Elsevier Editorial System(tm) for Journal of Petroleum Science and Engineering
Manuscript Draft

Manuscript Number: PETROL2963R2

Title: Review of explicit approximations to the Colebrook relation for flow friction

Article Type: Review Article

Keywords: Colebrook equation; Approximations; Fluid mechanics; Turbulent flow; Friction coefficient; Darcy friction factor; Hydraulic resistances; Excel spreadsheet

Corresponding Author: Dr Dejan Brkić, PhD in Petroleum Eng.

Corresponding Author's Institution: -

First Author: Dejan Brkić, PhD in Petroleum Eng.

Order of Authors: Dejan Brkić, PhD in Petroleum Eng.

Abstract: Because of Moody's chart as demonstrated applicability of the Colebrook equation over a very wide range of Reynolds number and relative roughness values, this equation become the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation suffers from being implicit in unknown friction factor and thus requires an iterative solution where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is main reason for attempting to develop a relationship that is a reasonable approximation for the Colebrook equation but which is explicit in friction factor. Review of existing explicit approximation of the implicit Colebrook equation with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative solution of implicit Colebrook equation is shown for entire range of turbulence where Moody diagram should be used as the reference. Finally, it can be concluded that most of available approximations of the Colebrook equation, with few exceptions, are very accurate with deviation of no more than few percentages.

Abstract DB

Click here to download Abstract: Abstract DB.doc

Because of Moody's chart as demonstrated applicability of the Colebrook equation over a very wide range of Reynolds number and relative roughness values, this equation become the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation suffers from being implicit in unknown friction factor and thus requires an iterative solution where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is main reason for attempting to develop a relationship that is a reasonable approximation for the Colebrook equation but which is explicit in friction factor. Review of existing explicit approximation of the implicit Colebrook equation with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative solution of implicit Colebrook equation is shown for entire range of turbulence where Moody diagram should be used as the reference. Finally, it can be concluded that most of available approximations of the Colebrook equation, with few exceptions, are very accurate with deviation of no more than few percentages.

I wish to thank reviewers and editors for valuable comments.

Reviewer 1:

1. I hope that my expression of English is now better.

Reviewer 2:

Missing

Reviewer 3:

Missing

Reviewer 4:

1. Abstract was rearranged now as you suggested (last two sentences). Please, see also Research highlights. My text has now graphical abstract also….

2. I disagree with your suggestion to reduce number of approximation shown in my review. Honestly, I have considered your suggestion seriously, but in the meantime I have found new approximations such as Papaevangelou et al (2010) and Vatankhah and Kouchakzadeh (2008, 2009). They are very accurate. Therefore, I have added these two approximations...

3. I agree with you that problem of range of applicability of certain approximation is very important. This problem is very complex. So to avoid any further speculations I have added diagram of accuracy of each approximation over the entire practical range of Reynolds number and relative roughness. According to these diagrams, one can choose will he/she use in certain case this particular approximation or not.

4. Detailed discussion with results is now presented in more appropriate way.

5. Conclusion is now rearranged after your suggestions.

Review of explicit approximations to the Colebrook relation for flow friction

Dejan Brkić,

Ph.D., Petroleum and Natural Gas Engineer, Strumička 88, 11050 Beograd, Serbia

Tel. +381642543668, e-mail: dejanrgf@tesla.rcub.bg.ac.rs

Abstract: Because of Moody's chart as demonstrated applicability of the Colebrook equation over a very wide range of Reynolds number and relative roughness values, this equation become the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation suffers from being implicit in unknown friction factor and thus requires an iterative solution where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is main reason for attempting to develop a relationship that is a reasonable approximation for the Colebrook equation but which is explicit in friction factor. Review of existing explicit approximation of the implicit Colebrook equation with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative solution of implicit Colebrook equation is shown for entire range of turbulence where Moody diagram should be used as the reference. Finally, it can be concluded that most of available approximations of the Colebrook equation, with few exceptions, are very accurate with deviation of no more than few percentages.

Keywords: Colebrook equation; Approximations; Fluid mechanics; Turbulent flow; Friction coefficient; Darcy friction factor; Hydraulic resistances; Excel spreadsheet

1. Introduction

Difficulty of solving turbulent flow problems in pipes lies in the fact that hydraulic friction factor is a complex function of relative surface roughness and Reynolds number. Precisely, hydraulic resistance depends on flow rate. Similar situation is with electrical resistance when diode is in a circuit. Furthermore, being more complex, widely used empirical Colebrook equation is transcendental which means that it cannot be solved by using only elementary functions and basic arithmetic operations in definitive form. Problem is that, since the Colebrook equation is implicit (i.e. both the right and left-hand terms contain friction factor), containing the unknown friction factor in implicit form, the Reynolds number and the pipe roughness, it has to be solved iteratively. Even today in the era of advance computer technology, explicit approximations of the implicit Colebrook relation is very often used for calculation of friction factor in pipes. The reason that so many researchers propose approximate solutions to Colebrook equation is that these correlations are necessary to calculate the pressure drop and average velocity in conduits in one step. Friction factor can be derived using logarithmic or power law formulation (Zagarola et al 1997). Colebrook equation belongs to the logarithmic law. The laws of resistances to fluid flow through rough pipe are of great significance. Colebrook equation is valid for turbulent regime in rough pipes including so called rough and smooth turbulent regime with special accent on transient regime between them. There is no perfectly smooth inner surface of pipe. All pipe walls have physically rough surfaces (Figure 1). Degree of roughness varies depending on the manufacturing process, surface finish, type of pipe material (Hammad 1999), age, conditions of exploitation, etc. In turbulent flow, thin layer of fluid very close to inner pipe surface in which flow is laminar is called “laminar sub-layer”. If the pipe roughness (protrusions of inner pipe surface) is completely covered by the sub-layer, the surface is smooth from the hydraulic point of view. With increasing of Reynolds number, thickness of laminar sub-layer decreases baring protrusions and fluid flow through pipe become consequently rough from the hydraulic point of view. At very low Reynolds number, relative roughness does not have influence on friction factor and it depends only on value of Reynolds number. But, on the contrary, at very high Reynolds number, Reynolds number does not have influence on friction factor and it depends only on relative roughness. Between these two opposite regime friction factor depends on both, Reynolds number and relative roughness. Colebrook equation was developed to cover this transient zone of turbulence, but it also covers completely smooth and completely rough regime. In laminar flow, all pipes behave as smooth but Colebrook equation is not valid for this regime. One of the presented approximations in this paper includes laminar regime (Churchill 1977).

Figure 1. Hydraulic regimes; A) Hydraulically “smooth”, B) Partially turbulent, and C) Turbulent (rough)

The Colebrook equation is widely used in the petroleum industry for calculations of oil and gas pipelines, in civil engineering for calculation of water distribution systems, for drainage systems, ventilation systems, in chemical engineering, and in all fields of engineering where fluid flow can be occurred.

2. On Colebrook equation for flow friction

Before 1939. when Colebrook equation was published, for turbulent regime in smooth pipes widely was used Prandtl equation also implicit in friction factor (Colebrook 1939). Prandtl derived a formula from the logarithmic velocity profile and experimental data on smooth pipes (1):

$$\frac{1}{\sqrt{\lambda}} = 2 \cdot \log_{10}\left(\frac{\mathrm{Re} \cdot \sqrt{\lambda}}{2.51}\right) = 2 \cdot \log_{10}\left(\mathrm{Re} \cdot \sqrt{\lambda}\right) - 0.8 \qquad (1)$$

The development of approximate equations for the calculation of friction factor in rough pipes began with Nikuradse's turbulent pipe flow investigations in 1932 and 1933 (Hager and Liiv 2008). For turbulent regime in rough pipes widely was used von Karman's relation (2):

$$\frac{1}{\sqrt{\lambda}} = 2 \cdot \log_{10}\left(\frac{3.71 \cdot \mathrm{D}}{\varepsilon}\right) = 1.74 - 2 \cdot \log_{10}\left(\frac{2 \cdot \varepsilon}{\mathrm{D}}\right) = 1.14 - 2 \cdot \log_{10}\left(\frac{\varepsilon}{\mathrm{D}}\right) \qquad (2)$$

Greek ε is the equivalent Nikuradse's sand-grain roughness value for the inner surface of pipe (or so called uniform roughness). Prandtl's and von Karman's relations are also known as NPK (Nikuradse-Prandtl-Karman) equations. Colebrook later performed experiments on sixteen spun concrete-lined pipes and six spun bitumastic-lined pipes ranging in diameter from 101.6 mm to 1524 mm with average surface roughness values between 0.04318 mm and 0.254 mm (Taylor et al 2006).

In an attempt to classify the data available at the time and those from experiment conducted by himself and his colleague White (Colebrook and White 1937), Colebrook (1939) developed a curve fit to describe transitional roughness (3):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(\frac{2.51}{\mathrm{Re} \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.71 \cdot \mathrm{D}}\right) \qquad (3)$$

Colebrook equation also can be noted as (3a):

$$\frac{1}{\sqrt{\lambda}} = 1.14 - 2 \cdot \log_{10}\left(\frac{\varepsilon}{\mathrm{D}} + \frac{9.35}{\mathrm{Re}\sqrt{\lambda}}\right) \qquad (3a)$$

Colebrook equation describes a monotonic change in the friction factor from smooth to fully rough (Figure 2). It is valid especially for commercial steel pipes. Strictly mathematically is incorrect what Colebrook had done, i.e. log(A+B)≠log(A)+log(B), but physically this relation gives good results. Problem can be treated as inverse; according to logarithmic rules equally is incorrect to split the Colebrook relation into two pieces.

Figure 2. Colebrook relation make transitional curve between hydraulically “smooth” regime described by Prandtl (1), and turbulent (rough) regime described von Karman (2)

Colebrook equation is also basis for Rouse (1943) and widely used Moody (1944) chart. Many people seem to believe that the Moody diagram has surprisingly good properties. In fact, all it is a plot of solutions of the nonlinear transcendental Colebrook equation. In principle, Moody diagram is used for solution of three types of problems, i.e. problem in which head loss is unknown, in which volume flow rate is unknown and in which diameter is unknown. Solving for unknown head loss with Moody diagram is relative straightforward but use of implicit Colebrook formula complicate solving all three types of problems.

Many researchers adopt a modification of the Colebrook equation (4) recommended by the American Gas Association (AGA) in case of natural gas pipelines calculations, using 2.825 constant instead of 2.51 (Haaland 1983, Coelho and Pinho 2007). This procedure produces maximal deviation up to 3.2% (Figure 3).

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(\frac{2.825}{\mathrm{Re} \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.71 \cdot \mathrm{D}}\right) \tag{4}$$

Figure 3. Distribution of estimated deviation of implicit Colebrook equation modified by AGA (4) compared with standard implicit Colebrook equation (3)

Figure 3 is three-dimensional. Similar figures in further text are two-dimensional.

Also, some researchers use Fanning factor which is not the same as the Darcy friction factor (here noted as λ). Darcy friction factor is 4 times greater than the Fanning friction factor, but physical meaning is equal. Darcy, Darcy-Weisbach and Moody friction factors are synonyms.

Colebrook equation is somewhere known as Colebrook-White equation (CW equation). White was not actually a co-author of the paper in which this equation was presented (Colebrook 1939). But, Colebrook made a special point of acknowledging important contribution of White to the development of the equation (Colebrook and White 1937, Colebrook 1939). Letter W has additional symbolic value because alternate explicit reformulations of Colebrook equation with Lambert W-function involved exist (Brkić 2011a, Clamond 2009, Goudar and Sonnad 2003, Keady 1998, More 2006, Nandakumar 2007, Sonnad and Goudar 2004, 2005, 2006, 2007). Approximations proposed by Brkić (2011a) were also developed using Lambert W-function and its solution proposed by Barry et al (2000). Further about Lambert-W function, readers can see in paper of Hayes (2005).

3. Available explicit approximations of the Colebrook equation with analysis of their estimated accuracy

How well Colebrook equation fits the experimental data is beyond the scope of presented approximations. Perhaps one of these equations even fits the available data better than the Colebrook equation. Until the comparison is made with real, measured values, however, this will not be known. According to Cipra (1996), some of the key formulas of turbulence are off by as much as 65%. Yoo and Singh (2004, 2010) found that the Colebrook equation produced an average error of more than 11% while the roughness height of commercial pipes varied quite significantly, depending on the pipe size and type.

As it will be shown, there were some early expressions of Colebrook equation in explicit form which were not particularly accurate, but in the years 1973-1984 there was a flurry of activity obtaining more accurate approximations that appeared mainly in the chemical engineering literature. Note that some of presented approximations exist in several versions. Here has to be very careful because typographical errors are always possible (Concha 2008, Brkić 2009a). Approximations will be presented starting from the oldest. Estimated accuracy compared with iterative solution of implicit Colebrook equation will be shown in figures 4-23 for entire range of turbulence where Moody diagram should be used as the reference.

3.1 Moody approximation

Approximation proposed by Moody (1947) is the oldest approximation of implicit Colebrook relation (5):

$$\lambda \approx 0.0055 \cdot \left(1 + \left(2 \cdot 10^4 \cdot \frac{\varepsilon}{D} + \frac{10^6}{Re}\right)^{\frac{1}{3}}\right) \quad (5)$$

With error up to 21.49% compared with implicit Colebrook equation (Figure 4), it has today only historical value.

Figure 4. Distribution of estimated error of Moody approximation compared with implicit Colebrook equation

3.2 Wood approximation

Approximation proposed by Wood (1966) is, equal as those by Moody (1947), power-law type equation (6). Its accuracy is not improved compared with Moody approximation.

$$\lambda \approx 0.094 \cdot \left(\frac{\varepsilon}{D}\right)^{0.225} + 0.53 \cdot \left(\frac{\varepsilon}{D}\right) + 88 \cdot \left(\frac{\varepsilon}{D}\right)^{0.44} \cdot Re^{-V} \qquad (6)$$

Where V is (7):

$$V = 1.62 \cdot \left(\frac{\varepsilon}{D}\right)^{0.134} \qquad (7)$$

Estimated error of Wood approximation is up to 23.79% compared with implicit Colebrook equation (Figure 5).

Figure 5. Distribution of estimated error of Wood approximation compared with implicit Colebrook equation

3.3 Eck approximation

Approximation proposed by Eck (1973) is most simple but not very accurate (8), but better than those by Moody (1947) and Wood (1966).

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.715 \cdot D} + \frac{15}{Re}\right) \qquad (8)$$

Estimated error of Eck approximation is up to 8.2% compared with implicit Colebrook equation (Figure 6).

Figure 6. Distribution of estimated error of Eck approximation compared with implicit Colebrook equation

3.4 Churchill approximation (only for turbulent regime)

Approximation proposed by Churchill (1973) is very similar with approximations proposed by Swamee and Jain (1976) and Jain (1976). It is first approximation with improved accuracy (9):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.71 \cdot D} + \left(\frac{7}{Re}\right)^{0.9}\right) \quad (9)$$

Estimated error of Churchill approximation (valid only for turbulent regime) is up to 2.18% compared with implicit Colebrook equation (Figure 7).

Figure 7. Distribution of estimated error of approximation by Churchill (valid only for turbulent regime), Swamee and Jain, approximation by Jain and Churchill approximation (valid for full range of flow) compared with implicit Colebrook equation

3.5 Swamee and Jain approximation

Approximation proposed by Swamee and Jain (1976) (10) with error up to 2.04% is almost identical as those proposed by Churchill (1973). Distribution of estimated error of Swamee and Jain approximation over turbulent part of Moody's chart is shown in figure 7.

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{5.74}{Re^{0.9}}\right) \quad (10)$$

Some further details on this approximation readers can see in paper of Swamee and Rathie (2007).

3.6 Jain approximation

Approximation proposed by Jain (1976) (11) with error up to 2.05% is comparable with those proposed by Churchill (1973) and Swamee and Jain (1976):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.715 \cdot D} + \left(\frac{6.943}{Re}\right)^{0.9}\right) \tag{11}$$

Distribution of estimated error of Jain approximation over turbulent part of Moody's chart is shown in figure 7.

3.7 Churchill approximation (full range of turbulence including laminar regime)

Approximation proposed by Churchill (1977) covers entire laminar and turbulent regime (including unstable zone between them) with error up to 2.19% (12):

$$\lambda \approx 8 \cdot \left(\left(\frac{8}{Re}\right)^{12} + \frac{1}{\left(C_1 + C_2\right)^{1.5}}\right)^{\frac{1}{12}} \tag{12}$$

Where C_1 is (13):

$$C_1 = \left(2.457 \cdot \ln \frac{1}{\left(\frac{7}{Re}\right)^{0.9} + 0.27 \cdot \frac{\varepsilon}{D}}\right)^{16} \tag{13}$$

And C_2 is (14):

$$C_2 = \left(\frac{37530}{Re}\right)^{16} \tag{14}$$

Distribution of estimated error of Churchill approximation (full range of turbulence including laminar regime and unstable zone between them) over turbulent part of Moody's chart is shown in figure 7.

3.8 Chen approximation

First, really accurate approximation (15) was developed by Chen (1979).

$$\frac{1}{\sqrt{\lambda}} \approx -2.0 \cdot \log_{10}\left(\frac{\varepsilon}{3.7065 \cdot D} - \frac{5.0452}{Re} \cdot \log_{10}\left(\frac{1}{2.8257} \cdot \left(\frac{\varepsilon}{D}\right)^{1.1098} + \frac{5.8506}{Re^{0.8981}}\right)\right) \qquad (15)$$

Estimated error of Chen approximation is up to 0.35% compared with implicit Colebrook equation (Figure 8).

Figure 8. Distribution of estimated error of Chen approximation compared with implicit Colebrook equation

For some details on Chen (1979) approximation readers can see discussion by Schorle et al (1980) and closure of Chen (1980).

3.9 Round approximation

Approximation proposed by Round (1980) is relative simple but not very accurate (16):

$$\frac{1}{\sqrt{\lambda}} \approx 1.8 \cdot \log_{10}\left(\frac{Re}{0.135 \cdot Re \cdot \left(\frac{\varepsilon}{D}\right) + 6.5}\right) \qquad (16)$$

Estimated error of Round approximation is up to 10.92% compared with implicit Colebrook equation (Figure 9).

Figure 9. Distribution of estimated error of Round approximation compared with implicit Colebrook equation

3.10 Barr approximation

Approximation proposed by Barr (1981) was very accurate for the time when it was developed (17). It does not require internal iterative calculus.

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{4.518 \cdot \log_{10}\left(\frac{Re}{7}\right)}{Re \cdot \left(1 + \frac{Re^{0.52}}{29} \cdot \left(\frac{\varepsilon}{D}\right)^{0.7}\right)}\right) \qquad (17)$$

Estimated error of Barr approximation is up to 0.27% compared with implicit Colebrook equation (Figure 10).

Figure 10. Distribution of estimated error of Barr approximation compared with implicit Colebrook equation

3.11 Zigrang and Sylvester approximations

Approximations proposed by Zigrang and Sylvester (1982) use internal iterative procedure to achieve high accuracy (18, 19):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{13}{Re}\right)\right) \qquad (18)$$

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{13}{Re}\right)\right)\right) \tag{19}$$

Form of approximation by Zigrang and Sylvester (1982) (18) is less accurate than (19), since the first one is based on two internal iterations while the second one uses three internal iterations. Estimated error of more complex but also more accurate approximation by Zigrang and Sylvester (1982) is up to 0.13% compared with implicit Colebrook equation (Figure 11). For simpler form of Zigrang and Sylvester approximation error is up to 1% (Figure 11).

Figure 11. Distribution of estimated error of Zigrang and Sylvester approximations compared with implicit Colebrook equation

3.12 Haaland approximation

Approximation proposed by Haaland (1983) is very accurate and simple. It was first one, designed equally for calculation of friction factor for liquid and gaseous flow (20):

$$\frac{1}{\sqrt{\lambda}} \approx -\frac{1.8}{n} \cdot \log_{10}\left(\left(\frac{\varepsilon}{3.7 \cdot D}\right)^{1.11 \cdot n} + \left(\frac{6.9}{Re}\right)^{n}\right) \tag{20}$$

For n=1, Haaland equation (20) is valid for flow of liquid. Haaland (1983) suggested that n=3 yields friction factors in consonance with those recommended for use in gas transmission lines.

Estimated error of approximation by Haaland (1983) valid for liquid flow is up to 1.4% compared with implicit Colebrook equation (Figure 12).

Figure 12. Distribution of estimated error of Haaland approximation for liquid flow compared with implicit Colebrook equation

Same comparison is done for Haaland approximation for gaseous flow (Figure 13). This was done in comparisons with standard implicit Colebrook equation and with modified implicit Colebrook equation rearranged by AGA.

Figure 13. Distribution of estimated error of Haaland approximation for gaseous flow compared with implicit Colebrook equation (upper) and with modified implicit Colebrook equation rearranged by AGA (lower)

3.13 Serghides approximations

Approximations proposed by Serghides (1984)[1]. are accurate but with internal iterative calculus (21, 22):

$$\lambda \approx \left(S_1 - \frac{(S_2 - S_1)^2}{S_3 - 2 \cdot S_2 + S_1} \right)^{-2} \tag{21}$$

$$\lambda \approx \left(4.781 - \frac{(S_1 - 4.781)^2}{S_2 - 2 \cdot S_1 + 4.781} \right)^{-2} \tag{22}$$

Where S_1 is (23):

$$S_1 = -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{12}{Re} \right) \tag{23}$$

Where S_2 is (24):

[1] Cronologically, after approximation by Serghides (1984) are approximations proposed by Chen (1984, 1985). But these approximations are similar with Altshul, Russian power-law equation from Soviet practice and therefore they will be presented with this equation later in the text. Note also that author of Chen approximation from 1979 is Chen N.H. (Chen 1979), while the author of Chen approximations from 1984 is Chen J.J.J. (Chen 1984).

$$S_2 = -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51 \cdot S_1}{Re}\right) \quad (24)$$

Where S_3 is (25):

$$S_3 = -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51 \cdot S_2}{Re}\right) \quad (25)$$

Form of approximation by Serghides (1984) (21) is more accurate than (22), since the first one is based on three internal steps while the second one use two internal steps. More complex version is with accuracy up to 0.13% while less complex one is up to 0.35% (Figure 14).

Figure 14. Distribution of estimated error of Serghides approximations compared with implicit Colebrook equation

3.14 Manadilli approximation

Approximation by Manadilli (1997) contains signomial terms (26).

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} + \frac{95}{Re^{0.983}} - \frac{96.82}{Re}\right) \quad (26)$$

A special group of functions appearing in mathematical models of many processes is the signomial functions. A signomial function is defined as the sum of signomial terms, which in turn are products of power functions multiplied with a real constant. Estimated error of approximation by Manadilli (1997) is up to 2.06% compared with implicit Colebrook equation (Figure 15).

Figure 15. Distribution of estimated error of Manadilli approximation compared with implicit Colebrook equation

3.15 Romeo, Royo and Monzón approximation

Approximation proposed by Romeo et al (2002) has three internal iterations (27):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7065 \cdot D} - \frac{5.0272}{Re} \cdot \log_{10}\left(\frac{\varepsilon}{3.827 \cdot D} - \frac{4.567}{Re} \cdot \log_{10}\left(\left(\frac{\varepsilon}{7.7918 \cdot D}\right)^{0.9924} + \left(\frac{5.3326}{208.815 + Re}\right)^{0.9345}\right)\right)\right) \quad (27)$$

The calculation of the parameters of Romeo et al (2002) approximation was done through non-linear multivariable regression. Estimated error of approximation by Romeo, Royo and Monzón is up to 0.13% compared with implicit Colebrook equation (Figure 16).

