Equivelar octahedron of genus 3 in 3-space

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ABSTRACT. Building up a toroidal polyhedron of genus 3, consisting of 8 nine-sided faces, is given. From the point of view of topology, a polyhedron can be considered as an embedding of a cubic graph with 24 vertices and 36 edges in a surface of genus 3. This polyhedron is a contender for the maximal genus among octahedrons in 3-space.

1. Introduction

This solution can be attributed to the problem of determining the maximal genus of polyhedra with the number of faces - f . As is known, at least 7 faces are required for a polyhedron of genus $g = 1$. For cases $f \geq 8$, there are currently few examples.

If all faces of the toroidal polyhedron are p -gons and all vertices are q -valence ($q \geq 3$), such polyhedra are called either locally regular or simply equivelar [1]. The characteristics of polyhedra are abbreviated as $\{p, q; g\}$ [1].

2. Polyhedron $\{9, 3; 3\}$ V1.

The paper considers building up a polyhedron $\{9, 3; 3\}$ in 3-space. The faces of a polyhedron are non-convex flat 9-gons without self-intersections. The polyhedron is symmetric when rotated through 180° around the z axis (Fig. 3).

One of the features of this polyhedron is that any face has two pairs with which it borders two edges. The polyhedron also has a ratio of angles and faces - $f = p - 1$.

To describe polyhedra with similar characteristics ($f = p - 1$) we use the Euler formula $v - e + f = \chi = 2 - 2g$, where χ is the Euler characteristic. Since $q = 3$, the equality $3v = 2e$ holds true. Therefore, the number of faces and genus:

$$g = \frac{f \cdot (f + 1)}{12} - \frac{f}{2} + 1 \quad (1)$$

$$f = \frac{\sqrt{48g - 23} + 5}{2} \quad (2)$$

Table 1 lists some of the first polyhedra that could possibly have similar characteristics for $f \geq 8$. At present, it is difficult to answer whether it is possible to realize $g = 4$ and $g = 8$.

Table 1

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p
3	8	24	36	9
4	9	30	45	10
8	12	52	78	13

Fig. 1. General views {9, 3; 3} V1

Fig. 2. Layout of faces on the plane {9, 3; 3} V1

Fig. 3. Coordinate axis {9, 3; 3} V1

Vertex coordinates {9, 3; 3} V1

Coordinates are represented in the Cartesian system. The polyhedron faces lie in the planes bounding the faces of 2 tetrahedrons (outer and inner faces). Below are the equations of the planes.

Planes 1 (Fig. 2). Tetrahedron 1

$$\begin{aligned}y + z + 100 &= 0 \\x - z + 100 &= 0 \\y - z - 100 &= 0 \\x + z - 100 &= 0\end{aligned}$$

Planes 2 (Fig. 2). Tetrahedron 2

$$\begin{aligned}x - 7y - 4z - 384 &= 0 \\7x + y - 4z + 384 &= 0 \\x - 7y + 4z + 384 &= 0 \\7x + y + 4z - 384 &= 0\end{aligned}$$

Coordinates of polyhedron vertices {9, 3; 3} V1

Table 2

№	x - coordinates	y - coordinates	z - coordinates
1	-308	-108	-208
2	308	108	-208
3	16	0	-100
4	-16	0	-100
5	28	4	-96
6	-28	-4	-96
7	-4272 / 41	-448 / 41	-3652 / 41
8	4272 / 41	448 / 41	-3652 / 41
9	-182	-18	-82
10	182	18	-82
11	-1216 / 13	-336 / 13	-964 / 13
12	1216 / 13	336 / 13	-964 / 13
13	336 / 13	-1216 / 13	964 / 13
14	-336 / 13	1216 / 13	964 / 13
15	18	-182	82
16	-18	182	82
17	448 / 41	-4272 / 41	3652 / 41
18	-448 / 41	4272 / 41	3652 / 41
19	-4	28	96
20	4	-28	96
21	0	-16	100
22	0	16	100
23	-108	308	208
24	108	-308	208

Coordinates of face vertexes {9, 3; 3} V1

Table 3

Face number	Vertex number in table 2								
	1	2	3	4	5	6	7	8	9
1	1	3	5	6	7	9	12	14	18
2	2	10	13	15	17	20	21	22	23
3	8	12	13	16	18	19	20	21	23
4	2	4	5	6	8	10	11	13	17
5	1	4	3	5	8	10	12	16	23
6	1	9	14	16	18	19	22	21	24
7	7	11	14	15	17	19	20	22	24
8	2	3	4	6	7	9	11	15	24

Polyhedron graph {9, 3; 3} V1

A polyhedron can be described as an embedding of a cubic graph in a surface of genus 3 in 3-space. More detailed characteristics of the graph are given in table 4.

