



Periodicity of the Rational Numbers

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November 12, 2019

Abstract

In this article, we prove that α is a rational number if, and only if, its decimal representation possesses a period.

keywords: rational numbers, periodicity, irrational numbers.

Preliminaries

1. A real number α is called rational if we can write $\alpha = \frac{p}{q}$ for some integers p and $q \neq 0$.
2. There are infinitely many real numbers that are not rational. We call them irrational.
3. For example, $\sqrt{2}$ and π are irrational numbers (see [1], Theorem 6.18).
4. Consider $\alpha = a_0.a_1a_2a_3\cdots$ the decimal representation of some real number α , where a_0 is an integer and a_1, a_2, \dots belong to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

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5. The representation (4) possesses a *period* if there exist two natural numbers n and m such that

$$a_n a_{n+1} \cdots a_{n+m} = a_{n+m+1} a_{n+m+2} \cdots a_{n+2m+1} = a_{n+2m+2} a_{n+2m+3} \cdots a_{n+3m+2} = \cdots.$$

6. Here, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of the natural numbers.
7. The numbers $0.12334433443344\dots$ and $6.1222\dots$ possess a period.
8. The number $0.112123123412345\dots$, in which its decimal places are given by 1, 12, 123, 1234, 12345 and so on, does not possess a period.
9. Finite decimal representations always have a period.

Proof of the Main Result

10. *Every real number that possesses a period is rational.*
11. Indeed, let $\alpha = a_0.a_1a_2a_3\dots$ be a nonnegative real number, as in (4), that possesses a period.
12. From definition (5), there exist two natural numbers n and m such that

$$a_n a_{n+1} \cdots a_{n+m} = a_{n+m+1} a_{n+m+2} \cdots a_{n+2m+1} = \cdots.$$

13. Therefore, we write

$$\alpha = a_0 + 0.a_1a_2a_3\dots a_{n-1} + 0.0\dots 0a_n a_{n+1} \cdots a_{n+m} \cdots.$$

14. Applying hypothesis (11) in (13), we get

$$\alpha = a_0 + 0.a_1a_2\dots a_{n-1} + a_n a_{n+1} \cdots a_{n+m} \left(\frac{1}{10^{n+m}} + \frac{1}{10^{n+2m+1}} + \cdots \right).$$

15. It holds $\left(\frac{1}{10^{n+m}} + \frac{1}{10^{n+2m+1}} + \cdots\right) = \frac{1}{10^{n+m}} \left(1 + \frac{1}{10^{m+1}} + \frac{1}{10^{2m+2}} + \cdots\right)$
 $= \frac{1}{10^{n+m}} \left[1 + \frac{1}{10^{m+1}} + \frac{1}{(10^{m+1})^2} + \frac{1}{(10^{m+1})^3} + \cdots\right] \in \mathbb{Q}.$
16. Clearly, (15) proves (10).
17. ***Conversely, every rational number $\alpha = \frac{p}{q}$ possesses a period.***
18. In fact, without loss of generality, we may assume $0 < p < q$.
19. There exists a natural number k_1 such that $10^{k_1-1}p < q \leq 10^{k_1}p$.
20. To prove (19), let $X = \{k \in \mathbb{N}; 10^{k-1}p < q\} \subset \mathbb{N}$.
21. So, since $p < q$, one has $1 \in X$. Therefore, $X \neq \emptyset$.
22. We also have that X is finite.
23. By the Principle of Mathematical Induction (see Exercise 2.4 in Chapter 1 from [2]), one can show that there exists $k_1 \in X$ satisfying (19).
24. Write $\frac{p}{q} = \frac{1}{10^{k_1}} \frac{10^{k_1}p}{q}$.
25. By Euclidean Algorithm (see [3]), there are integers r_1 and q_1 , with $0 \leq r_1 < q$, such that $10^{k_1}p = q_1q + r_1$.
26. By (24), $\frac{p}{q} = \frac{1}{10^{k_1}} \left(q_1 + \frac{r_1}{q}\right).$
27. If $r_1 = 0$, there is nothing to do and (17) is easily true.
28. Otherwise, there is another natural number k_2 such that $10^{k_2-1}r_1 < q \leq 10^{k_2}r_1$.

29. We have $\frac{p}{q} = \frac{1}{10^{k_1}} \left(q_1 + \frac{r_1}{q} \right) = \frac{1}{10^{k_1}} \left(q_1 + \frac{10^{k_2} r_1}{q} \frac{1}{10^{k_2}} \right)$.
30. One more time, by Euclidean Algorithm, there are integers r_2 and q_2 , with $0 \leq r_2 < q$, such that $10^{k_2} r_1 = q_2 q + r_2$.
31. By (29), $\frac{p}{q} = \frac{1}{10^{k_1}} \left(q_1 + \frac{1}{10^{k_2}} \left(q_2 + \frac{r_2}{q} \right) \right)$.
32. If $r_2 = 0$, then (17) is easily true.
33. Otherwise, we repeat the arguments as from (28) to ensure the existence of a sequence of integer numbers k_n, q_n and r_n such that
- $$\frac{p}{q} = \frac{1}{10^{k_1}} \left[q_1 + \frac{1}{10^{k_2}} \left(q_2 + \frac{1}{10^{k_3}} \left[q_3 + \cdots + q_{n-1} + \frac{1}{10^{k_n}} \left(q_n + \frac{r_n}{q} \right) \right] \right) \right].$$
34. If we repeat $n = q + 1$ times the arguments done in (33), since $0 \leq r_n < q$ for all n , there exist at least two different natural numbers $i < j$, such that $r_i = r_j$.
35. So, the process from r_j will repeat all the correspondent values of k_s, q_s and r_s that previously appear since $r_i = r_j$.
36. (35) shows exactly the period of $\alpha = \frac{p}{q}$ and proves (17).

Final Remarks

37. The number 0.1234567891011121314... is not rational because it does not have a period.
38. $\pi = 3.141592653589\cdots$ does not have a period because it is not a rational number.
39. A didactic illustration of the proof of a particular case of the claim (17) is the following.

$$\begin{array}{c}
p0 \\
r_1 0 \dots 0 \\
r_2 0 \dots 0 \\
\vdots \\
r_n 0 \dots 0 \\
r_1 0 \dots 0 \\
r_2 0 \dots 0 \\
\vdots
\end{array}
\begin{array}{c}
\hline q \\
0, r_1 0 \dots 0 q_2 0 \dots 0 \dots q_n 0 \dots 0 r_1 0 \dots 0 q_2 \dots
\end{array}$$

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Ethical conduct of research

This original work was pre-registered under the OSF Preprints [4], please cite it accordingly [5]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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