Figure 16. Distribution of estimated error of approximation by Romeo, Royo and Monzón compared with implicit Colebrook equation

3.16 Sonnad and Goudar approximation

Approximation by Sonnad and Goudar (2006) was developed using Lambert W-function (28):

$$\frac{1}{\sqrt{\lambda}} \approx 0.8686 \cdot \ln\left(\frac{0.4587 \cdot Re}{G^{G/(G+1)}}\right) \quad (28)$$

Where G is (29):

$$G = 0.124 \cdot Re \cdot \frac{\varepsilon}{D} + \ln(0.4587 \cdot Re) \quad (29)$$

For some details on Sonnad and Goudar (2006) approximation readers can see discussions by Vatankhah and Kouchakzadeh (2008, 2009) and Yıldırım (2008). Paper of Sonnad and Goudar (2007) should be also recommended as reference. Approximation by Sonnad and Goudar (2006) is not suitable for all range of relative roughness and Reynolds numbers (Sonnad and Goudar 2004). Estimated error of approximation by Sonnad and Goudar (2006) is up to 0.8% compared with implicit Colebrook equation (Figure 17).

Figure 17. Distribution of estimated error of Sonnad and Goudar approximation compared with implicit Colebrook equation

Vatankhah and Kouchakzadeh (2008, 2009) rearranged approximation proposed by Sonnad and Goudar (2006) as (29a, 29b), to increase its accuracy with error up to 0.15% (Figure 18). These two improved equations here will be noted as approximations by Vatankhah and Kouchakzadeh (2008, 2009).

$$\frac{1}{\sqrt{\lambda}} \approx 0.8686 \cdot \ln\left(\frac{0.4587 \cdot \mathrm{Re}}{(\mathrm{G}-0.31)^{\mathrm{G}/(\mathrm{G}+0.9633)}}\right) \quad (29a)$$

$$\lambda \approx \left(0.8686 \cdot \ln\left(\frac{0.4587 \cdot \mathrm{Re}}{(\mathrm{G}-0.28)^{\mathrm{G}/(\mathrm{G}+0.98)}}\right)\right)^{-2} \quad (29b)$$

Figure 18. Distribution of estimated error of Vatankhah and Kouchakzadeh approximation compared with implicit Colebrook equation

Parameter G in Vatankhah and Kouchakzadeh approximation is actually parameter G from Sonnad and Goudar approximation

3.17 Rao and Kumar approximation

Approximation by Rao and Kumar (2007) cannot be recommended to be used because of its inaccuracy (30). Of course this inaccuracy is valid only apropos standard Colebrook equation.

$$\frac{1}{\sqrt{\lambda}} \approx 2 \cdot \log_{10}\left(\frac{\left(2 \cdot \frac{\varepsilon}{\mathrm{D}}\right)^{-1}}{\left(\frac{0.444+0.135 \cdot \mathrm{Re}}{\mathrm{Re}}\right) \cdot \Phi(\mathrm{Re})}\right) \quad (30)$$

Where $\Phi(\mathrm{Re})$ is (31):

$$\Phi(\mathrm{Re}) = 1 - 0.55 e^{-0.33 \cdot \left(\ln\left(\frac{\mathrm{Re}}{6.5}\right)\right)^2} \tag{31}$$

Φ(Re) can be neglected in most cases. Some additional details can be seen in paper of Rao and Kumar (2009). Estimated error of Rao and Kumar approximation is up to 82% compared with implicit Colebrook equation (Figure 19).

Figure 19. Distribution of estimated error of Rao and Kumar approximation compared with implicit Colebrook equation

3.18 Buzzelli approximation

Buzzelli (2008) proposed one up to date among most accurate and also relatively simple approximation (32):

$$\frac{1}{\sqrt{\lambda}} = B_1 - \left(\frac{B_1 + 2 \cdot \log_{10}\left(\frac{B_2}{\mathrm{Re}}\right)}{1 + \frac{2.18}{B_2}} \right) \tag{32}$$

Where B is (33):

$$B_1 = \frac{(0.774 \cdot \ln(\mathrm{Re})) - 1.41}{\left(1 + 1.32 \cdot \sqrt{\frac{\varepsilon}{D}}\right)} \tag{33}$$

and B_2 is (34):

$$B_2 = \frac{\varepsilon}{3.7 \cdot D} \cdot \mathrm{Re} + 2.51 \cdot B_1 \tag{34}$$

Estimated error of approximation by Buzzelli (2008) is up to 0.13% compared with implicit Colebrook equation (Figure 20).

Figure 20. Distribution of estimated error of Buzzelli approximation compared with implicit Colebrook equation

3.19 Avci and Karagoz approximation

Formula by Avci and Karagoz (2009) have been recently developed from the experimental Princeton super-pipe data (35):

$$\lambda = \frac{6.4}{\left(\ln(\mathrm{Re}) - \ln\left(1 + 0.01 \cdot \mathrm{Re} \cdot \frac{\varepsilon}{D} \cdot \left(1 + 10 \cdot \sqrt{\frac{\varepsilon}{D}}\right)\right)\right)^{2.4}} \qquad (35)$$

Estimated error of approximation by Avci and Karagoz (2009) is up to 4.7% compared with implicit Colebrook equation (Figure 21).

Figure 21. Distribution of estimated error of Avci and Karagoz approximation compared with implicit Colebrook equation

3.20 Papaevangelou, Evangelides and Tzimopoulos approximation

Papaevangelou et al (2010) noticed that error values tended to “bend” to negative values in an exponential way for Re lower than 10^6. According to that they chose parameters in their equation (36):

$$\lambda = \frac{0.2479 - 0.0000947 \cdot \left(7 - \log_{10} \mathrm{Re}\right)^4}{\left(\log_{10}\left(\frac{\varepsilon}{3.615 \cdot D} + \frac{7.366}{\mathrm{Re}^{0.9142}}\right)\right)^2} \qquad (36)$$

Estimated error of approximation by Papaevangelou, Evangelides and Tzimopoulos is up to 0.85% compared with implicit Colebrook equation (Figure 22).

Figure 22. Distribution of estimated error of approximation by Papaevangelou, Evangelides and Tzimopoulos compared with implicit Colebrook equation

3.21 Brkić approximation

Brkić (2011a) approximations were developed using Lambert W-function (37, 38):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(10^{-0.4343\beta} + \frac{\varepsilon}{3.71 \cdot D}\right) \quad (37)$$

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(\frac{2.18 \cdot \beta}{Re} + \frac{\varepsilon}{3.71 \cdot D}\right) \quad (38)$$

Where β is (39):

$$\beta = \ln \frac{Re}{1.816 \cdot \ln\left(\frac{1.1 \cdot Re}{\ln(1 + 1.1 \cdot Re)}\right)} \quad (39)$$

Estimated error of Brkić approximation is up to 2.3% compared with implicit Colebrook equation (Figure 23).

Figure 23. Distribution of estimated error of Brkić approximation compared with implicit Colebrook equation

Additionally (Barry et al. 2000), error can be reduced using (39a) or (39b):

$$\beta_1 \approx 1.4586887 \cdot \beta - 0.4586887 \cdot \ln\left(\frac{Re \cdot 0.917365}{\ln(1 + Re \cdot 0.917365)}\right) \quad (39a)$$

$$\beta_1 \approx \ln\left(0.488 \cdot Re \cdot \left[\ln\left(\frac{Re}{\beta}\right)\right]^{-1}\right) \quad (39b)$$

For parameter β (39), solution for Lambert W-function by Barry et al. (2000) is used. Procedure by Winitzki (2003) gives parameter β with similar accuracy (40):

$$\beta \approx \ln(1+0.458\cdot \mathrm{Re})\cdot\left(1-\frac{\ln(1+\ln(1+0.458\cdot \mathrm{Re}))}{2+\ln(1+0.458\cdot \mathrm{Re})}\right) \tag{40}$$

Parameter β calculated by using procedures by Barry et al. (2000) and by Winitzki (2003) gives similar results in accuracy.

4. Comparative analysis and complexity of available explicit approximations of the Colebrook equation

Although the Colebrook formula itself is not very accurate (Cipra 1996), its accurate resolution is nonetheless an issue for numerical simulations because a too crude resolution may affect the repeatability and comparisons of calculation (Clamond 2009).

There is no special explanation for different coefficients in some equations (e.g. 3.7065 or 3.707 instead of 3.71 etc). Only reasonable explanation can be that this changed coefficients maybe better fit experimental data. Examples for this are e.g. Churchill (1973) approximation (9) and Jain (1976) approximation (11) with slightly different coefficients.

Churchill (1977) relation (12) holds for all values of Reynolds and relative roughness, including laminar regime (Figure 24).

Figure 24. Churchill (1977) approximation includes laminar and highly unstable transient zone from laminar to turbulent

Since Churchill (1977) relation (12) is a continuous function for Reynolds numbers above 0, it also lets one calculate a friction factor in the transient zone, between laminar and smooth turbulent regime. Of course the accuracy of such a friction factor probably cannot be determined, but it will be reasonable. Regarding this issue, readers also can see paper of Swamee and Swamee (2007).

In his recent paper, Yıldırım (2009, 2011) compared 12 of here presented more than 20 approximations of the implicit Colebrook equation. Ouyang and Aziz (1996) made similar research for the approximations available in that time. Data of Ouyang and Aziz (1996) are also available from paper of Abdolahi et al (2007). Similar comparisons are available from the papers of Gregory and Fogarasi (1985), Zigrang and Sylvester (1985) and papers of Goudar and Sonnad (2007, 2008).

For the analysis of relative error distribution, 20 points (16 for relative comparisons shown in figures 25-30) are used for relative roughness (ε/D) and 37 points for Reynolds number (Re). Points for relative roughness (ε/D) used for comparative study are shown in figures 25-30. Points for Reynolds number are between $1 \cdot 10^4$-$1 \cdot 10^8$. This means that grid with 592 check points is formed for comparative study (Figures 25-30) and with 740 check points for error distribution study (Figures 4-23). This comparative study is based on some particular conditions of test grid points which means that presented relative error could be different using different check points. But here presented analysis with 592 and 740 check points give good general picture of accuracy for the presented approximations. Relative error is not distributed systematically over the entire range of Reynolds number and relative roughness which means that real maximal relative error can be slightly above here reported. MS Excel file is available as electronic appendix[2] to this paper and in this file any other values for Reynolds number and for relative roughness can be used as input parameters. Modern software, especially in the case of spreadsheets, can be conveniently programmed to solve any implicit equation, with a minimum of programming and details. Some of MS Excel effectiveness for numerical computations comes from a module 'Solver'. It was originally designed for optimization problems, where one has to find values of a number of different parameters such that some quantity is minimized, usually the sum of errors of a number of equations. With this tool one can find such optimal solutions, or solutions of one or many equations, even if they are nonlinear. In more details, to allow iterative computations in MS Excel 2007, one has to choose 'Excel options', and then in 'Formulas' to tick box 'Enable iterative calculation'. In this case, Excel is set to terminate the calculation after maximum $3\cdot10^4$ iterations or $1\cdot10^{-7}$ difference or less between the values of two successive iterations. Consequently, deviations involving explicit equations have been reported to $1\cdot10^{-5}$%. This means that real relative error is presented by sum of calculated relative error and deviation. So deviation in 'Excel options' has to be set to be significantly smaller compared to estimated error of observed approximations ('Maximum Change' in 'Formulas').

Maximal percentage (relative) error of presented approximations over the entire range of applicability of Colebrook equation is shown in table 1.

Table 1. Maximal relative error for available approximations for test check points

[2] MS Excel can be also successfully used in other engineering fields; see electronic appendix in Brkić and Tanasković (2008)

Approximation by Rao and Kumar (2007) is extremely inaccurate compared with standard Colebrook equation (Figure 25). Approximations proposed by Moody (1947), Wood (1966), Eck (1973) and Round (1980) should not be used because they produce significant relative error (Figure 26). Moderate accurate approximations such as proposed by Jain (1976), Swamee and Jain (1976), Churchill (1973, 1977), Manadilli (1997), Brkić (2011a) and Avzi and Karagoz (2009), can be used since they made maximal relative error up to 5% (Figure 27). Approximations by Zigrang and Sylvester (1982) (18), Haaland (1983), Sonnad and Goudar (2006) and Papaevangelou et al (2010) produce maximal error up to 1.5% (Figure 28). Very accurate (Figure 29), with estimated error up to 0.5%, are approximations by Chen (1979), Barr (1981), Zigrang and Sylvester (1982) (19), Serghides (1984), Romeo et al (2002), Buzzelli (2008), and approximations proposed by Vatankhah and Kouchakzadeh approximations (2008, 2009). Note that approximations proposed by Vatankhah and Kouchakzadeh (2008, 2009) are actually very successfully improved approximation by Sonnad and Goudar (2006).

Figure 25. Inaccuracy of approximation by Rao and Kumar (2007)

Figure 26. Non-advisable approximations

Figure 27. Approximations with estimated error up to 5% (less accurate approximations)

Figure 28. Approximations with estimated error up to 1.5% (moderate accurate approximations)

Figure 29. Approximations with estimated error up to 0.5% (very accurate approximations)

Extremely accurate approximations from figure 29 can be seen in figure 30 in better resolution.

Figure 30. Extremely accurate approximations presented in higher resolution

Numerical solutions for friction factors based on the Colebrook equation can be obtained by to any desired degree of precision. For many applications, the simpler but less accurate explicit equation will be sufficed. Sometime, simplicity is sacrificed for excessive accuracy. To find balance between these two extremes it is appropriate to introduce concept of complexity or complexity index of explicit approximations. Zigrang and Sylvester (1985) defined complexity as the number of algebraic notation calculator key strokes required to solve the equation for $Re=10^5$ and $\varepsilon/D=0.001$ (Table 2). Complexity index is defined as quotient of key strokes required for an observed approximation and the least complex one (Table 2).

Table 2. Complexity and complexity index of available explicit approximations

Eck (1973) approximation is the least complex with only 27 key strokes required for solution and hence this equation has complexity index 1 (Table 2).

5. A note on power-law formulas valid for the same range of turbulent flow as standard Colebrook equation

Derivation of relations for the friction factor is mostly based on the logarithmic or power law formulation of velocity profiles in boundary layers (Zagarola et al 1997). As mentioned in introduction, Colebrook equation is based on the logarithmic formulation. Example of the power-law formula, valuable for the same flow regimes as Colebrook's is old Altshul equation from the Soviet era (Figure 31) (41, 42):

$$\lambda = 0.11 \cdot \left(\frac{\varepsilon}{D} + \frac{68}{Re} \right)^{\frac{1}{4}} \tag{41}$$

$$\lambda = 0.1 \cdot \left(\frac{1.46 \cdot \varepsilon}{D} + \frac{100}{Re} \right)^{\frac{1}{4}} \tag{42}$$

Altshul formula was eliminated from the last wording of the Russian norms. However, it is used as before since other recommendations were not proposed (Sukharev et al 2005).

Figure 31. Distribution of deviation of Altshul formula (41) compared with implicit Colebrook equation

Similar are formulas proposed by Chen (1984) (43, 44):

$$\lambda = 0.3164 \cdot \left(\frac{1}{Re^{0.83}} + 0.11 \cdot \left(\frac{\varepsilon}{D} \right) \right)^{0.3} \tag{43}$$

$$\lambda = 0.184 \cdot \left(\frac{1}{Re^{0.67}} + 0.7 \cdot \left(\frac{\varepsilon}{D} \right) \right)^{0.3} \tag{44}$$

Deviation of presented power-law formulas from Russian practice and by Chen (1984) is shown in figure 32.

Figure 32. Power law formulas as substitution for implicit Colebrook equation

Readers also can see paper by Chen (1985). In general, approximations by Moody (1947) and by Wood (1966) also belong to power-law formulas.

6. Some remarks on further developments

Today, different approach can be used for determination of friction factor. Good example for the era of computerization is approach of Özger and Yildlrim (2009). They use adaptive neuro-fuzzy computing technique for determination of turbulent flow friction coefficient. In the paper of Yoo and Singh (2005) are shown two new methods for the computation of commercial pipe friction factor. Today, main problem is not how to calculate friction factor. Problem is how to measure or estimate roughness of pipe (Farshad et al 2001). Most pipes usually have rough inner pipe surface. Resistance to fluid flow offered by rough boundaries is larger than that for smooth one due to the formation of eddies behind protrusions. Colebrook equation is valid for both, smooth and rough turbulent regime including transient zone between them. In principle, a system of partial differential equations known as Navier-Stokes equations describes the exact behavior of the fluid flow in so-called boundary layer, but solving these equations remains beyond current theory and computations. Sletfjerding and Gudmundsson (2003) proposed also methodology for determination of friction factor directly from roughness measurements. In that way they eliminated roughness as a parameter in Colebrook equation (only Reynolds number and pipe diameter are necessary as input parameters). Using a similar approach to that of in Nikuradse's experiment, Sletfjerding and Gudmundsson (2003) related measured roughness values with friction factor, but their equation is implicit in friction factor. In formulation given by Sletfjerding and Gudmundsson (2003) equation is implicit and valid for average steel pipe (45):

$$\frac{1}{\sqrt{\lambda}} = -1.89 \cdot \log_{10}\left(\frac{1.55}{\mathrm{Re}\cdot\sqrt{\lambda}} + \left(\frac{4}{D}\right)^{1.03}\right) \tag{45}$$

For other materials of pipes readers can consult paper of Sletfjerding and Gudmundsson (2003).

7. Conclusion

Maybe, it is difficult for many to recall for the time as recently as the 1970's where there were no personal computers or even calculators that could do much more than add or subtract. In that environment an implicit relationship such as Colebrook (1939), which was well-known then, was impractical and some simplification was essential. Today, it is not difficult to solve single Colebrook equation by iteration. But solution of complex looped pipeline problem in such case requires double iterative procedure where first is for the standard implicit Colebrook equation while second one is for Hardy Cross method or similar iterative method used to solve simulation problem in a looped pipe network (Brkić 2009b, 2011b). This double procedure can be serious burden even for today very powerful computers.

All shown approximate equations give the friction factor explicitly as a function of Reynolds number and relative roughness. Comparative analysis indicates that almost all approximate equations give a very good prediction of the friction factor and can reproduce the Colebrook equation and its Rouse and Moody plot. Therefore, these approximations for the friction factor provide a rational, accurate, and practically useful method over the entire range of the Moody chart in terms of Reynolds number and relative roughness. Apropos relative complexity at first sight, these approximations can be very easily implemented in a computer code.

Most available approximations of the Colebrook equation are very accurate. Exceptions are Round (1980), Eck (1973), Moody (1947), Wood (1966), and Rao and Kumar (2007) approximations. The average error of almost all explicit approximations of the Colebrook (1939) relation is up to 3%.

Acknowledgment

Author received his PhD in Petroleum and Natural Gas engineering in Serbia in March 2010. He is currently finishing master degree studies in Mechanical engineering in Serbia financed by TEMPUS. Author will appreciate future comments from readers. Approximations presented in this paper are sometimes very complex and hence typographical errors are possible. These potential errors are by author himself and not by original authors of approximations. Therefore, author regrets because of any future inconvenience regarding this matter.

Appendix. Supporting Information

Supplementary data associated with this article can be found in the online version at doi:

MS Excel spreadsheet file is accompanied as electronic annex with on-line version of this paper. Note that you have to install Office 2007 (Enterprise edition) to inspect this file. File contains all formulas presented in the text. Readers can change values of Reynolds number and relative roughness to calculate Darcy friction factor. To allow necessary implicit calculation in MS Excel 2007 which is used for calculation of implicit Colebrook's relation, the 'Office button' at the upper-left corner of the Excel screen have to be pressed, and in the 'Excel options', 'Formulas' has to be chosen and finally box ‘Enable iterative calculation’ have to be ticked. This allows implementation of so called ‘Circular references’ into a calculation.

References:

Abdolahi, F., Mesbah, A., Boozarjomehry, R.B., Svrcek, W.Y. 2007. The effect of major parameters on simulation results of gas pipelines. Int. J. Mech. Sci. 49(8), 989–1000.

Avci, A., Karagoz, I. 2009. A novel explicit equation for friction factor in smooth and rough pipes. J. Fluid. Eng. ASME 131(6), 061203 (1-4).

Barr, D.I.H. 1981. Solutions of the Colebrook-White function for resistance to uniform turbulent flow. Proc. Inst. Civil. Eng. 71(2), 529–536.

Barry D.A., Parlange, J.-Y., Li, L., Prommer, H., Cunningham, C.J., Stagnitti F. 2000. Analytical approximations for real values of the Lambert W-function. Math. Comput. Simulat. 53(1-2), 95-103.

Brkić, D. 2009a. Comments on ‘Settling velocities of particulate systems 15: Velocities in turbulent Newtonian flows’. Int. J. Miner. Process. 92(3-4), 201-202.

Brkić, D. 2009b. An improvement of Hardy Cross method applied on looped spatial natural gas distribution networks. Appl. Energ. 86(7-8), 1290-1300.

Brkić, D. 2011a. An explicit approximation of the Colebrook equation for fluid flow friction factor. Petrol. Sci. Tech. 29(xx), xxx-xxx. accepted, in press

Brkić, D. 2011b. Gas distribution network hydraulic problem from practice, Petrol. Sci. Tech. 29(4), 366-377.

Brkić, D., Tanasković, T.I. 2008. Systematic approach to natural gas usage for domestic heating in urban areas. Energy 33(12), 1738-1753.

Buzzelli, D. 2008. Calculating friction in one step. Mach. Des. 80(12), 54-55.

Chen, J.J.J. 1984. A simple explicit formula for the estimation of pipe friction factor. Proc. Inst. Civil. Eng. 77, 49-55.

Chen, J.J.J. 1985. Systematic explicit solutions of the Prandtl and Colebrook-White equations for pipe flow. Proc. Inst. Civil. Eng. 79, 383-389.

Chen, N.H. 1979. An explicit equation for friction factor in pipes. Ind. Eng. Chem. Fundam. 18(3), 296–297.

Chen, N.H. 1980. Comments on: "An explicit equation for friction factor in pipe". Ind. Eng. Chem. Fundam. 19(2), 229-230.

Churchill, S.W. 1973. Empirical expressions for the shear stress in turbulent flow in commercial pipe. AIChE J. 19(2), 375–376.

Churchill, S.W. 1977. Friction-factor equation spans all fluid flow regimes. Chem. Eng. 84(24), 91–92.

Cipra, B. 1996. A new theory of turbulence causes a stir among experts. Science 272(5264), 951.

Clamond, D. 2009. Efficient resolution of the Colebrook equation. Ind. Eng. Chem. Res. 48(7), 3665–3671.

Coelho, P.M., Pinho, C. 2007. Considerations about equations for steady state flow in natural gas pipelines. J. Braz. Soc. Mech. Sci. Eng. 29(3), 262-273.

Colebrook, C.F. 1939. Turbulent flow in pipes with particular reference to the transition region between the smooth and rough pipe laws. J. Inst. Civil. Eng. (London) 11(4), 133-156.

Colebrook, C.F., White C.M. 1937. Experiments with fluid friction in roughened pipes. Proc. Roy. Soc. Ser. A Math. Phys. Sci. 161(906), 367-381.

Concha, F. 2008. Settling velocities of particulate systems 15: Velocities in turbulent Newtonian flows. Int. J. Miner. Process. 88(3-4), 89–93.

Eck, B. 1973. Technische Stromungslehre; Springer: New York.

Farshad, F., Rieke, H., Garber, J. 2001. New developments in surface roughness measurements, characterization, and modeling fluid flow in pipe. J. Petrol. Sci. Eng. 29(2), 139–150.

Goudar, C.T., Sonnad, J.R. 2003. Explicit friction factor correlation for turbulent flow in smooth pipes. Ind. Eng. Chem. Res. 42(12), 2878-2880.

Goudar, C.T., Sonnad, J.R. 2007. Explicit friction factor correlation for turbulent flow in rough pipe. Hydrocarb. Process. 86(4), 103-105.

Goudar, C.T., Sonnad, J.R. 2008. Comparison of the iterative approximations of the Colebrook-White equation. Hydrocarb. Process. 87(8), 79-83.

Gregory, G.A., Fogarasi, M. 1985. Alternate to standard friction factor equation. Oil. Gas. J. 83(13), 120, 125–127.

Haaland, S.E. 1983. Simple and explicit formulas for friction factor in turbulent pipe flow. J. Fluid. Eng. ASME 105(1), 89-90.

Hager, W.H., Liiv, U. 2008. Johann Nikuradse – Hydraulic experimenter. J. Hydraul. Res. IAHR 46(4), 435-444.

Hammad, M. 1999. Product Performance: Energy performance of plastic pipes. Polymer Testing 18(2), 111–122.

Hayes, B. 2005. Why W? Am. Sci. 93(2), 104-108.

