given a graph of maximum degree 3. So, the second part of the proposition holds.

Now, we go on the trail from the starting vertex to the ending vertex. If the trail visits a vertex v twice, there will be two cases to be considered. First, that vertex v is not the starting vertex, the trail must terminate there, meaning that v is the ending vertex. Because the first time our trail visits v , it uses 2 edges incident to v , recall that v is not the starting vertex. So, the second time our trail visits v , it meets its end there. Second, that vertex v is the starting vertex, the trail can still move on. Because in this case, the trail only uses one edge in the first time it visits v right at the very beginning of itself. Now, the trail has completed a

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vertex-disjoint cycle. If it does move on, it can only visit another vertex w for the second time in the settings of the first case. If that happens, it must also end there. So, in both cases, we see that our trail can visit at most two vertices twice. **Q.E.D**

Papadimitriou et al. [?] proves the hardness of the Hamiltonian cycle in cubic graph. We will reduce from this problem.

Proposition: $\text{HAM-CYCLE} \leq_p \text{LONG TRAIL}$

Proof: Given a HAM-CYCLE instance $G = (V, E)$, we reduce it to our problem by asking if there exist any trail of length $|V| + 1$ in G . If the given instance is a YES instance, from a Hamiltonian cycle of that instance, we add any unused edge (u, v) to the cycle. It is easy to see that from u we go on the Hamiltonian cycle back to u then take the edge (u, v) , we have constructed a $|V| + 1$ -long trail. Conversely, if there exists a $|V| + 1$ -long trail in G , by the aforementioned proposition, that trail will be exactly a Hamiltonian cycle plus one additional edge. Indeed, a $|V| + 1$ -long trail has $|V| + 2$ vertices, counting multiplicities. According to the proposition, it can visit at most two vertices twice. So, if we omit multiplicities, it still must have all $|V|$ vertices of G . As demonstrated in the proof of the proposition, it must fall into the second case. Therefore, the vertex-disjoint cycle part must be a Hamiltonian cycle. **Q.E.D**

REFERENCES

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