Table 4

Vertices	24
Edges	36
Radius	5
Diameter	5
Chromatic number	3
Chromatic index	3
Girth	4
Automorphisms [8]	8
Hamiltonian cycle	yes

Fig. 4. Basic view of the graph {9, 3; 3} V1

Fig. 5. View with Hamiltonian cycle {9, 3; 3} V1

3. Polyhedron $\{9, 3; 3\}$ V2.

The combinatorial structure of this polyhedron differs from $\{9, 3; 3\}$ V1. Each face has one pair with which it borders two edges, also any two faces have a common edge like the tetrahedron and Szilassi polyhedron.

Fig. 6. General views $\{9, 3; 3\}$ V2

Fig. 7. Layout of faces on the plane $\{9, 3; 3\}$ V2

Fig. 8. Coordinate axis $\{9, 3; 3\}$ V2

Vertex coordinates {9, 3; 3} V2

Planes 1 (Fig. 7). Tetrahedron 1

$$2y + z + 100 = 0$$

$$2x - z + 100 = 0$$

$$2y - z - 100 = 0$$

$$2x + z - 100 = 0$$

Planes 2 (Fig. 7). Tetrahedron 2

$$15x - 49y - 53z - 2120 = 0$$

$$49x + 15y - 53z + 2120 = 0$$

$$15x - 49y + 53z + 2120 = 0$$

$$49x + 15y + 53z - 2120 = 0$$

Coordinates of polyhedron vertices {9, 3; 3} V2

Table 5

№	x - coordinates	y - coordinates	z - coordinates
1	-212	0	-100
2	212	0	-100
3	-88	12	-76
4	88	-12	-76
5	-70	-150 / 7	-40
6	70	150 / 7	-40
7	-2120 / 33	-1060 / 33	-940 / 33
8	2120 / 33	1060 / 33	-940 / 33
9	-60	-40	-20
10	60	40	-20
11	-6360 / 163	-7420 / 163	-1460 / 163
12	6360 / 163	7420 / 163	-1460 / 163
13	-7420 / 163	6360 / 163	1460 / 163
14	7420 / 163	-6360 / 163	1460 / 163
15	-40	60	20
16	40	-60	20
17	1060 / 33	-2120 / 33	940 / 33
18	-1060 / 33	2120 / 33	940 / 33
19	-150 / 7	70	40
20	150 / 7	-70	40
21	12	88	76
22	-12	-88	76
23	0	212	100
24	0	-212	100

Coordinates of face vertexes {9, 3; 3} V2

Table 6

Face number	Vertex number in table 5								
	1	2	3	4	5	6	7	8	9
1	1	3	5	6	7	9	11	14	17
2	7	11	13	15	18	19	20	21	23
3	4	6	8	10	14	16	21	23	24
4	1	2	4	9	11	16	17	20	22
5	8	12	14	16	17	19	20	22	24
6	2	1	3	10	12	15	18	19	21
7	3	5	7	9	13	15	22	24	23
8	2	4	5	6	8	10	12	13	18

Polyhedron graph $\{9, 3; 3\}$ V2

The characteristics of the graph are shown in table 7.

Table 7

Vertices	24
Edges	36
Radius	4
Diameter	4
Chromatic number	3
Chromatic index	3
Girth	5
Automorphisms [8]	4
Hamiltonian cycle	yes

Fig. 9. Basic view of the graph $\{9, 3; 3\}$ V2

Fig. 10. View with Hamiltonian cycle $\{9, 3; 3\}$ V2

Notes

More images of the polyhedron are available at the link [11]

References

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