Jain, A.K. 1976. Accurate explicit equation for friction factor. J. Hydraul. Div. ASCE 102(HY5), 674–677.

Keady, G. 1998. Colebrook-White formulas for pipe flow. J. Hydraul. Eng. ASCE 124(1), 96-97.

Manadilli, G. 1997. Replace implicit equations with signomial functions. Chem. Eng. 104(8), 129–130.

Moody, L.F. 1944. Friction factors for pipe flow. Trans. ASME 66(8), 671-684.

Moody, L.F. 1947. An approximate formula for pipe friction factors. Trans. ASME 69(12), 1005–1011.

More A.A. 2006. Analytical solutions for the Colebrook and White equation and for pressure drop in ideal gas flow in pipes. Chem. Eng. Sci. 61(16), 5515-5519.

Nandakumar, V. 2007. Analytical calculation of Fanning friction factor. Hydrocarb. Process. 86(1), 97-100.

Ouyang, L.B., Aziz, K. 1996. Steady-state gas flow in pipes. J. Petrol. Sci. Eng. 14(3-4), 137–158.

Özger M., Yıldırım G. 2009. Determining turbulent flow friction coefficient using adaptive neuro-fuzzy computing technique. Adv. Eng. Softw. 40(4), 281-287.

Papaevangelou G, Evangelides C, Tzimopoulos C. 2010. A new explicit equation for the friction coefficient in the Darcy-Weisbach equation, Proceedings of the Tenth Conference on Protection and Restoration of the Environment: PRE10, July 6-9, 2010, Greece, Corfu, 166, pp. 1-7.

Rao, A.R., Kumar B. 2007. Friction factor for turbulent pipe flow. Division of Mechanical Science, Civil Engineering, Indian Institute of Science, Bangalore, India, ID Code 9587, http://eprints.iisc.ernet.in/9587/

Rao, A.R., Kumar B. 2009. Transition of turbulent pipe flow. J. Hydraul. Res. IAHR 47(4), 529-533.

Romeo, E., Royo, C., Monzon, A. 2002. Improved explicit equation for estimation of the friction factor in rough and smooth pipes. Chem. Eng. J. 86(3), 369–374.

Round, G.F. 1980. An explicit approximation for the friction factor-Reynolds number relation for rough and smooth pipes. Can. J. Chem. Eng. 58(1), 122–123.

Rouse, H. 1943. Evaluation of boundary roughness. Proc. 2nd Hydraulics Conf. University of Iowa Studies in Engineering, Bulletin 27, Wiley: New York, 105–116.

Serghides, T.K. 1984. Estimate friction factor accurately. Chem. Eng. 91(5), 63–64.

Schorle, B.J., Churchill, S.W., Shacham, M. 1980. Comments on: “An explicit equation for friction factor in pipe”. Ind. Eng. Chem. Fundam. 19(2), 228-229.

Sletfjerding, E., Gudmundsson, J.S. 2003. Friction factor directly from roughness measurements. J. Energ. Resour. ASME 125(2), 126-130.

Sonnad, J.R., Goudar, C.T. 2004. Constraints for using Lambert W function-based explicit Colebrook-White equation. J. Hydraul. Eng. ASCE 130(9), 929-931.

Sonnad, J.R., Goudar, C.T. 2005. Explicit friction factor correlation for pipe flow analysis. Hydrocarb. Process. 84(6), 103-105.

Sonnad, J.R., Goudar, C.T. 2006. Turbulent flow friction factor calculation using a mathematically exact alternative to the Colebrook-White equation. J. Hydraul. Eng. ASCE 132(8), 863–867.

Sonnad, J.R., Goudar, C.T. 2007. Explicit reformulation of the Colebrook−White equation for turbulent flow friction factor calculation. Ind. Eng. Chem. Res. 46(8), 2593–2600.

Sukharev, M.G., Karasevich, A.M., Samoilov, R.V., Tverskoi, I.V. 2005. Investigation of the hydraulic resistance in polyethylene pipelines. J. Eng. Phys. Thermophys. 78(2), 350-359.

Swamee, P.K., Jain A.K. 1976. Explicit equations for pipe flow problems. J. Hydraul. Div. ASCE 102(HY5), 657–664.

Swamee, P.K., Swamee, N. 2007. Full-range pipe-flow equations. J. Hydraul. Res. IAHR 45(6), 841-843.

Swamee, P.K., Rathie, P.N. 2007. Exact equations for pipe-flow problems. J. Hydraul. Res. IAHR 45(1), 131-134.

Taylor, J.B., Carrano, A.L., Kandlikar, S.G. 2006. Characterization of the effect of surface roughness and texture on fluid flow-past, present, and future. Int. J. Therm. Sc. 45(10), 962–968.

Vatankhah, A.R., Kouchakzadeh, S. 2008. Discussion of “Turbulent flow friction factor calculation using a mathematically exact alternative to the Colebrook-White equation” by Jagadeesh R. Sonnad and Chetan T. Goudar. J. Hydraul. Eng. ASCE 134(8), 1187.

Vatankhah, A.R., Kouchakzadeh, S. 2009. Discussion: Exact equations for pipe-flow problems, by P.K. Swamee and P.N. Rathie. J. Hydraul. Res. IAHR 47(7), 537-538.

Winitzki, S. 2003. Uniform approximations for transcendental functions. in: Kumar, V., Gavrilova, M.L., Jeng, C., Tan, K., L’Ecuyer, P. (Eds.), Lecture Notes in Computer Science, Vol. 2667 (Computational Science and Its Applications – ICCSA 2003, International Conference, Montreal, Canada, May 18-21, 2003 Proceedings, Part 1), Springer, Berlin/Heidelberg, pp. 780–789.

Wood, D.J. 1966. An explicit friction factor relationship. Civil. Eng. 36(12), 60–61.

Yıldırım, G. 2008. Discussion of “Turbulent flow friction factor calculation using a mathematically exact alternative to the Colebrook-White equation” by Jagadeesh R. Sonnad and Chetan T. Goudar. J. Hydraul. Eng. ASCE 134(8), 1185-1186.

Yıldırım, G. 2009. Computer-based analysis of explicit approximations to the implicit Colebrook–White equation in turbulent flow friction factor calculation. Adv. Eng. Softw. 40(11), 1183–1190.

Yıldırım, G. 2011. Closure to the discussion of "Computer-based analysis of explicit approximations to the implicit Colebrook–White equation in turbulent flow friction factor calculation" by Dejan Brkić. Adv. Eng. Softw. 42(xx), xxxx–xxxx. accepted, in press. doi. 10.1016/j.advengsoft.2010.10.015

Yoo, D.H., Singh, V.P. 2004. Explicit design of commercial pipes with no secondary losses. J. Irrig. Drain. Eng. ASCE 130(5), 437-440.

Yoo, D.H., Singh, V.P. 2005. Two new methods for the computation of commercial pipe friction factor. J. Hydraul. Eng. ASCE 131(8), 694-704.

Yoo, D.H., Singh, V.P. 2010. Explicit design of commercial pipes with secondary losses. J. Hydro-environment Res. 4(1), 37-45.

Zagarola, M.V., Perry, A.E., Smits, A.J. 1997. Log laws or power laws: The scaling in the overlap region. Phys. Fluids. 9(7), 2094-2100.

Zigrang, D.J., Sylvester, N.D. 1982. Explicit approximations to the solution of Colebrook friction factor equation. AIChE J. 28(3), 514–515.

Zigrang, D.J., Sylvester, N.D. 1985. A review of explicit friction factor equations. J. Energ. Resour. ASME 107(2), 280–283.

Nomenclature:

D-inner diameter of pipe (m)

Re-Reynolds number (-)

ε-absolute roughness (m)

λ-Darcy (i.e. Moody or Darcy-Weisbach) friction factor (-)

δ-relative error (%)

V-auxiliary term in Wood approximation

C_1, C_2-auxiliary terms in Churchil approximation

S_1, S_2, S_3-auxiliary terms in Serghides approximations

G-auxiliary term in Sonnad-Goudar and Vatankhah-Kouchakzadeh approximation

Φ(Re)-auxiliary term in Rao-Kumar approximation

B_1, B_2-auxiliary terms in Buzzelli approximation

β-auxiliary term in Brkić approximations

n-explained with Haaland approximation (can be n=1 or n=3, n=1 is valid for flow of liquid while n=3 is recommended for use in gas transmission lines)

List of figures

Figure 11. Distribution of estimated error of Zigrang and Sylvester approximations compared with implicit Colebrook equation

Figure 12. Distribution of estimated error of Haaland approximation for liquid flow compared with implicit Colebrook equation

Figure 13. Distribution of estimated error of Haaland approximation for gaseous flow compared with implicit Colebrook equation (upper) and with modified implicit Colebrook equation rearranged by AGA (lower)

Figure 14. Distribution of estimated error of Serghides approximations compared with implicit Colebrook equation

Figure 15. Distribution of estimated error of Manadilli approximation compared with implicit Colebrook equation

Figure 16. Distribution of estimated error of approximation by Romeo, Royo and Monzón compared with implicit Colebrook equation

Figure 17. Distribution of estimated error of Sonnad and Goudar approximation compared with implicit Colebrook equation

Figure 18. Distribution of estimated error of Vatankhah and Kouchakzadeh approximation compared with implicit Colebrook equation

Figure 19. Distribution of estimated error of Rao and Kumar approximation compared with implicit Colebrook equation

Figure 20. Distribution of estimated error of Buzzelli approximation compared with implicit Colebrook equation

Figure 21. Distribution of estimated error of Avci and Karagoz approximation compared with implicit Colebrook equation

Figure 22. Distribution of estimated error of approximation by Papaevangelou, Evangelides and Tzimopoulos compared with implicit Colebrook equation

Figure 23. Distribution of estimated error of Brkić approximation compared with implicit Colebrook equation

Figure 24. Churchill (1977) approximation includes laminar and highly unstable transient zone from laminar to turbulent

Figure 25. Inaccuracy of approximation by Rao and Kumar (2007)

Figure 26. Non-advisable approximations

Figure 27. Approximations with estimated error up to 5% (less accurate approximations)

Figure 28. Approximations with estimated error up to 1.5% (moderate accurate approximations)

Figure 29. Approximations with estimated error up to 0.5% (very accurate approximations)

Figure 30. Extremely accurate approximations presented in higher resolution

Figure 31. Distribution of deviation of Altshul formula (41) compared with implicit Colebrook equation

Figure 32. Power law formulas as substitution for implicit Colebrook equation

List of tables

Research highlights

-Empirical Colebrook equation is an accepted standard for calculation of flow friction factor.

-Colebrook equation is transcendental function implicit in unknown friction coefficient.

-Implicit Colebrook equation cannot be rearranged to derive friction factor directly.

-Colebrook equation has to be solved iteratively or using approximations.

-Iterative calculus can cause problem in simulation of flow in a pipe system.

-Error of almost all explicit approximations of the Colebrook relation is up to 3%.

-An explicit approximation of the Colebrook relation can be very complex but also it can be easily implemented in a computer code.

Graphical Abstract DB

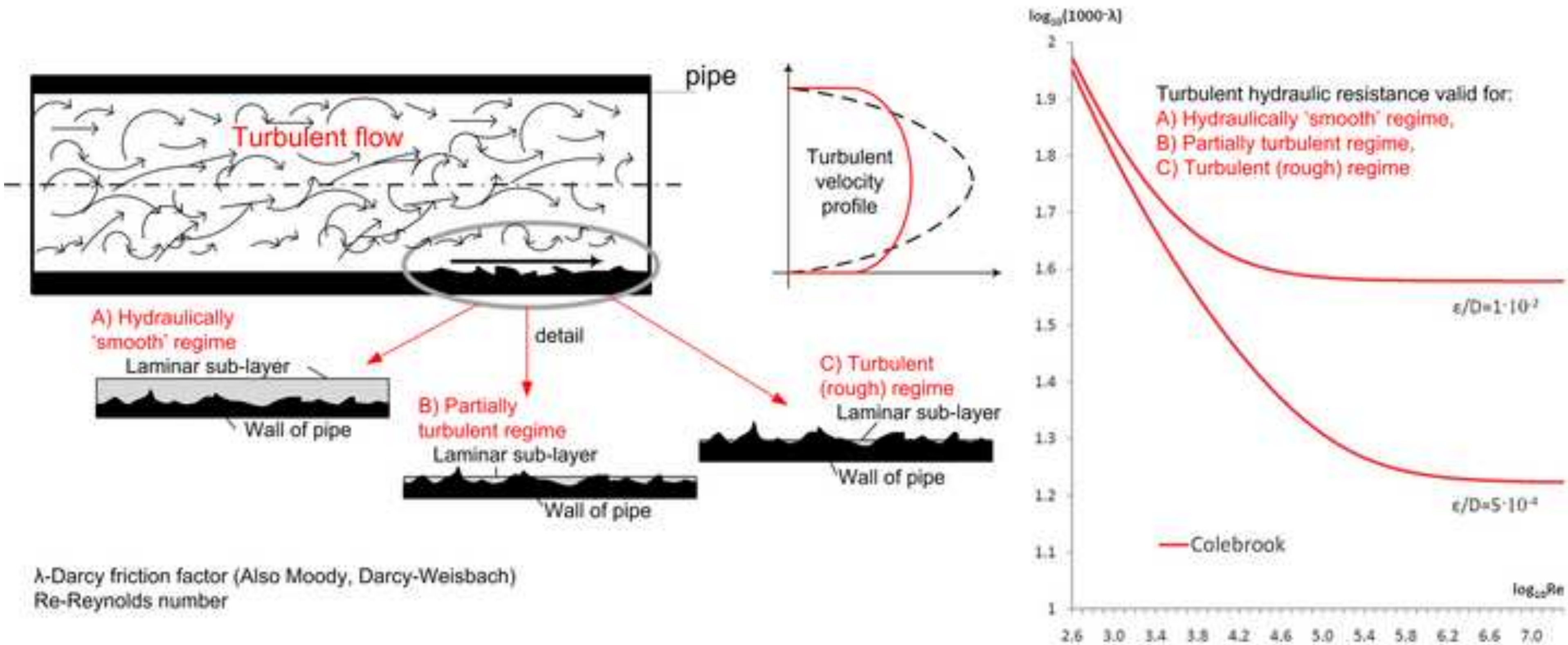


Figure 1 DB

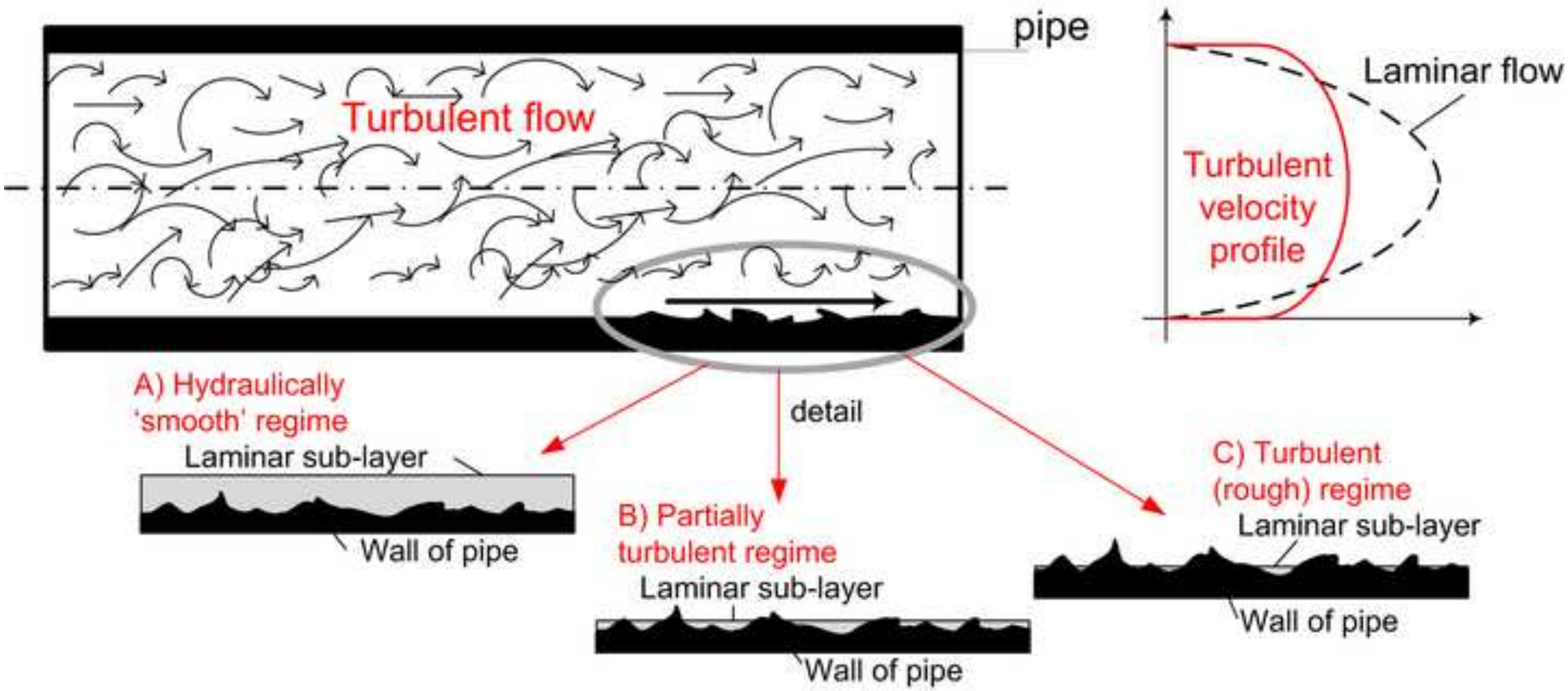

pipe
Turbulent flow
Laminar flow
Turbulent velocity profile
detail
A) Hydraulically 'smooth' regime
Laminar sub-layer
Wall of pipe
B) Partially turbulent regime
Laminar sub-layer
Wall of pipe
C) Turbulent (rough) regime
Laminar sub-layer
Wall of pipe

Figure 2 DB
Click here to download high resolution image

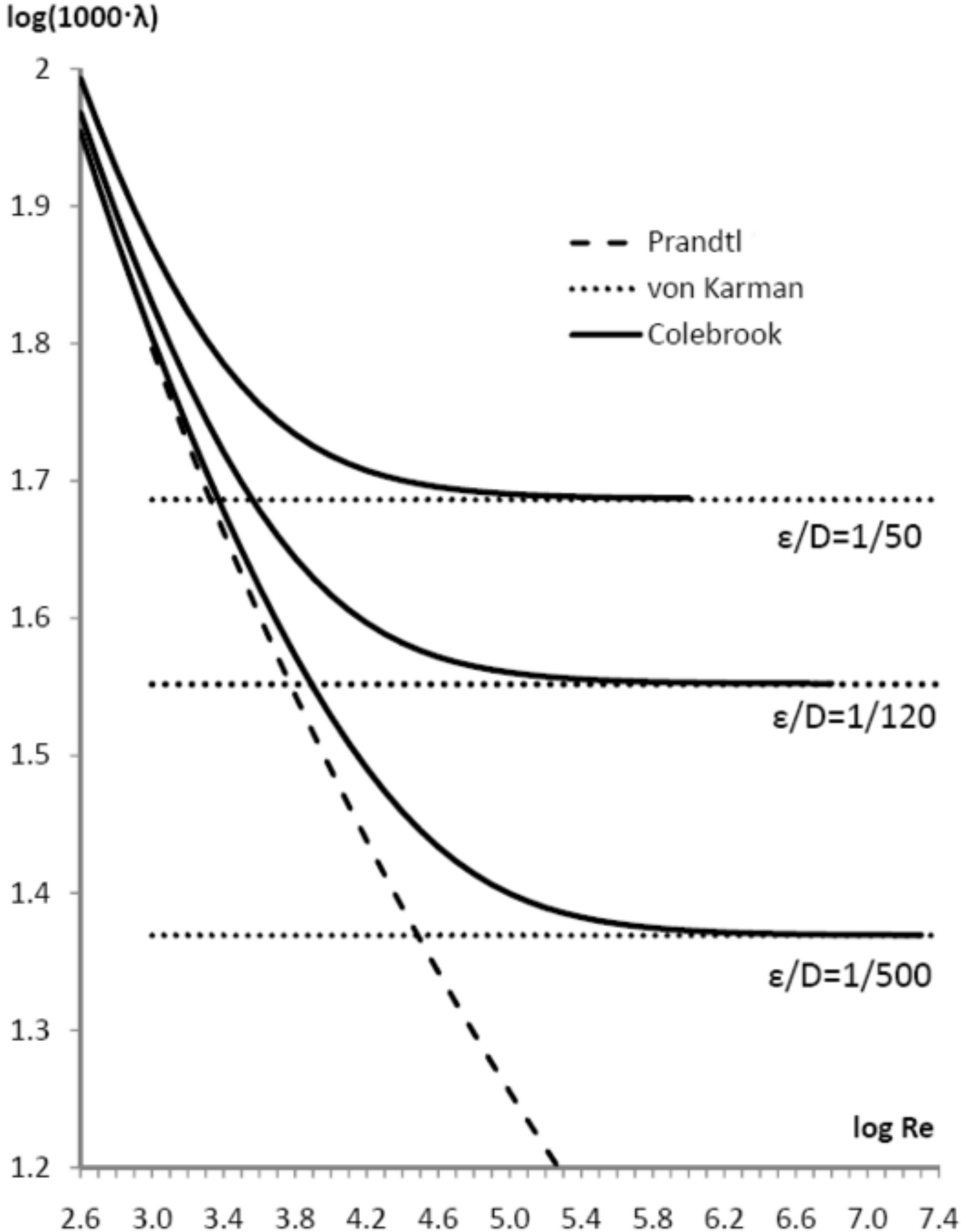


Figure 3 DB
Click here to download high resolution image

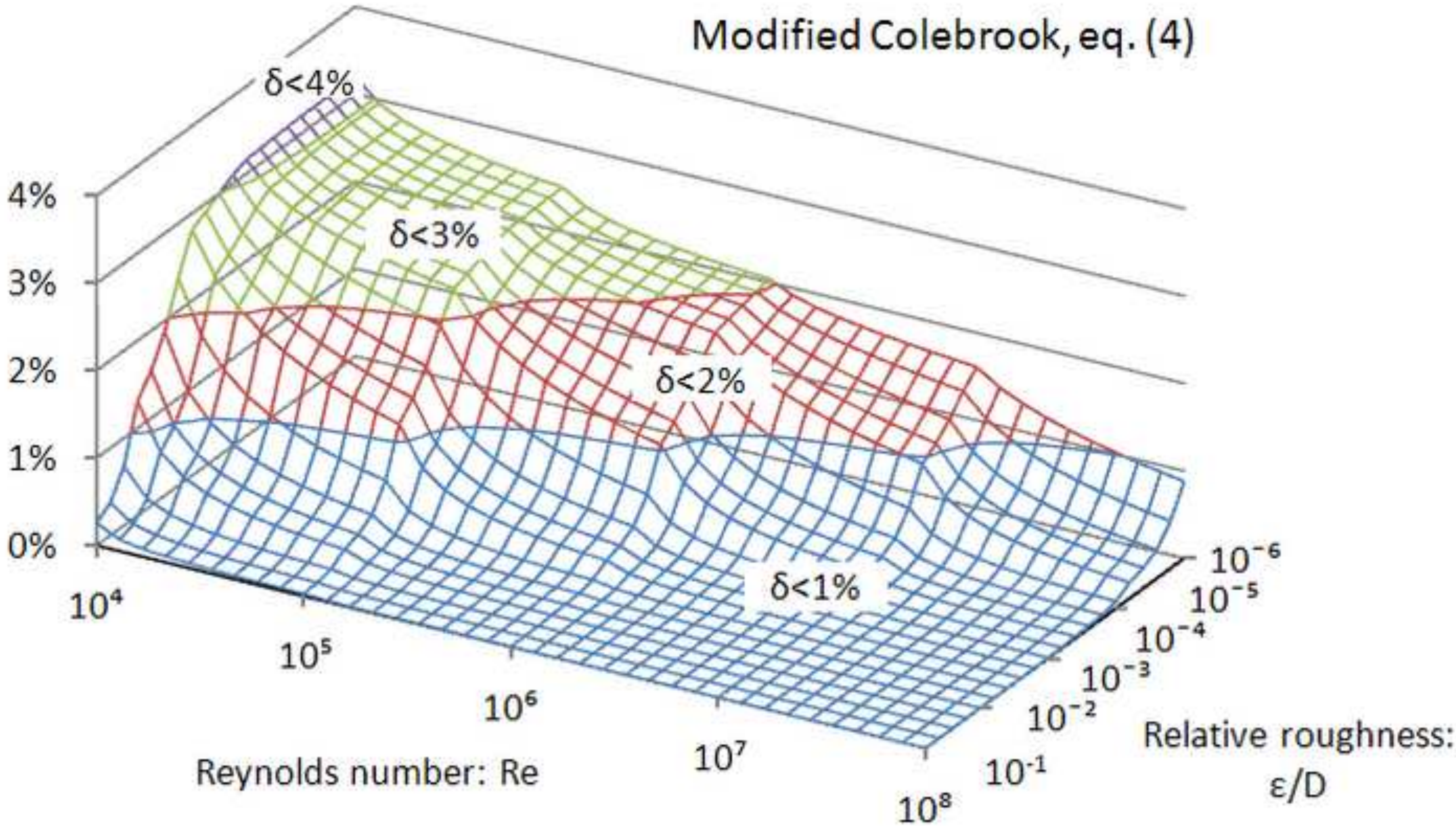


Figure 4 DB

Click here to download high resolution image

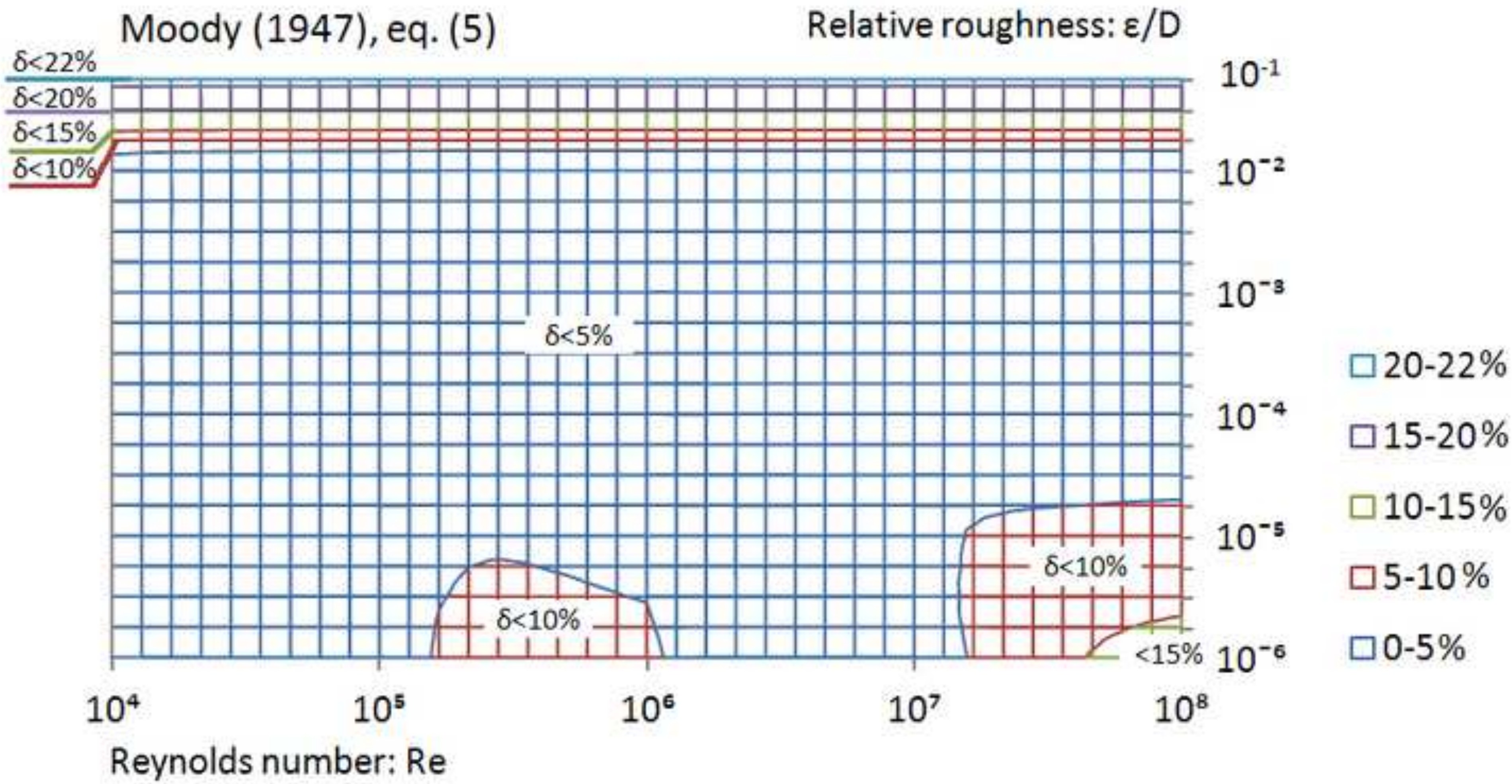


Figure 5 DB
Click here to download high resolution image

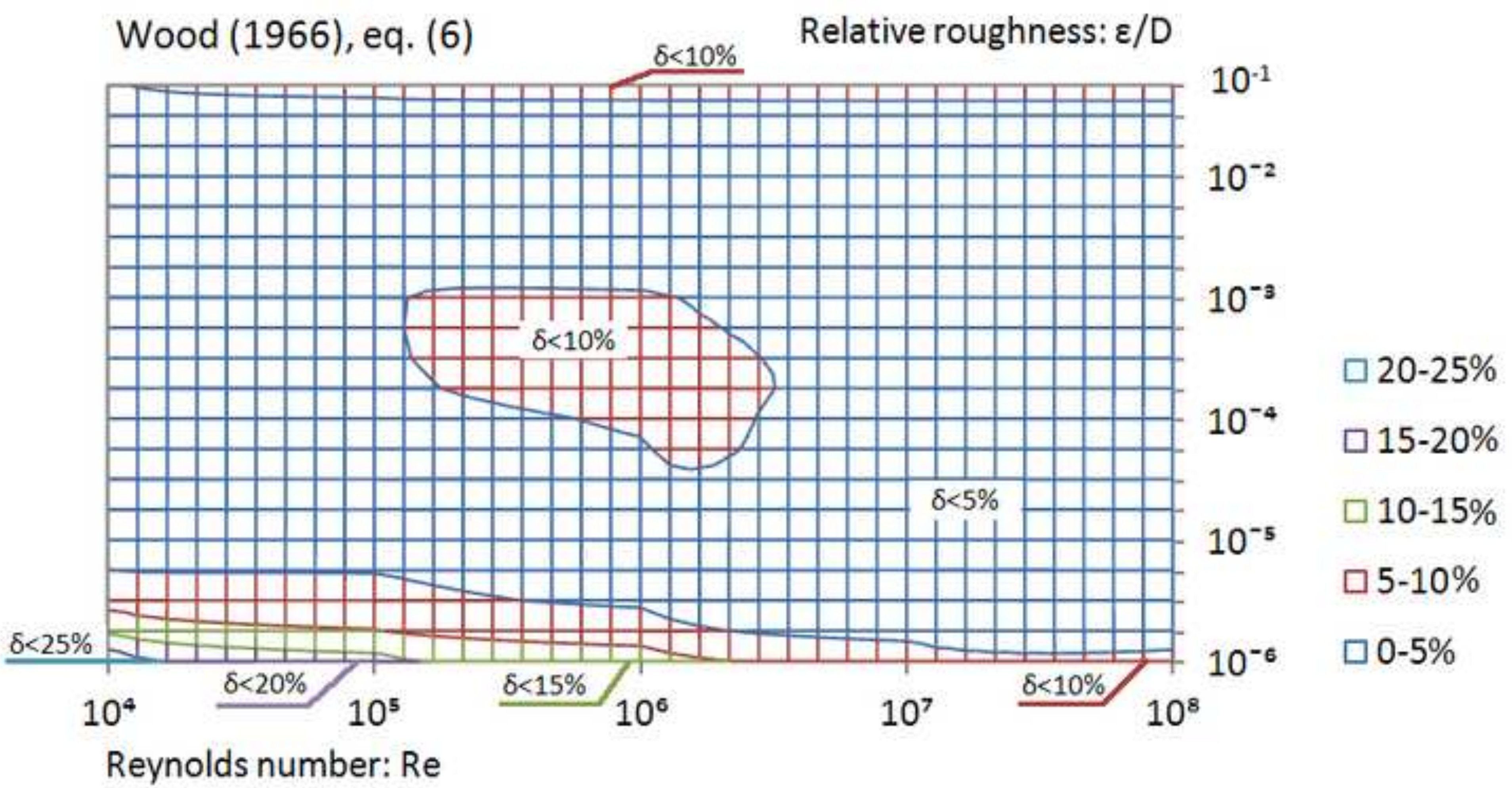


Figure 6 DB

Click here to download high resolution image

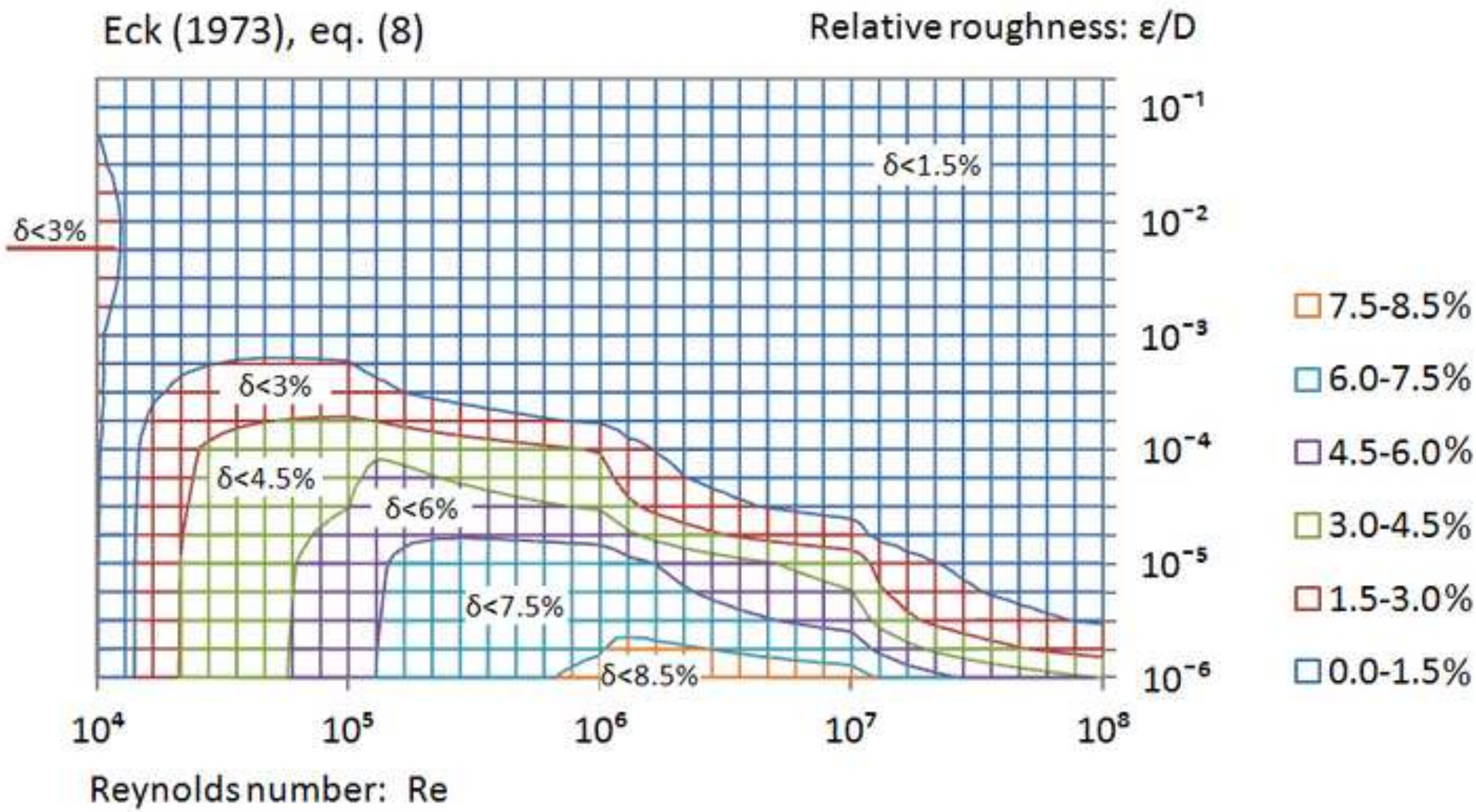


Figure 7 DB
Click here to download high resolution image

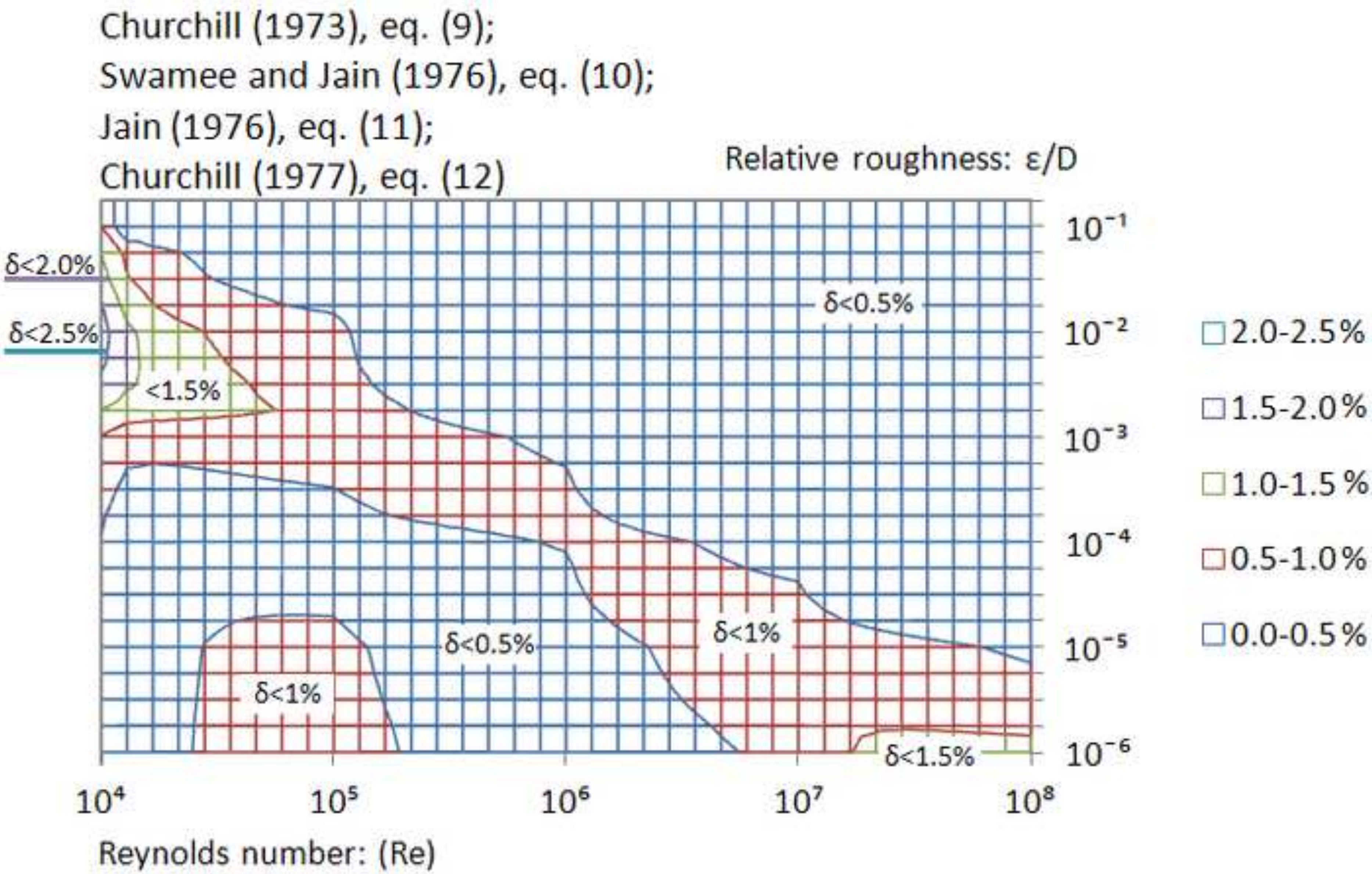


Figure 8 DB
Click here to download high resolution image

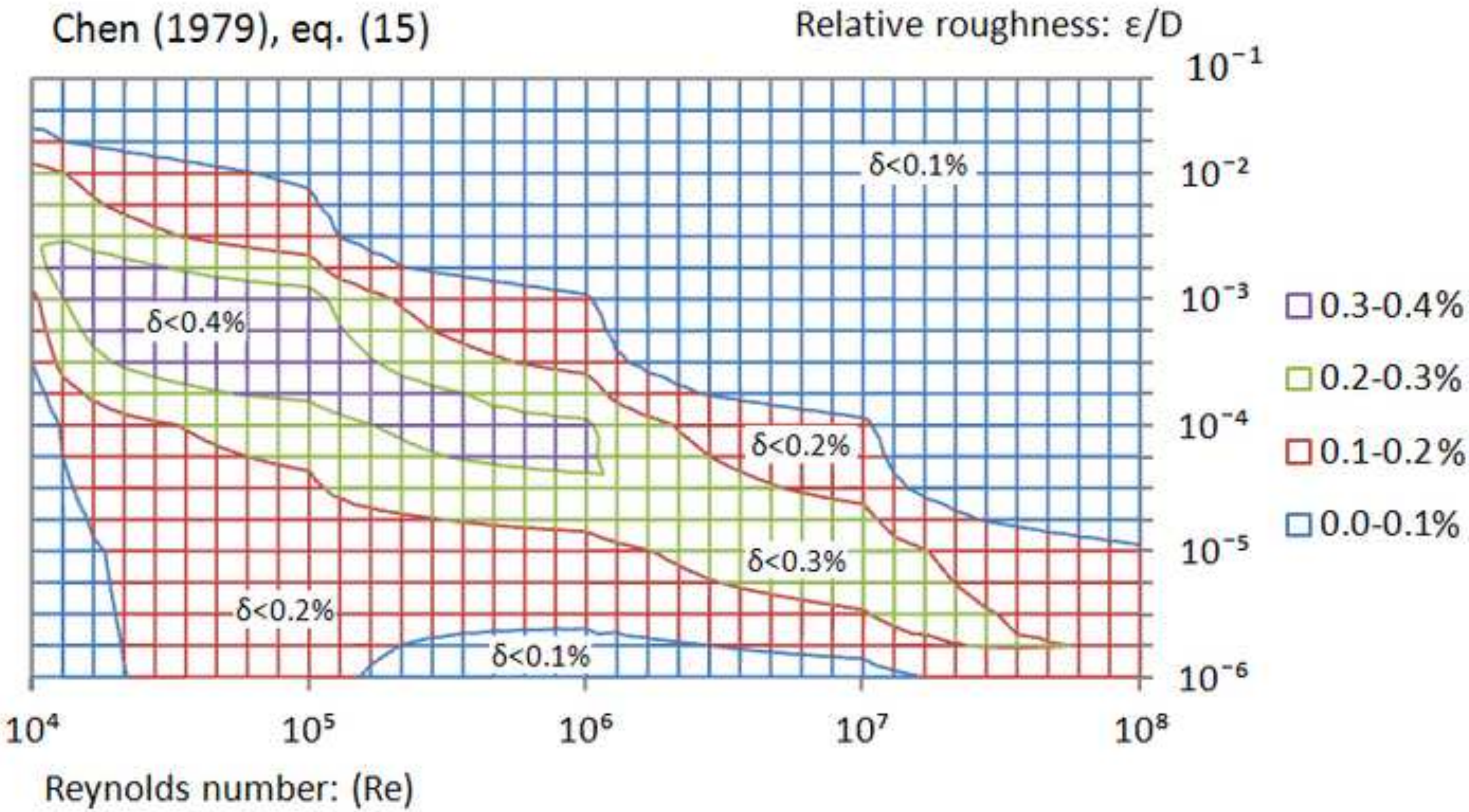


Figure 9 DB
Click here to download high resolution image

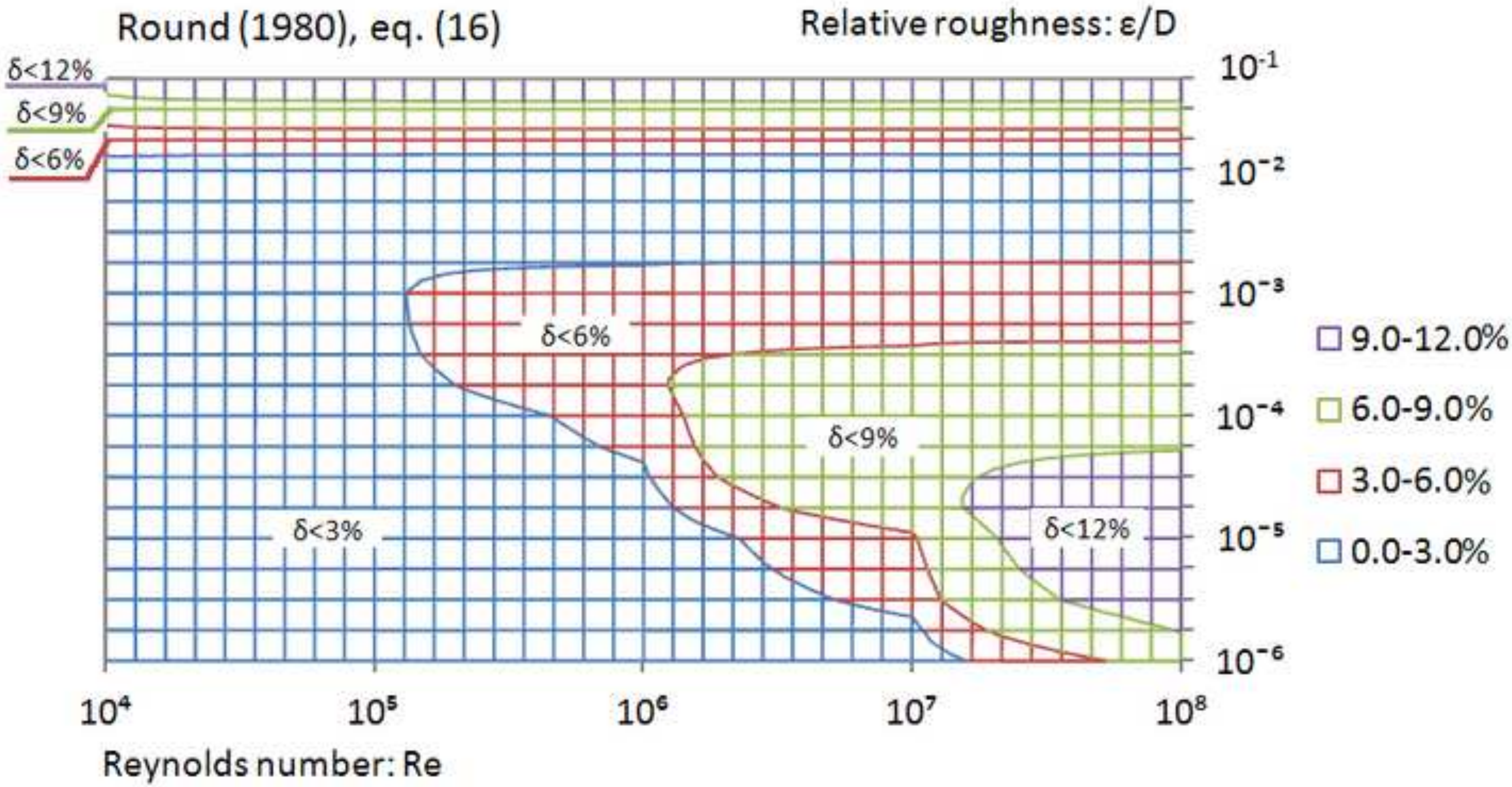


Figure 10 DB

Click here to download high resolution image

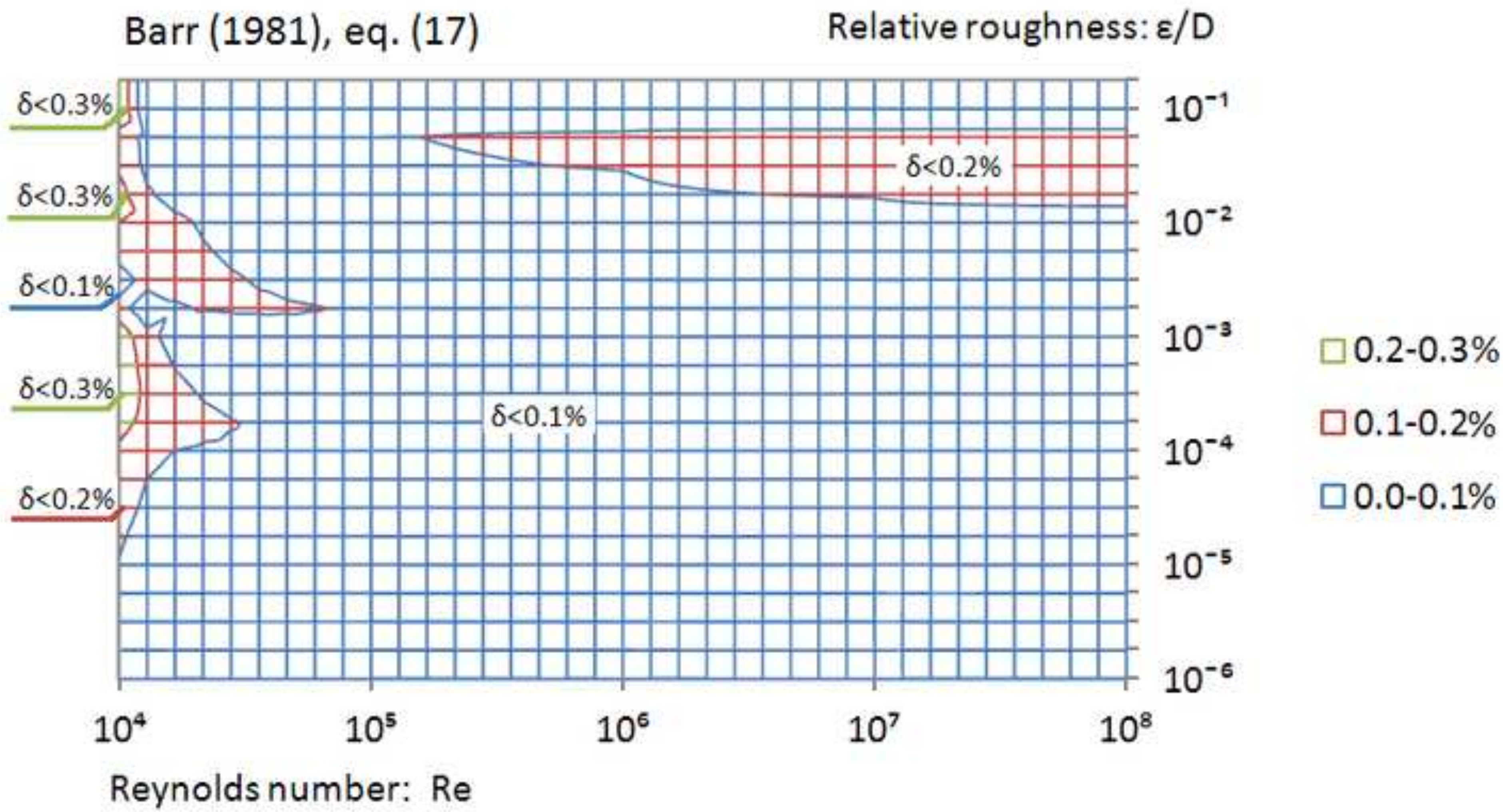


Figure 11 DB
Click here to download high resolution image

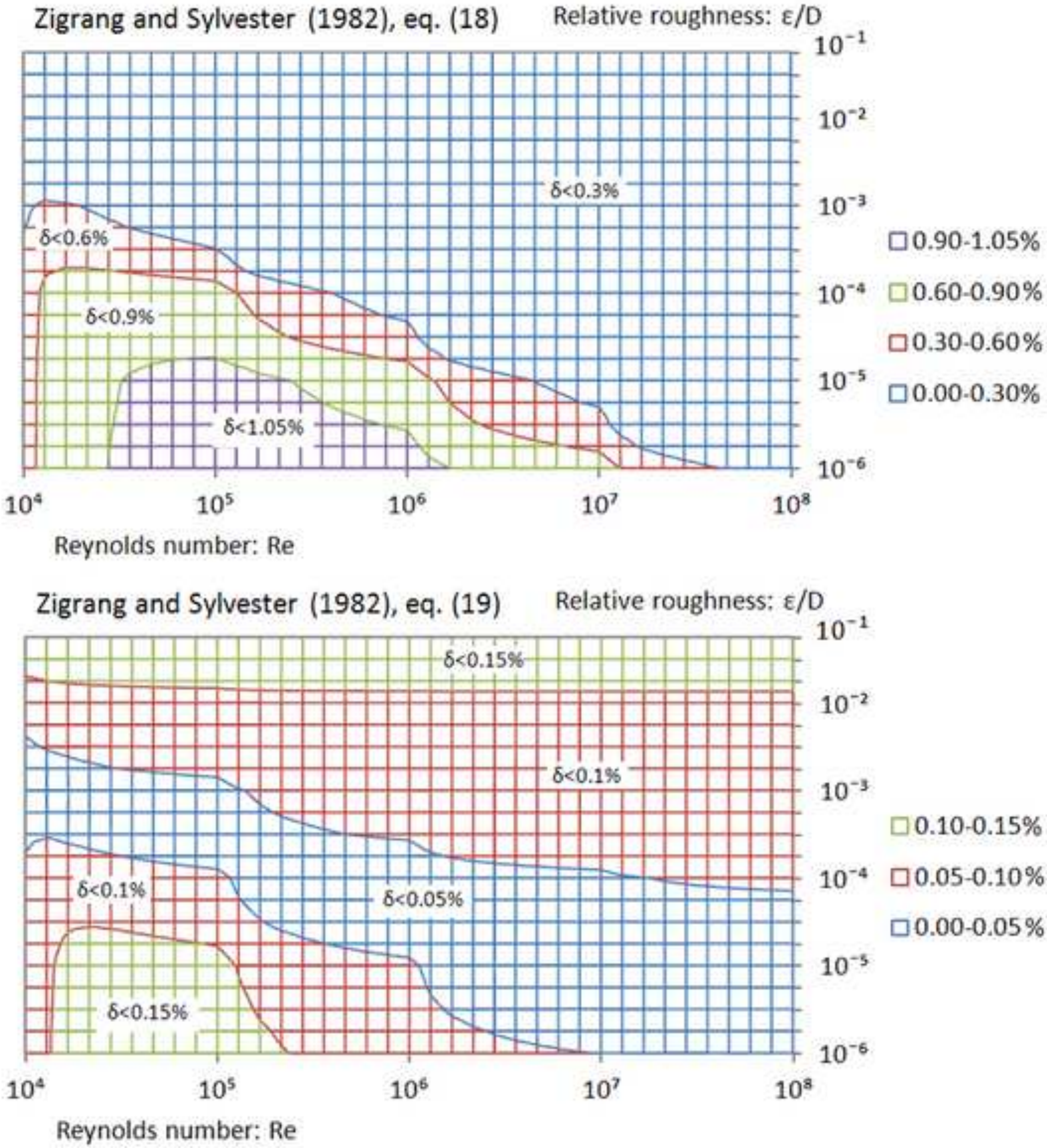


Figure 12 DB
Click here to download high resolution image

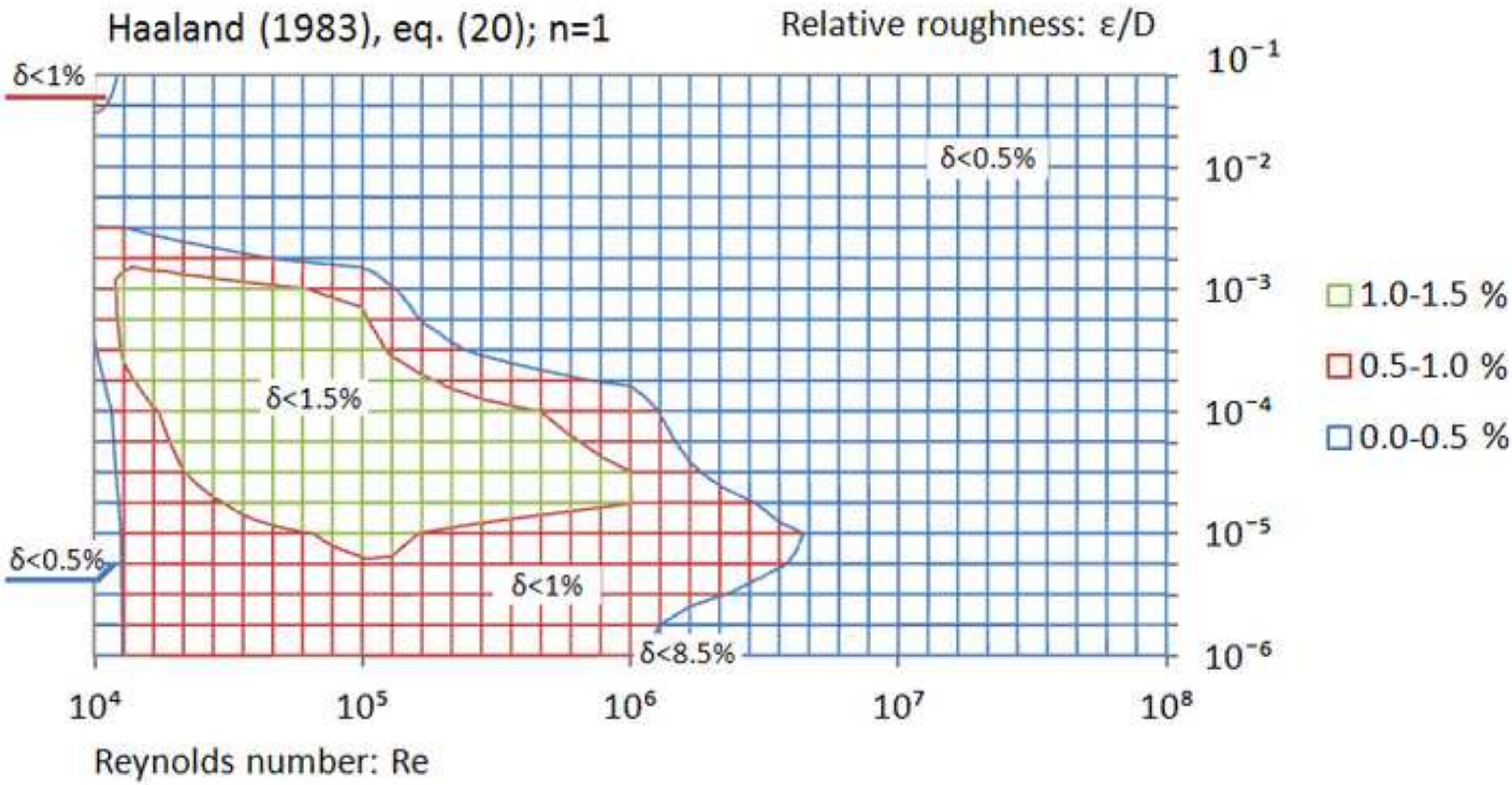
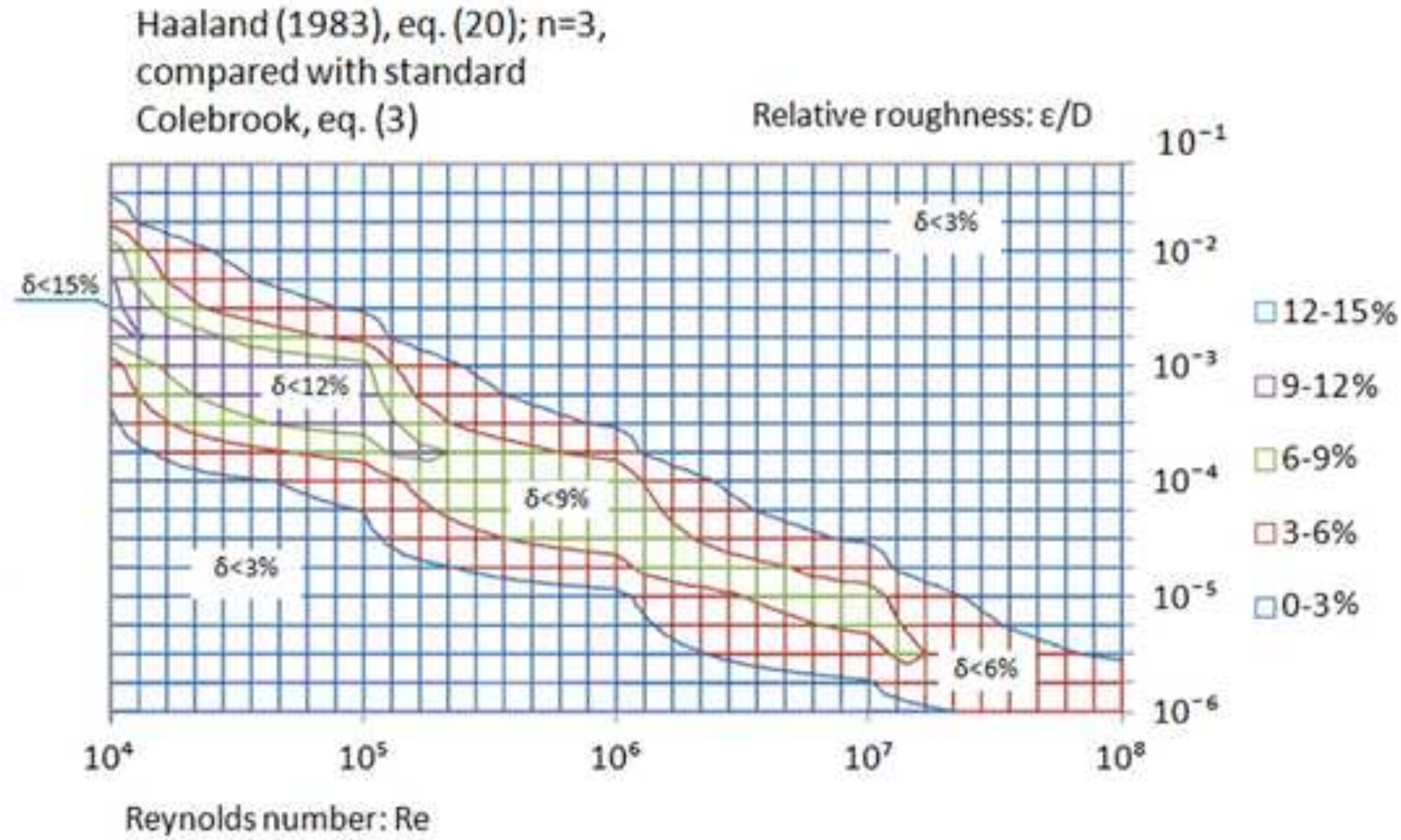


Figure 13 DB
Click here to download high resolution image



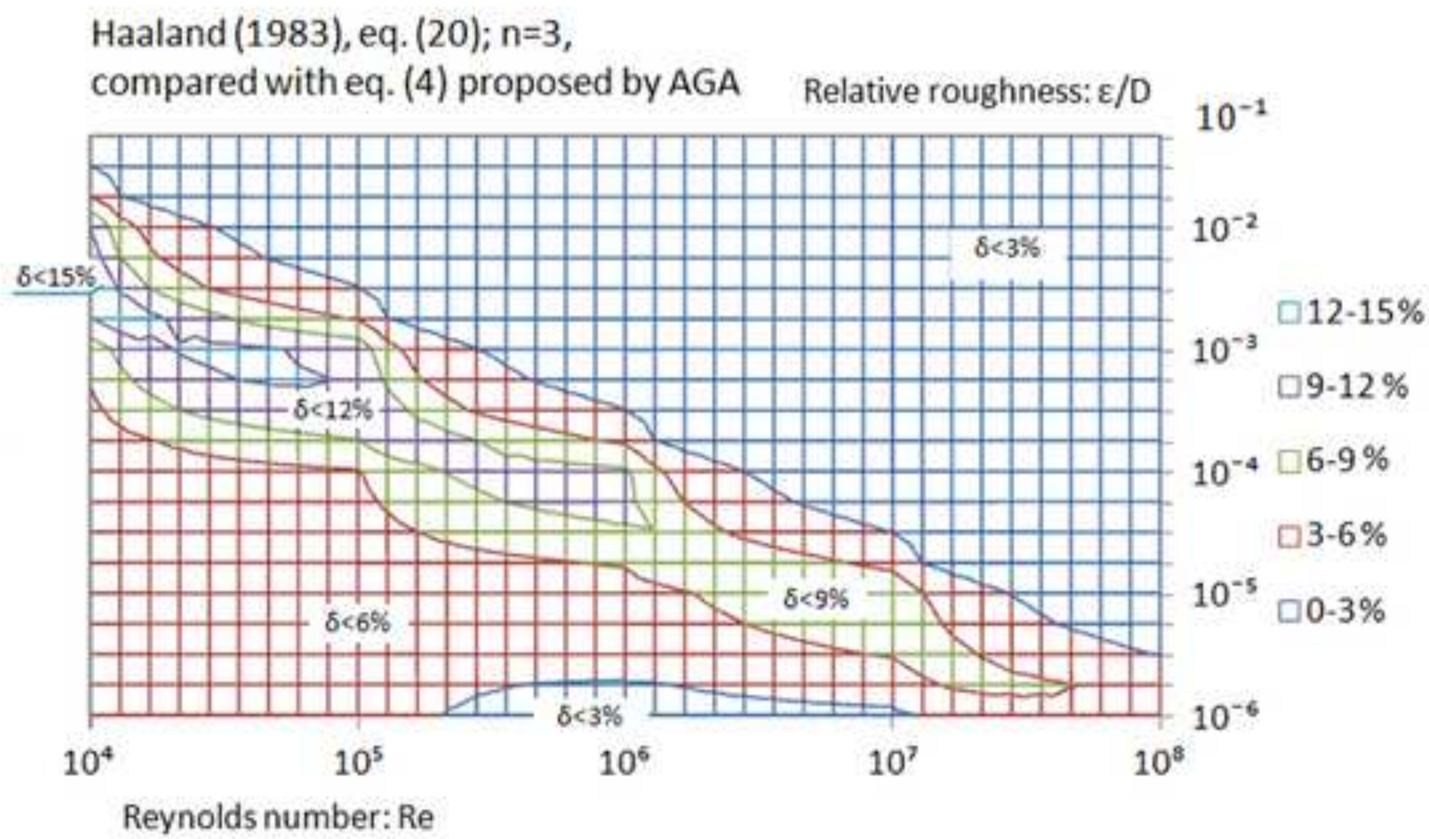


Figure 14 DB
Click here to download high resolution image

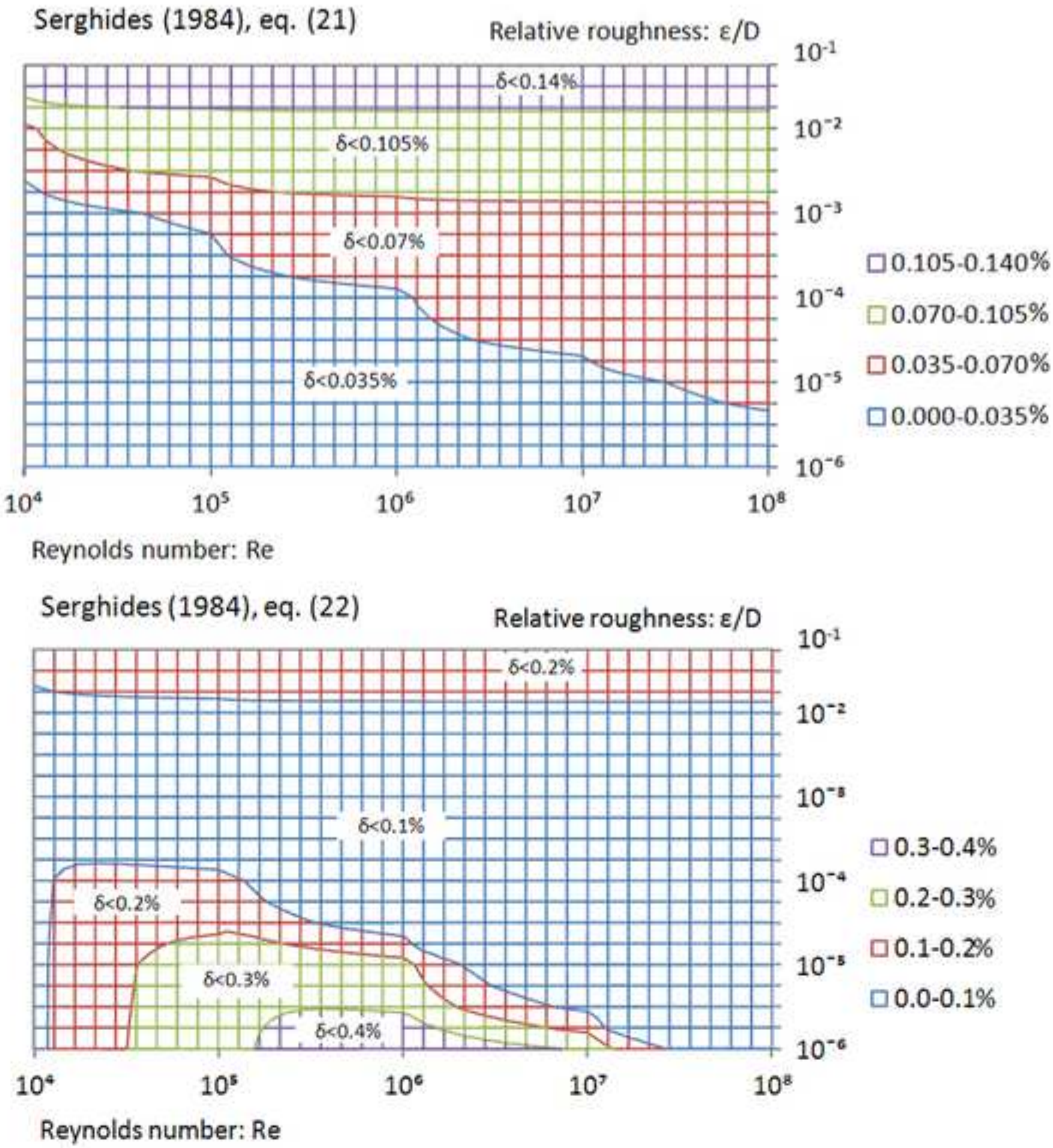


Figure 15 DB
Click here to download high resolution image

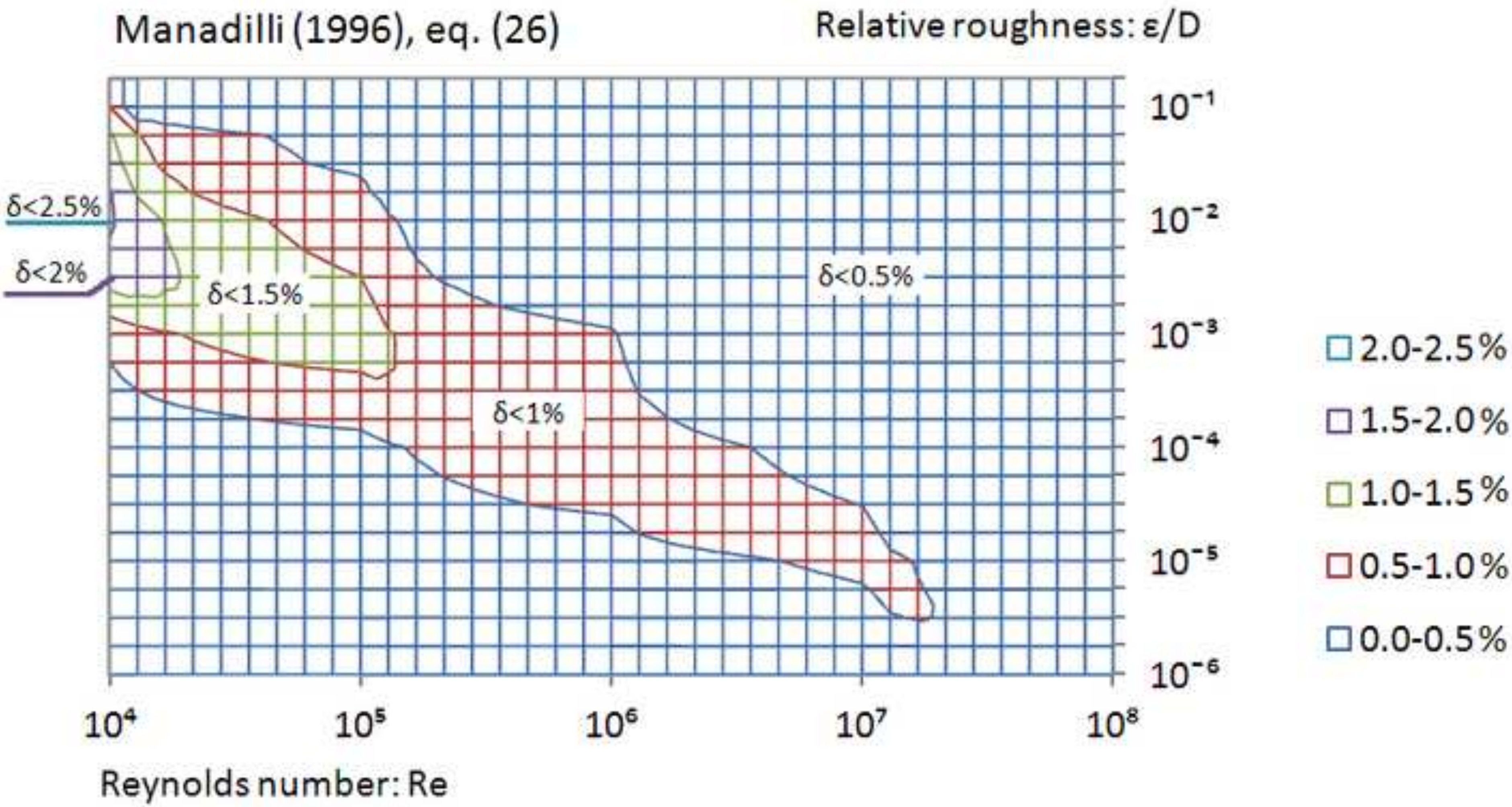


Figure 16 DB

Click here to download high resolution image

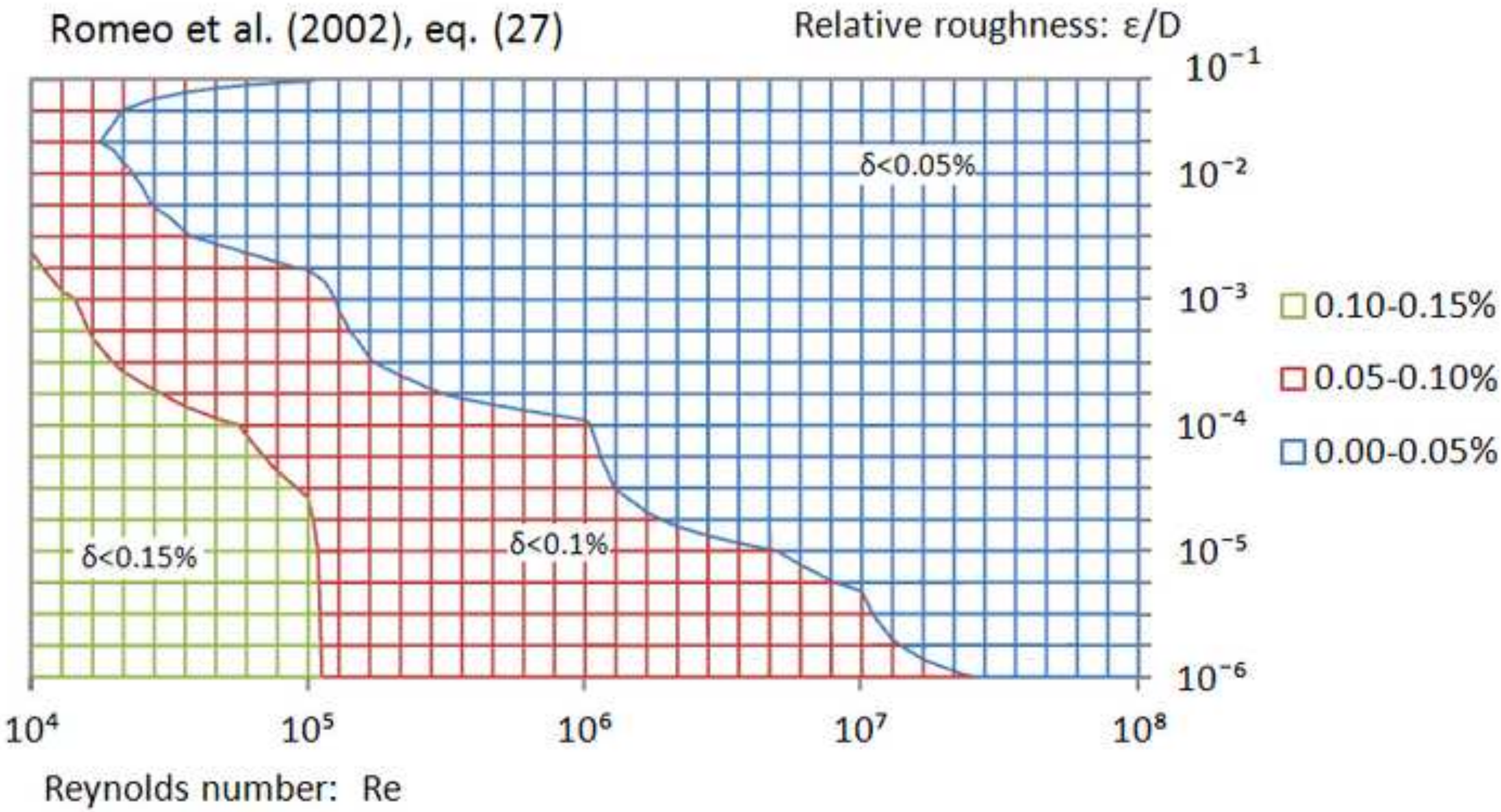


Figure 17 DB
Click here to download high resolution image

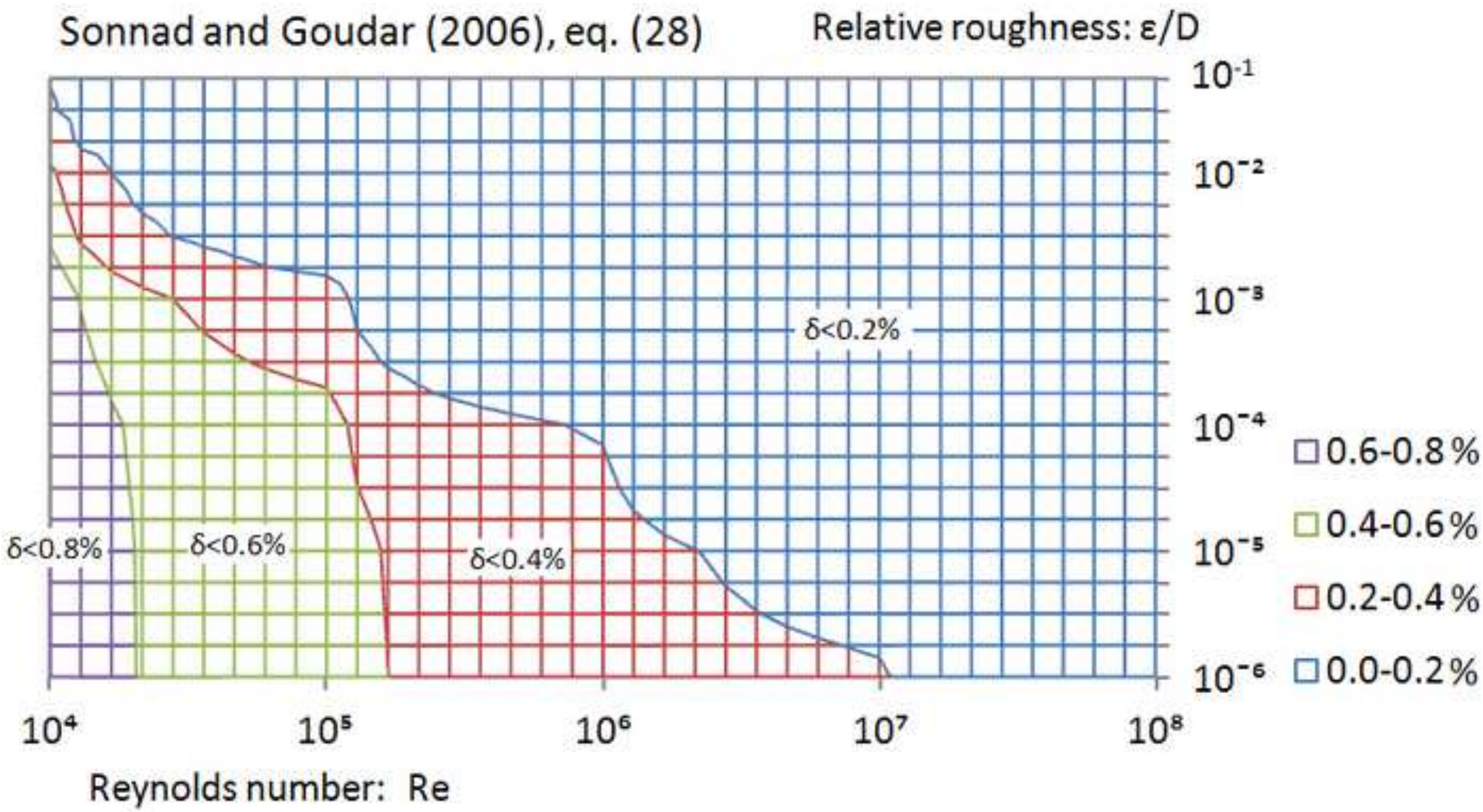


Figure 18 DB
Click here to download high resolution image

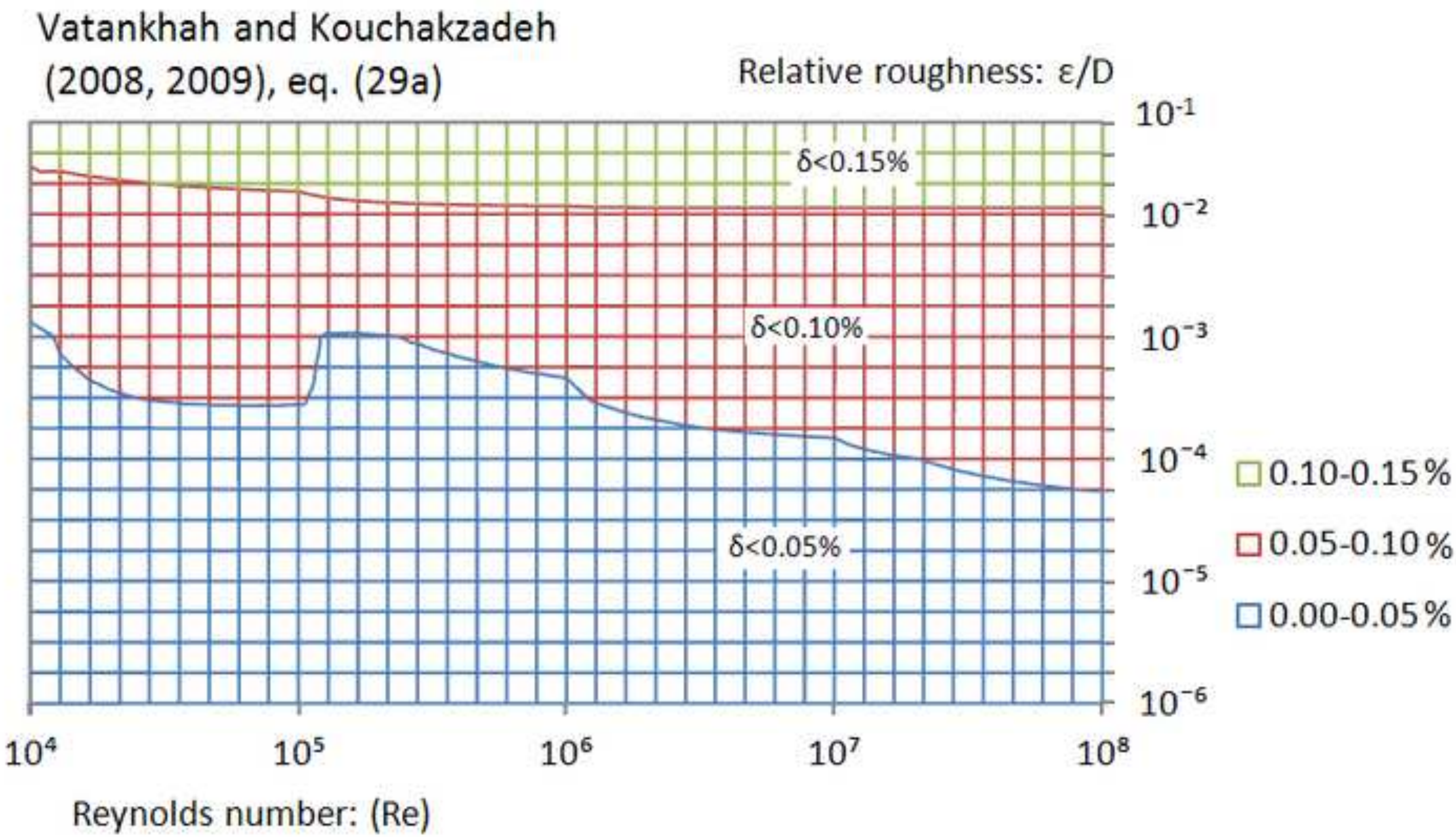


Figure 19 DB
Click here to download high resolution image

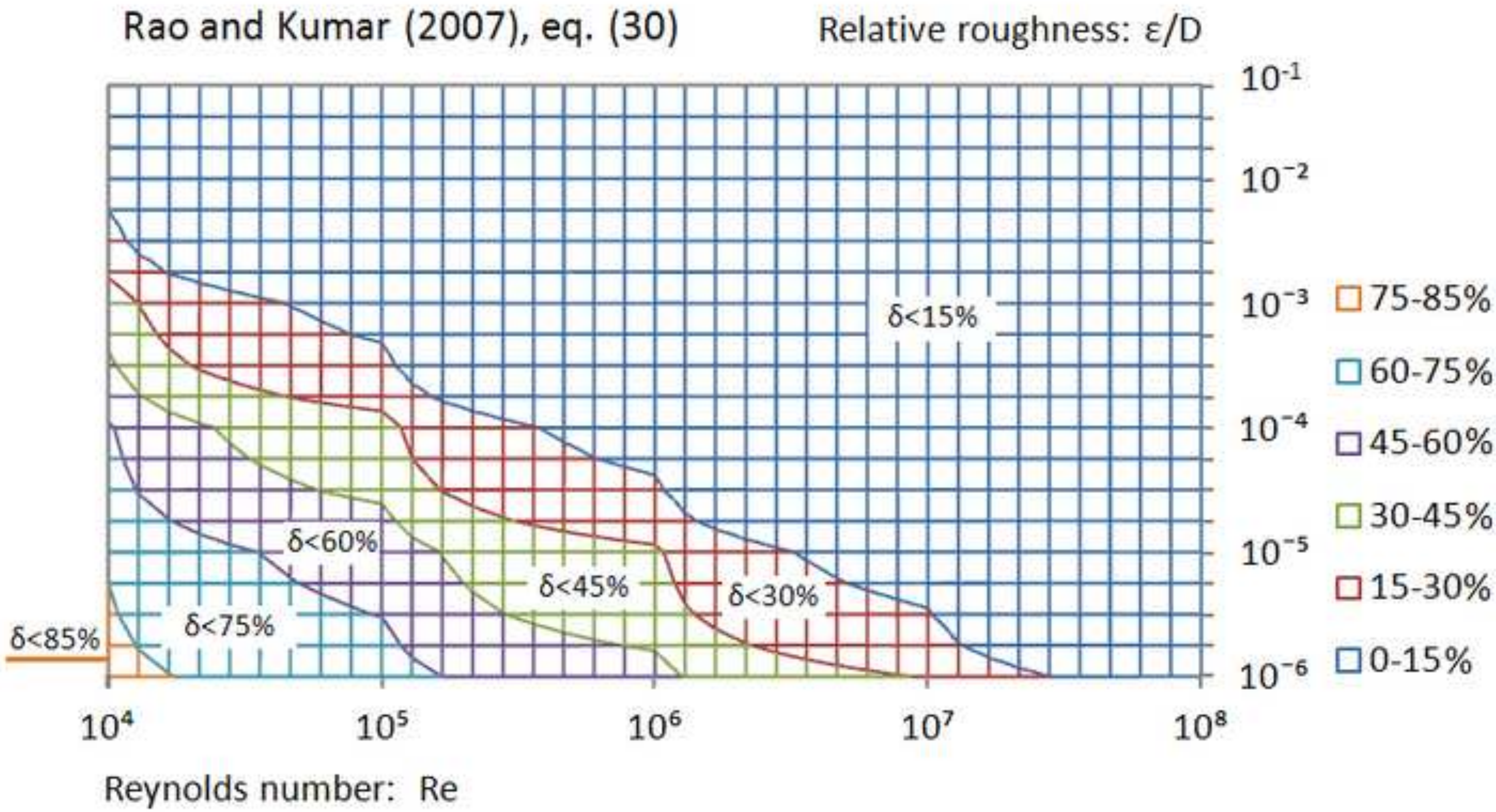


Figure 20 DB

Click here to download high resolution image

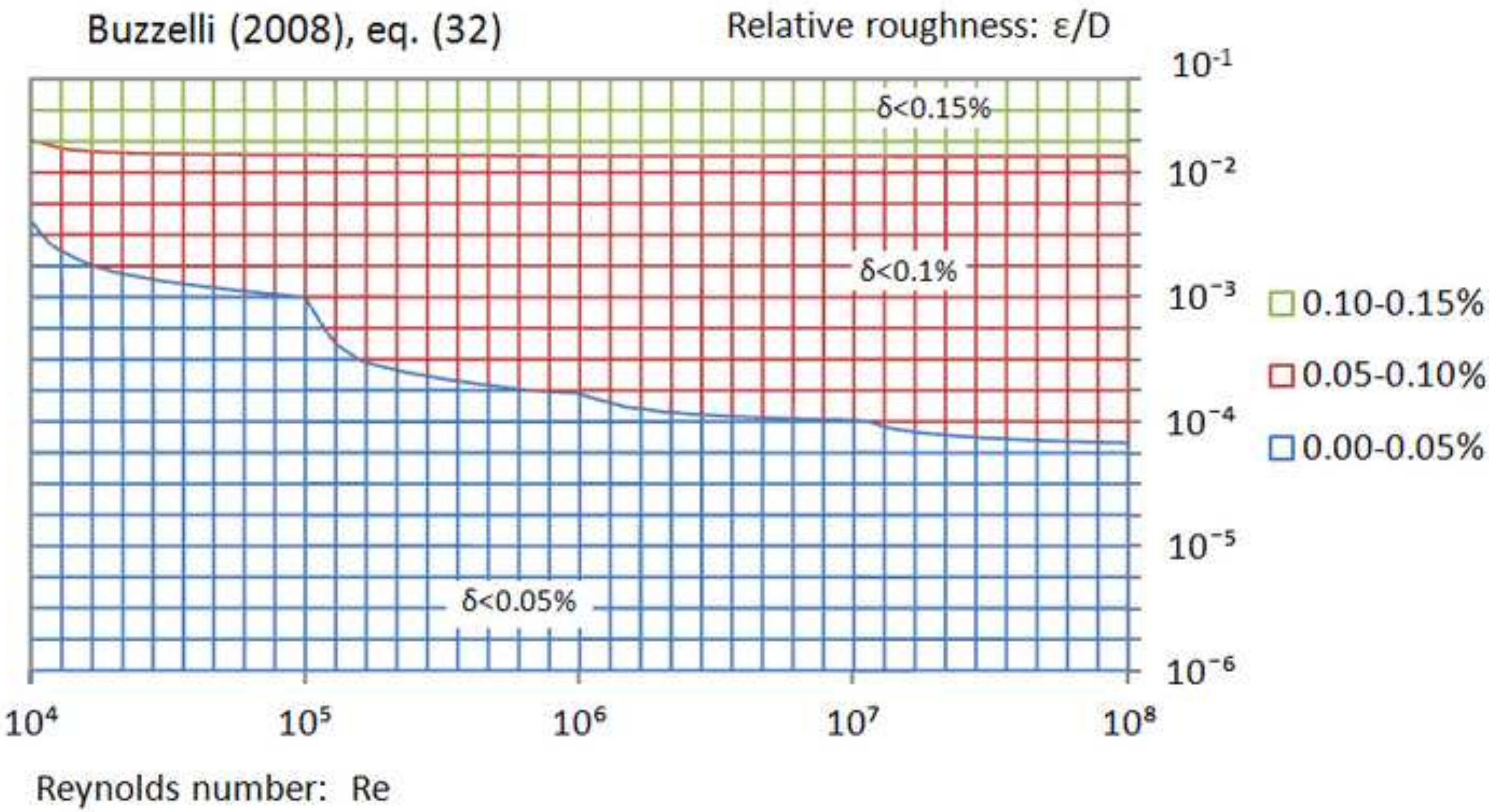


Figure 21 DB
Click here to download high resolution image

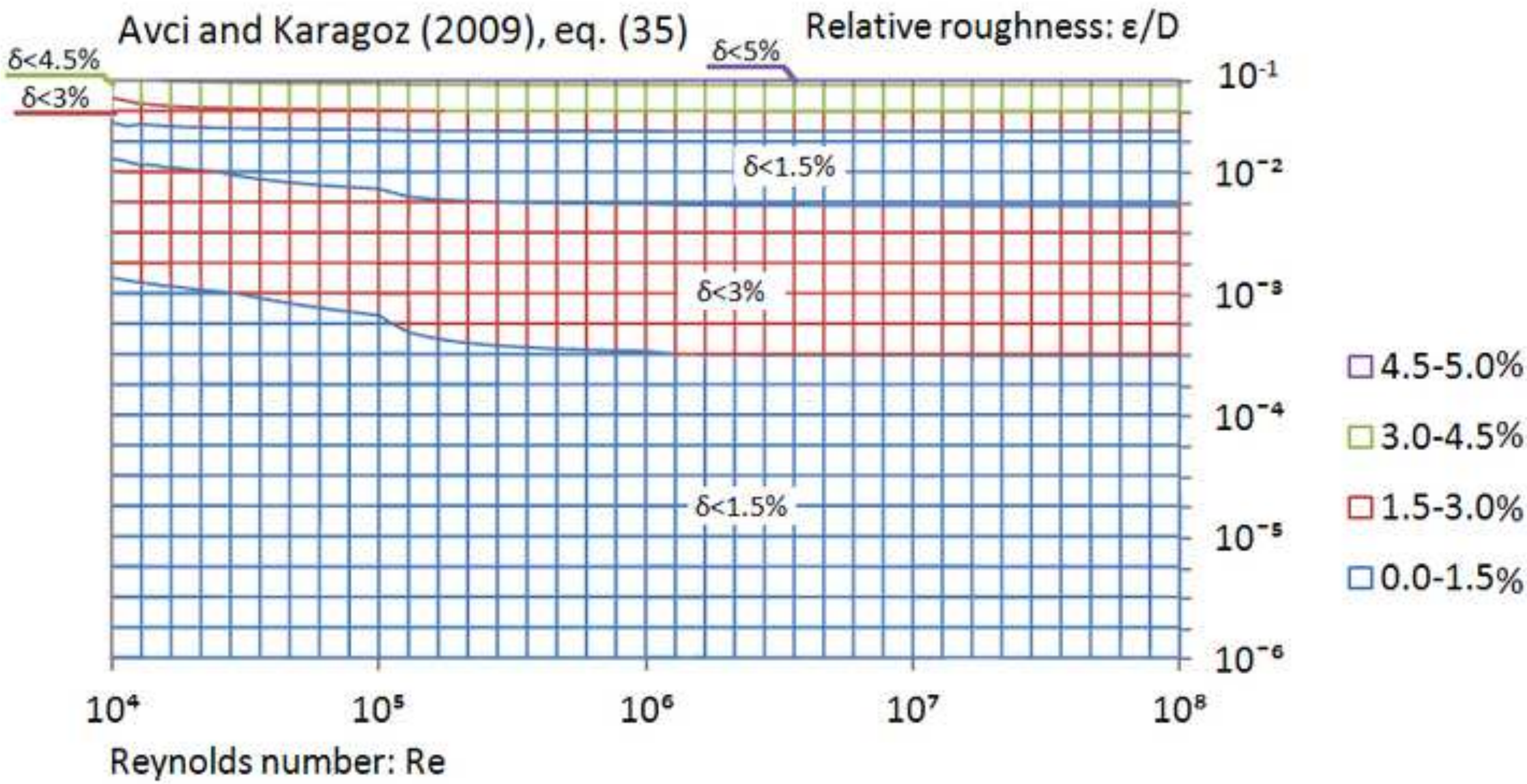


Figure 22 DB
Click here to download high resolution image

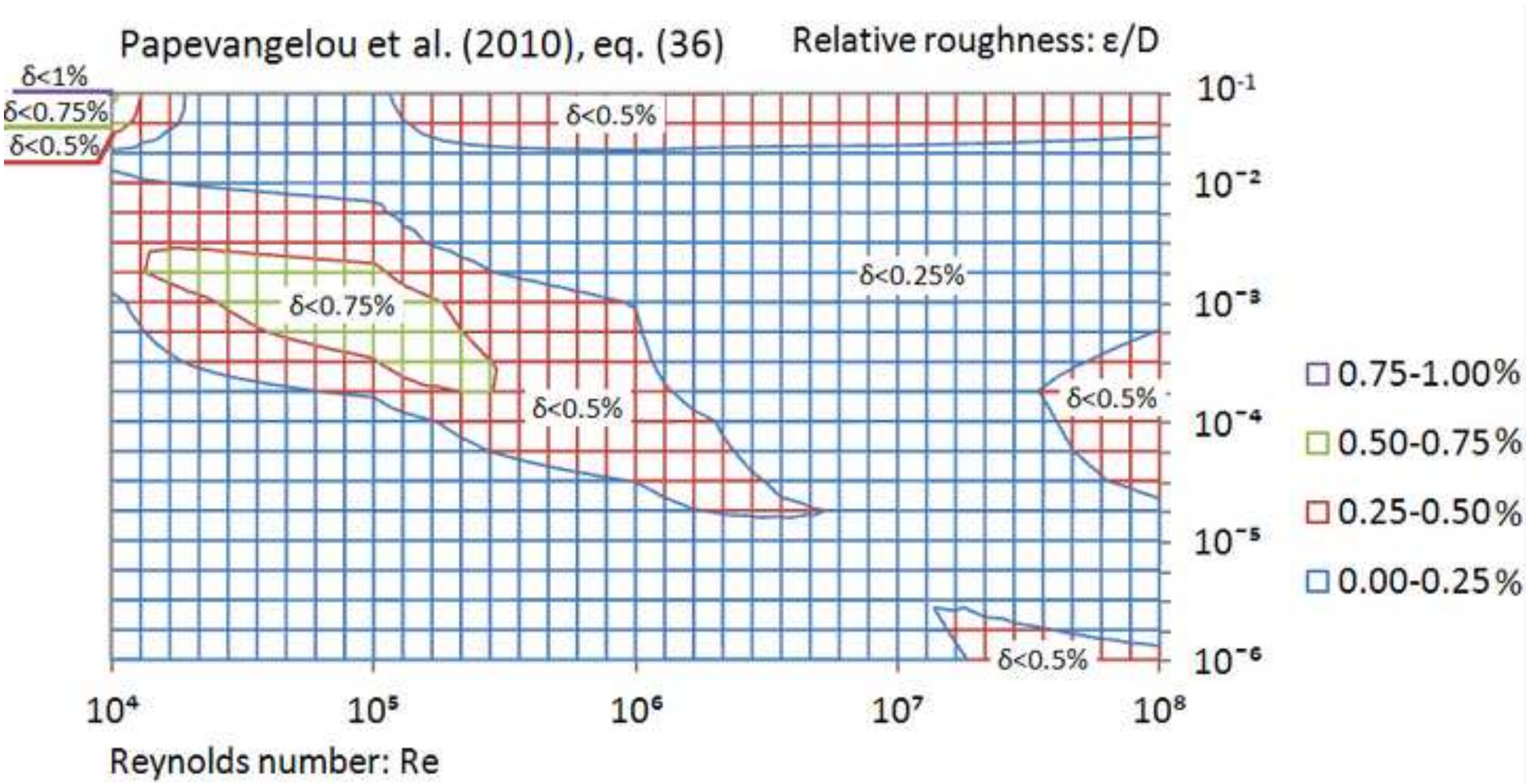


Figure 23 DB
Click here to download high resolution image

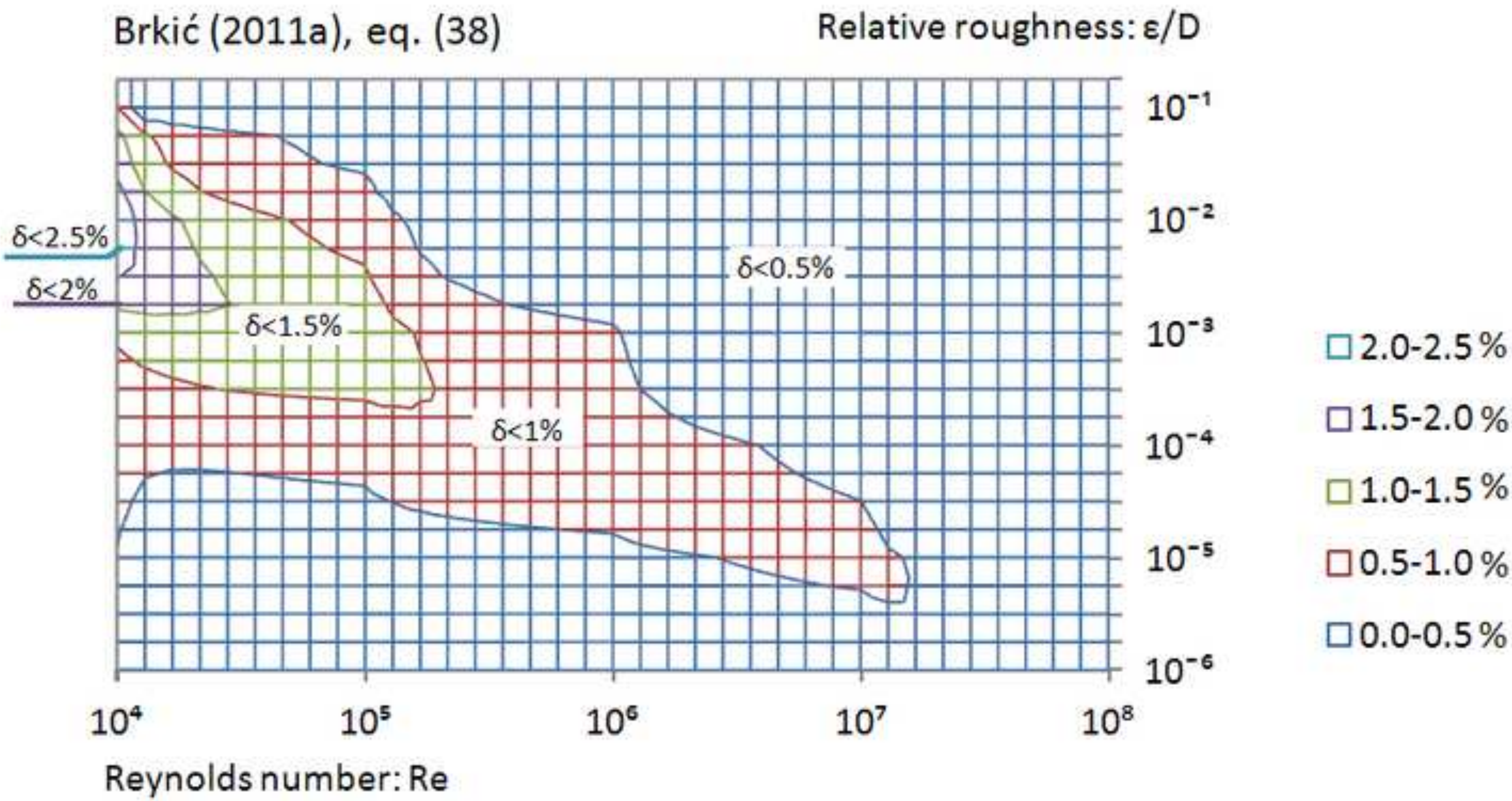


Figure 24 DB

Click here to download high resolution image

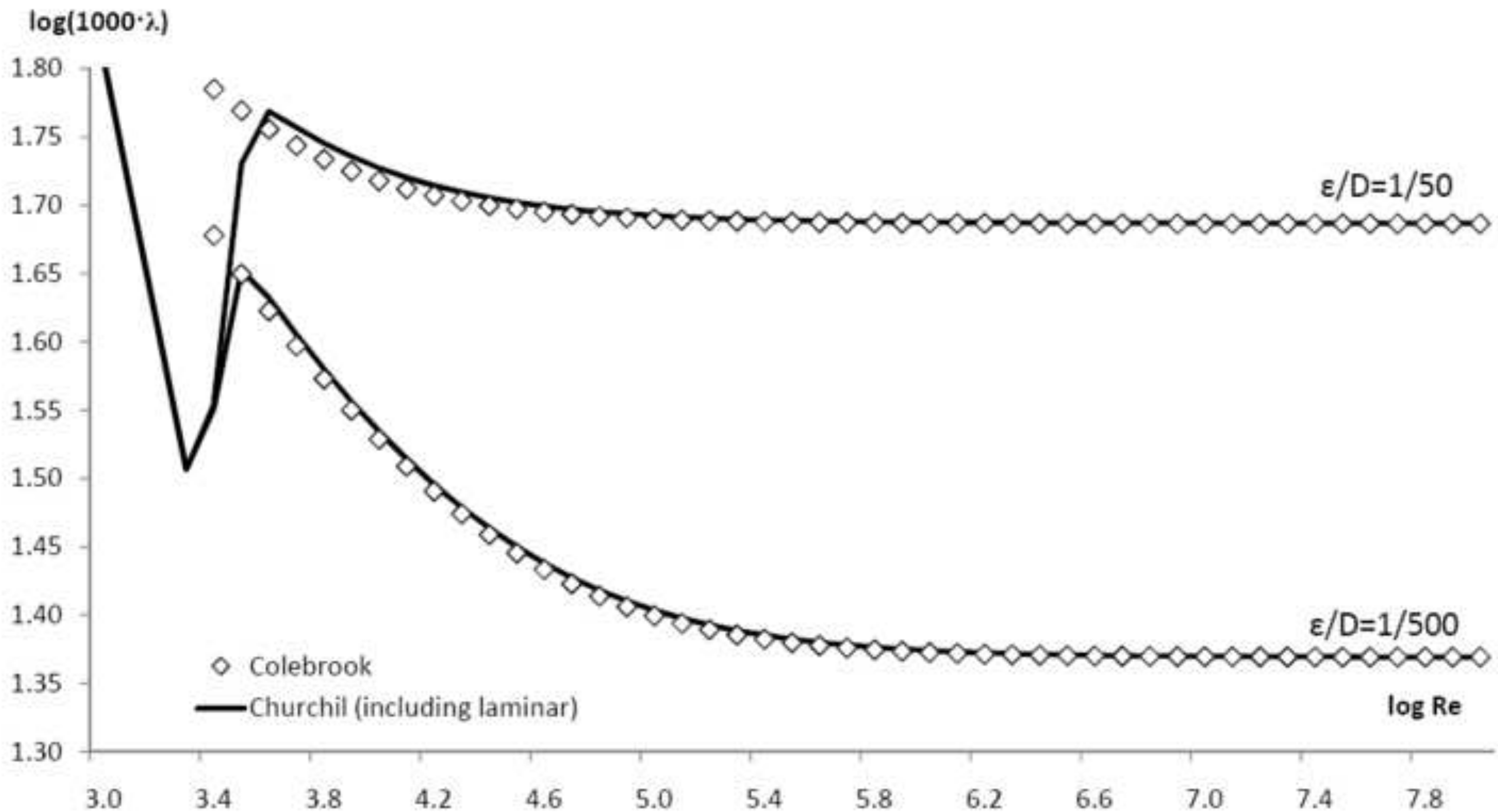


Figure 25 DB
Click here to download high resolution image

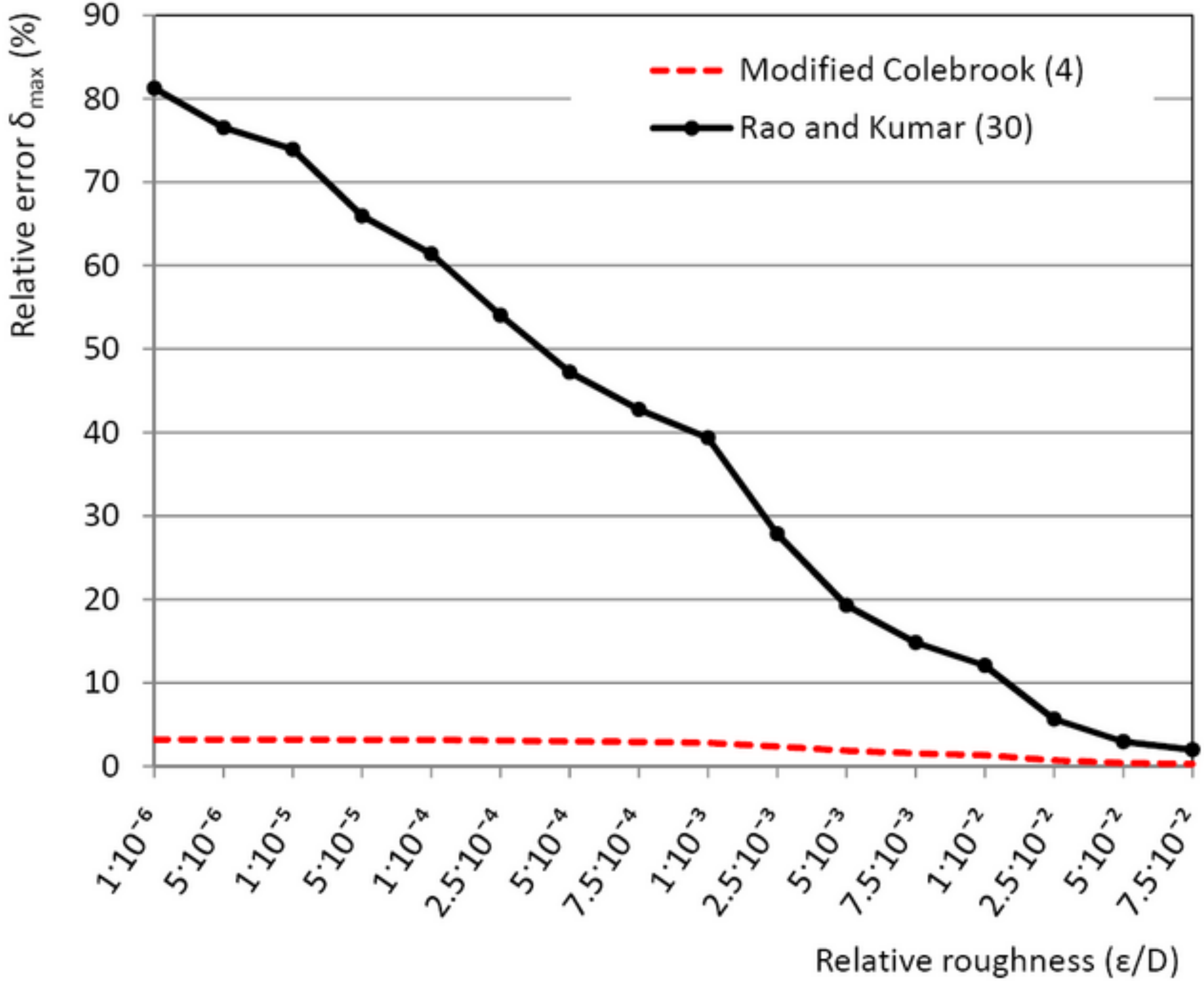


Figure 26 DB
Click here to download high resolution image

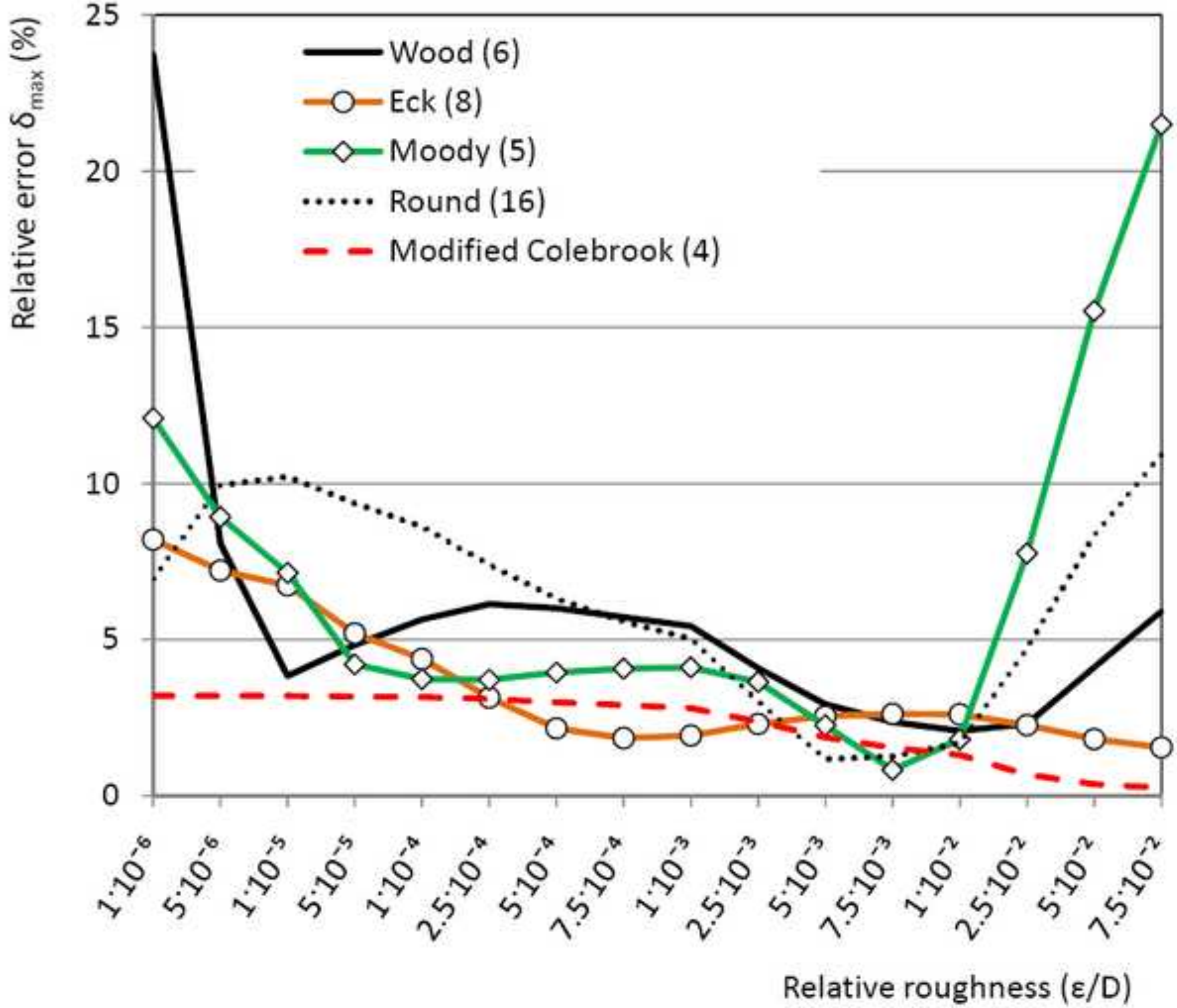


Figure 27 DB
Click here to download high resolution image

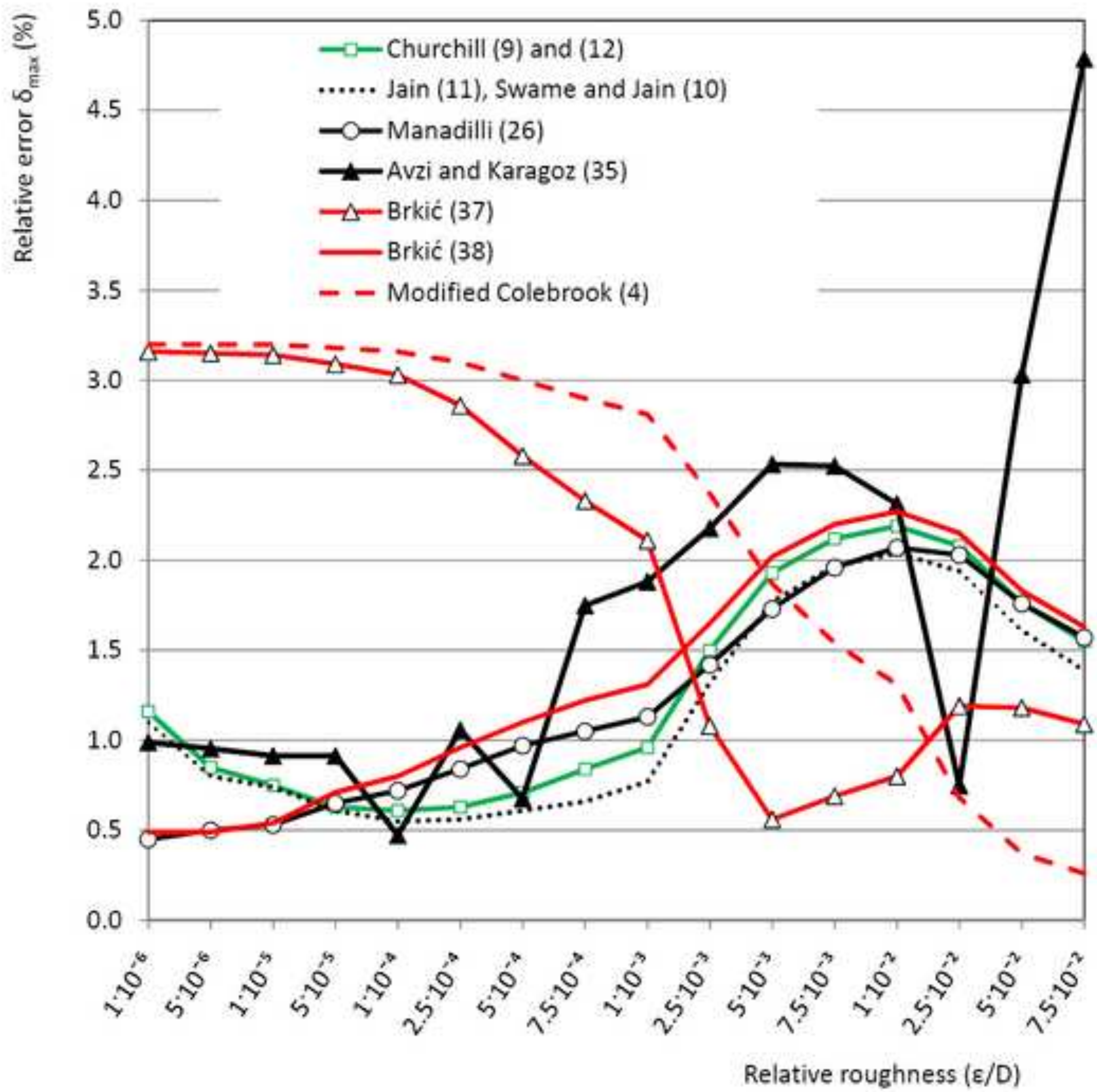


Figure 28 DB
Click here to download high resolution image

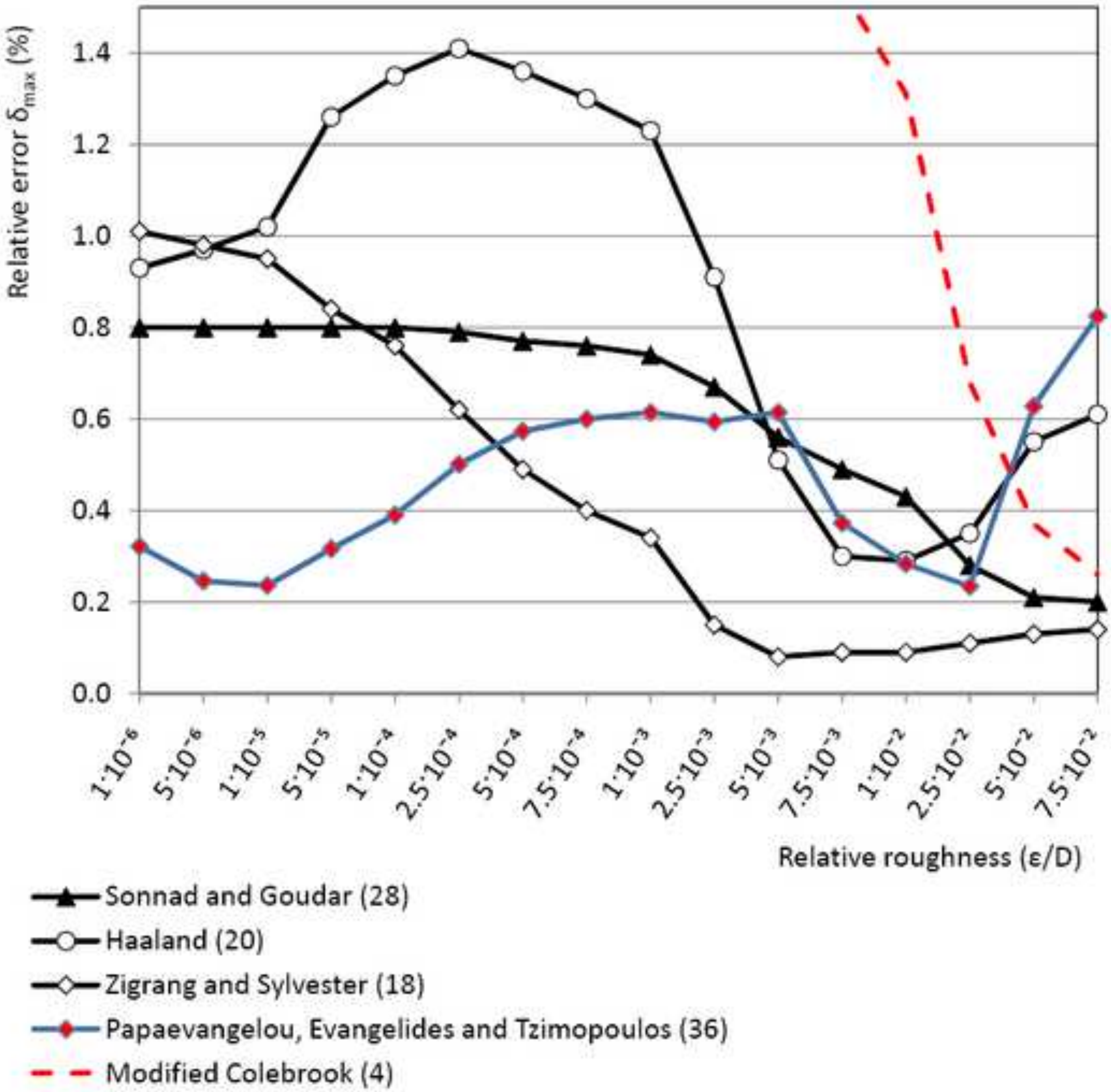


Figure 29 DB
Click here to download high resolution image

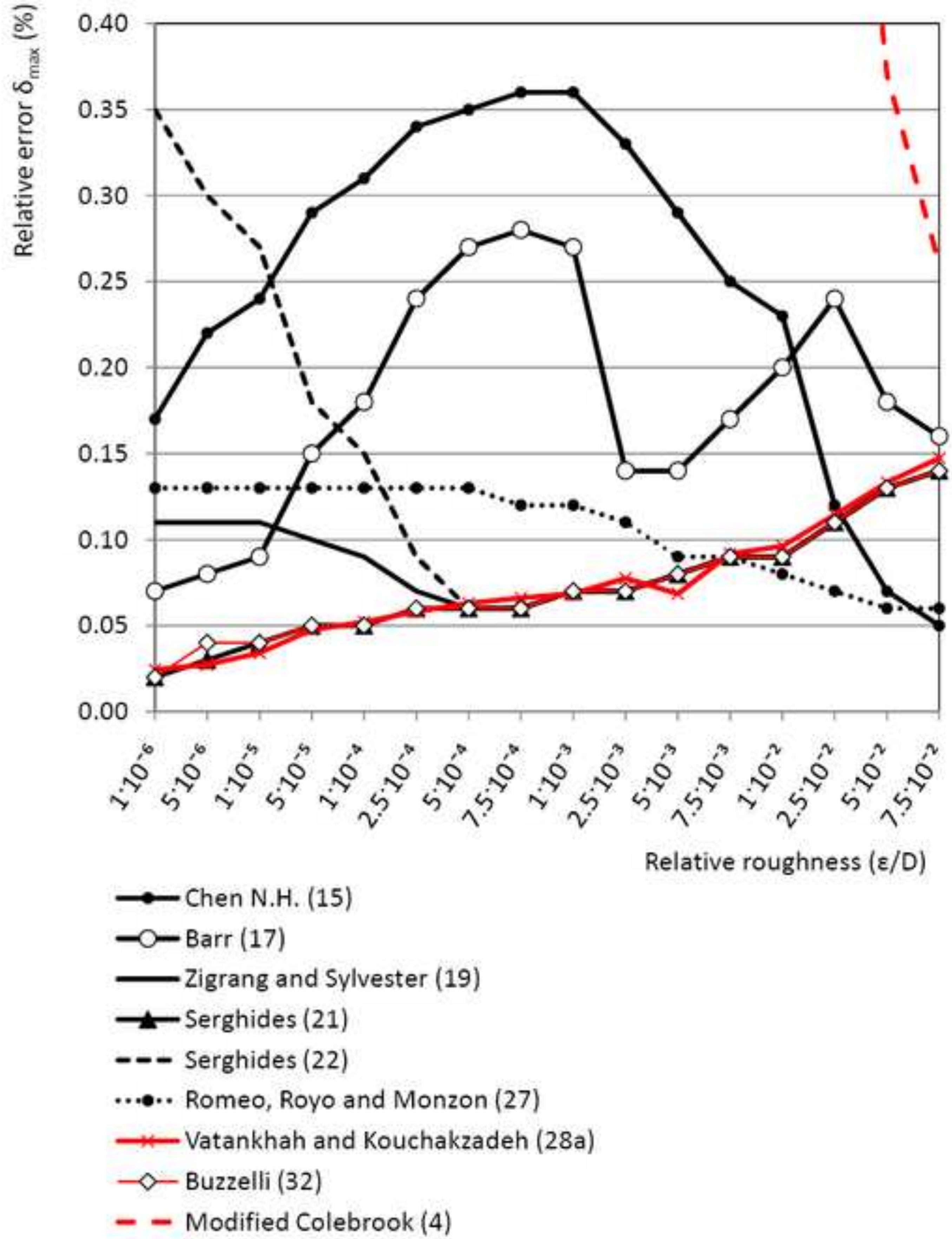


Figure 30 DB

Click here to download high resolution image

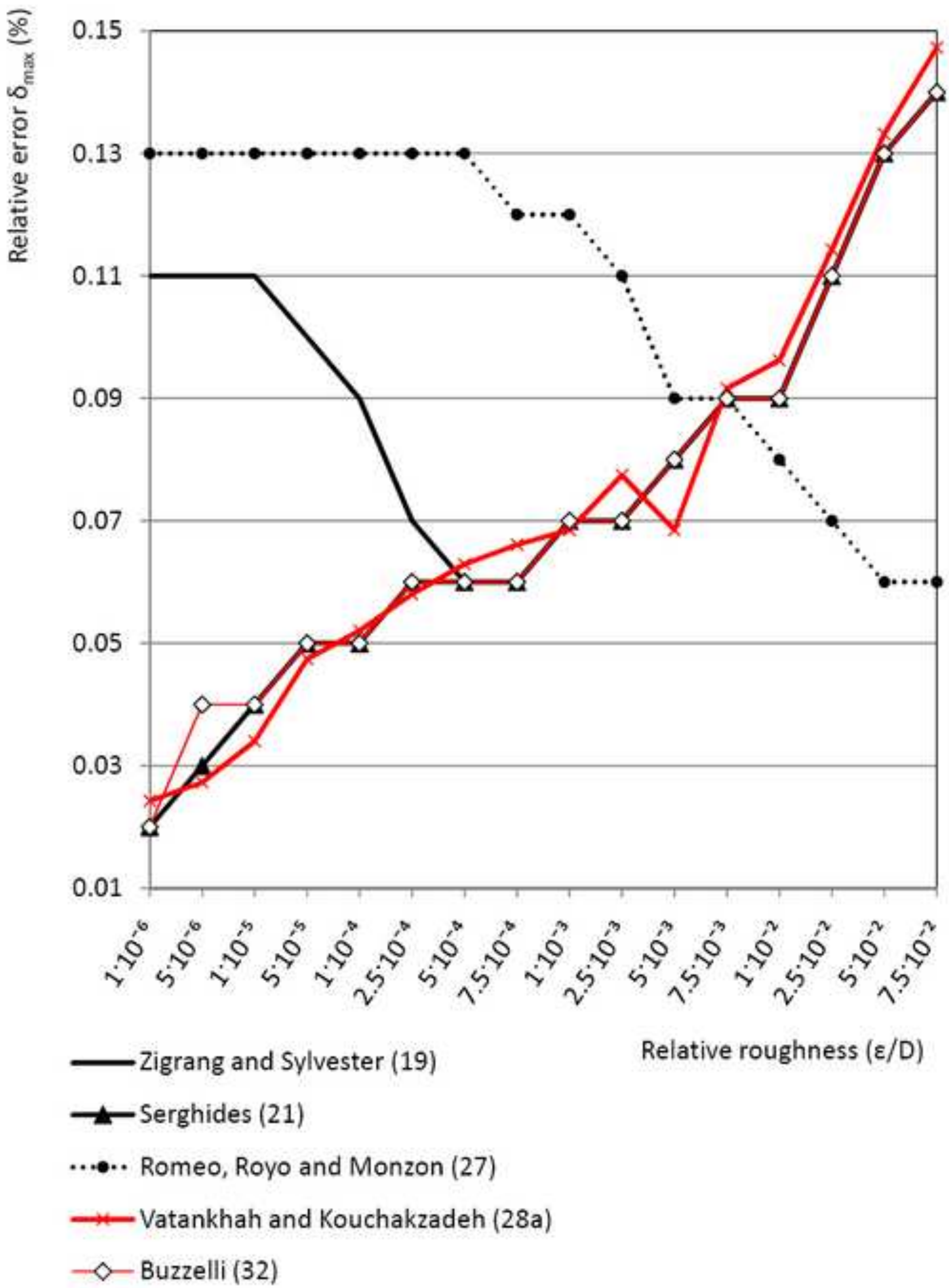


Figure 31 DB
Click here to download high resolution image

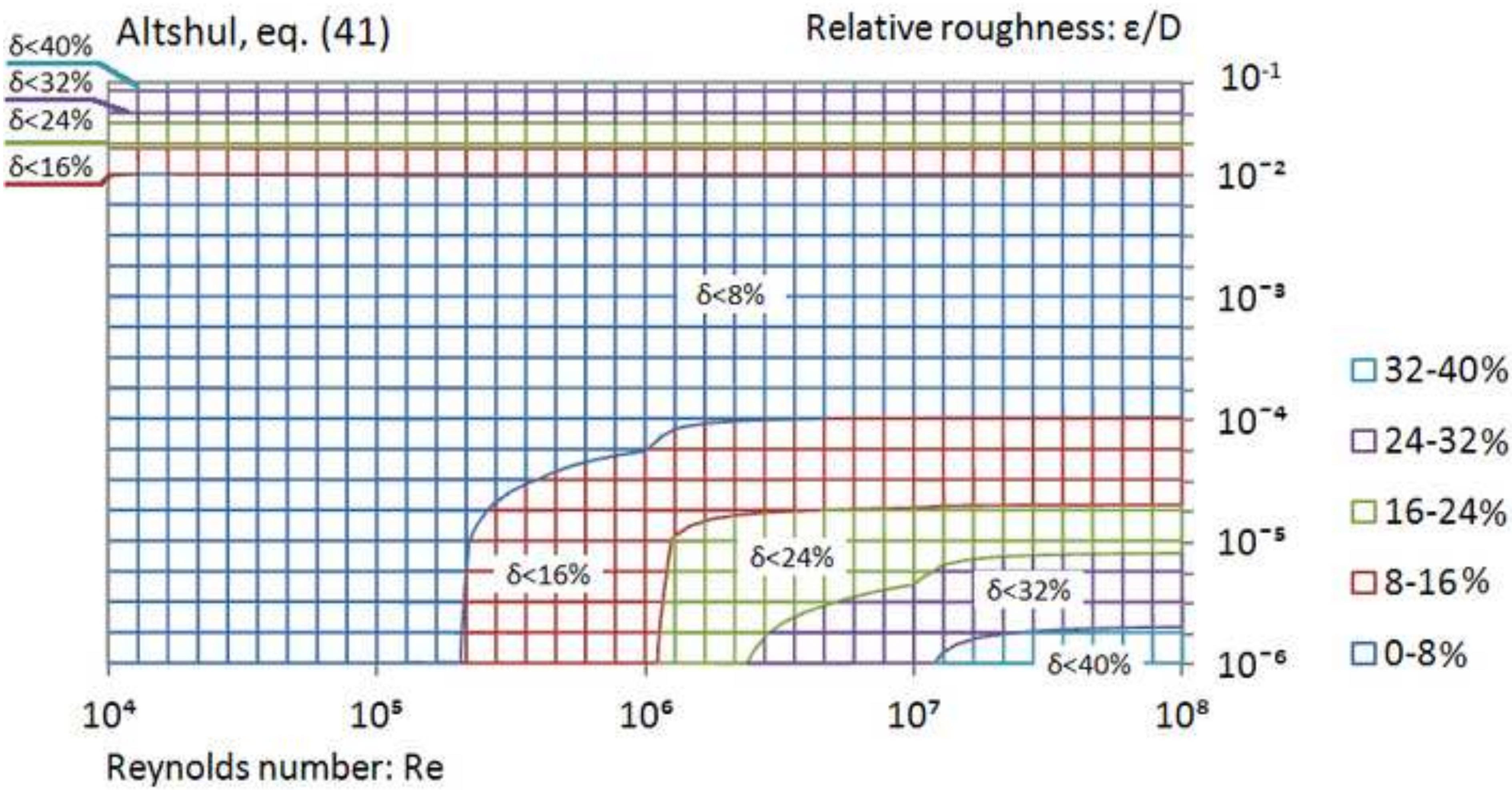


Figure 32 DB

Click here to download high resolution image

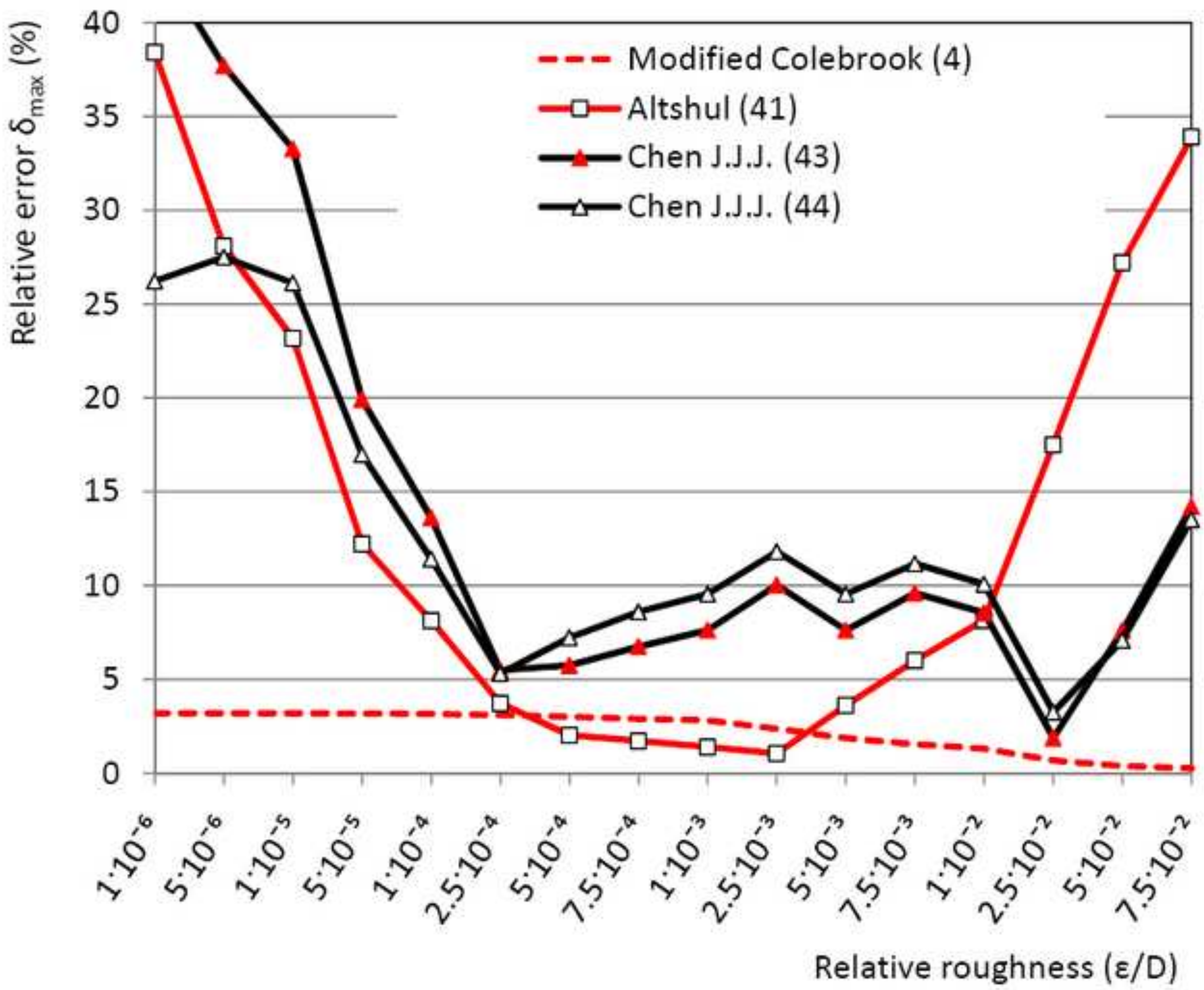


Table 1 DB
Click here to download Table: Table 1 DB.doc

Table 1. Maximal relative error for available approximations for test check points

	δ_{max} (%)	Relative roughness (ε/D)	Reynolds number (Re)
Romeo, Royo and Monzon (27)	-0.1345	$1\cdot10^{-6}$	$1\cdot10^{4}$
Buzzelli (32)	-0.1385	$7.5\cdot10^{-2}$	$1\cdot10^{8}$
Serghides (21)	-0.1385	$7.5\cdot10^{-2}$	$1\cdot10^{8}$
Zigrang and Sylvester (19)	-0.1385	$7.5\cdot10^{-2}$	$1\cdot10^{8}$
Vatankhah and Kouchakzadeh (29a)	0.1472	$7.5\cdot10^{-2}$	
Barr (17)	+0.2775	$7.5\cdot10^{-4}$	$1\cdot10^{4}$
Serghides (22)	+0.3544	$1\cdot10^{-6}$	$2\cdot10^{6}$
Chen (15)	-0.3556	$7.5\cdot10^{-4}$	$8\cdot10^{4}$
Sonnad and Goudar (28)	-0.8003	$1\cdot10^{-5}$	$1\cdot10^{4}$
Papaevangelou, Evangelides and Tzimopoulos (36)	-0.8247	$7.5\cdot10^{-2}$	$2\cdot10^{6}$
Zigrang and Sylvester (18)	-1.0074	$1\cdot10^{-6}$	$3\cdot10^{5}$
Haaland (20)	+1.4083	$2.5\cdot10^{-4}$	$9\cdot10^{4}$
Jain (11)	-2.0437	$1\cdot10^{-2}$	$1\cdot10^{4}$
Swame and Jain (10)	-2.0404	$1\cdot10^{-2}$	$1\cdot10^{4}$
Manadilli (26)	-2.0651	$1\cdot10^{-2}$	$1\cdot10^{4}$
Churchill (9)	-2.1718	$1\cdot10^{-2}$	$1\cdot10^{4}$
Churchill (12)	-2.1914	$1\cdot10^{-2}$	$1\cdot10^{4}$
Brkić (38)	-2.2719	$1\cdot10^{-2}$	$1\cdot10^{4}$
Brkić (37)	+3.1560	$1\cdot10^{-6}$	$1\cdot10^{4}$
[a]Modified Colebrook (4)	-3.2025	$1\cdot10^{-6}$	$1\cdot10^{4}$
Avzi and Karagoz (35)	-4.7857	$7.5\cdot10^{-2}$	$1\cdot10^{8}$
Eck (8)	+8.20	$1\cdot10^{-6}$	$5\cdot10^{6}$
Round (16)	+10.92	$7.5\cdot10^{-2}$	$1\cdot10^{8}$
Moody (5)	+21.49	$7.5\cdot10^{-2}$	$1\cdot10^{4}$
Wood (6)	+23.72	$1\cdot10^{-6}$	$1\cdot10^{4}$
Rao and Kumar (30)	+81.24	$1\cdot10^{-6}$	$1\cdot10^{4}$

[a]also in implicit form (constant 2.51 is replaced with 2.825, recommended by AGA-American Gas Association)

Table 2. Complexity and complexity index of available explicit approximations

	[a]Complexity	[b]Complexity index
[c]Eck (8)	27	1
Moody (5)	29	1.07
Churchil (9)	31	1.14
Haaland (20)	35	1.29
Jain (11)	35	1.29
Swame and Jain (10)	36	1.33
Round (16)	36	1.33
Manadilli (26)	44	1.62
Zigrang and Sylvester (18)	47	1.74
Avzi and Karagoz (35)	47	1.74
Rao and Kumar (30)	61	2.25
Brkić (38)	67	2.48
Sonnad and Goudar (28)	67	2.48
Papaevangelou, Evangelides and Tzimopoulos (36)	67	2.48
Brkić (37)	69	2.55
Zigrang and Sylvester (19)	69	2.55
Vatankhah and Kouchakzadeh (28a)	77	2.85
Barr (17)	80	2.96
Chen (15)	91	3.37
Wood (6)	98	3.62
Buzzelli (32)	104	3.85
Churchil (12)	106	3.92
Serghides (22)	107	3.96
Romeo, Royo and Monzon (27)	125	4.62
Serghides (21)	144	5.33

[a]number of estimated algebraic notation calculator key strokes required to solve the approximation, i.e. to find value of friction factor λ (estimated, i.e. average number of strokes)
[b]Complexity index is defined as quotient of key strokes required for an observed approximation and the least complex one; here Eck (8)
[c]the least complex approximation presented here

MS Excel DB

Click here to download Background dataset for online publication only: el. appendix DB.xlsx