TWO VALUES ARE SUFFICIENT FOR CHOICE CONSISTENCY

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1. INTRODUCTION

People often depart from the narrow sense of rationality traditionally assumed in Economics, but it is not always due to logical errors (while evaluating the choices) or behavioral reasons (like lack of self control, etc.). In many situations, people make choices which are optimal under some social and moral rules. So how can an outside observer, with no access to such private rules, make sense of their behavior? One way to understand choices is by assigning multiple values (or meanings) to the alternatives depending on the context. The number of additional values can then be viewed as a measure of the background information involved while making the choice. It is reasonable to expect that the number of valuations of an alternative increases with the size and complexity of the menu.

In this paper, we show that *only* two values (dominant and recessive) are sufficient (for each alternative) to make any choice behavior consistent. The size of the menu does not matter.

Does the understanding of choice, when one selects from a large pool of alternatives, really involve new conceptual difficulties (that aren't present when choosing from short menus), apart from the obvious informational overload? In fact many complexities of choice behavior (in the literature) can be captured through simple examples with a person pondering over only 3 or 4 alternatives.

A priori there are multiple ways to assign values to different alternatives so that the choice behavior is seen as an act of maximization. Notice however that an outsider can observe only the order of preferences, based on which alternatives are selected from different menus. Therefore, in this paper, we give a simple procedure, allowing two values to only those alternatives that are revealed through choice (Theorem 3.2). Though only two mutant forms (for each alternative) are sufficient to invoke menu-consistency, they may not be sufficient to explain choice behavior. In other words, the consistency constructed in the paper may not align with the decision maker's thought process. However, such constructions can be useful to find the problematic alternatives and help in redesigning the menu (by adding a mutant of an alternative) so that future choices reveal the intentions of the decision makers more clearly.

The result applies to the general case, where one is allowed to choose multiple items from a menu. One corollary is that the Background Information can never be greater than the information captured by the choice function. (Theorem 3.4)

Section 2 is a discussion on some basic conditions of inter-menu consistency and how choice is affected by the background set of alternatives. Section 3 gives a procedure to transform a menu into a consistent one. We show that at most two values are enough for consistency and give a measure for the amount of background information that the choice function fails to capture. We will also see that the information captured by the choice functions doesn't increase when two menus are combined.

2. PATTERNS OF CHOICE

Definition 2.1. *Given a variable menu S , the decision maker (through some rationale depending on the background information available to him about S) chooses a nonempty subset $C(S) \subseteq S$, called the choice/decision set of S . The function C is referred to as the choice function.*

Two of the basic conditions of choice consistency are the following :

- Property α : Let $x \in S \subseteq T$. Then $x \in C(T) \implies x \in C(S)$.
- Expansion : $C(S) \cap C(T) \subseteq C(S \cup T)$

Theorem 2.2. *Suppose for every S , the set $C(S)$ consists of exactly one element of S . Then α implies Expansion.*

Proof. Let $\{x\} = C(S \cup T)$ and $x \in S$. Then it follows from property α , that $x \in C(S)$. Since $C(S)$ contains only one element (by our hypothesis), we have $\{x\} = C(S)$.

If $x \in T$, we similarly have $\{x\} = C(T)$. Otherwise $C(S) \cap C(T) = \emptyset$ and there is nothing left to prove. $\square$

In the rest of the article, we will use (α) as our criterion for internal consistency. It is misleading to take the above conditions as *rules* of rationality. As argued by Amartya Sen in [7], one can't rely on such consistency axioms to separate rational from irrational behavior. In fact one can find many instances of sensible behavior which fail to satisfy the above conditions. Sen (ibid.) mentions 3 broad categories of such departures :

- (1) *Positional choice* : A person might not pick the largest piece of cake at a birthday party, (due to social norms) but looks for the second largest piece. This is still an act of maximization with a social constraint and fails to satisfy property α .
- (2) *Epistemic value of the menu* : The meaning of the alternatives can change depending on what options are available. To illustrate, given the choice between having tea at a colleague's home (x), and not going there (y), a person who chooses to have tea (x), may nevertheless choose to go away (y), if offered a choice between tea(x), going away (y) and having some cocaine(z).
- (3) *Freedom to reject* : One selects an alternative (x) not for its outcome, but to reject another alternative (y). Sen illustrates this as follows :

“The point of fasting in the form of not eating (y), given the possibility of eating well (z), may become less clear when the only alternative is to be partly famished anyway(x)”.

One can easily find simple examples for each of the above categories, where α is not satisfied because

$$C(\{x, y\}) = \{x\} \text{ and } C(\{x, y, z\}) = \{y\}.$$

In all of the above examples, the value of an alternative depends on the other items in the menu. A valuation of alternatives dependent on a background set may not hint at the real external motivations of the decision maker. However, given that only two values (for each alternative) are sufficient for choice consistency (Theorem 3.2), a background-set dependent valuation may be a useful approach to examine choice, especially in menus of large size and complexity.

Remark 2.3. A general axiomatic choice model where preferences depend on a background set was introduced by Stig Kanger in [2] and also described in Sen [9]. In Kanger’s model, the background set need not coincide with the menu set.

To give another illustration of how choices are influenced by the background set, here is an example (inspired by a scene from Pushkin’s Eugene Onegin).

Eugene Onegin : g has fallen in love with b but b rejects her and wants to avoid meeting her. So b is extremely irritated with all the other boys (b_i) for dragging him to a party he has no interest in. He knows that b_i likes g_i and intends to have some fun at their expense. On the other hand, g ’s smoking habits have gone out of control. A few people join her (outside the building). Our hero b , as you might guess, is eager to dance.

Let S denote the set of people present at the table. The girl g_i is a better dancer than g_j if $i < j$.

Definition 2.4. A girl $g_i \in S$ is said to be paired if $b_i \in S$ and S is said to be paired if all the girls in S are paired.

Choice Procedure of b : If none of the girls at the table are paired, then b asks the best dancer among them. Otherwise, he will ask the best dancer g_i among the girls who are paired, annoying b_i who is also present at the table.

Let us identify the pair $\{g_i, b_i\}$ as a single alternative when both g_i, b_i are present at the table. Let T denote the set of alternatives chosen from $\{g_1, \dots, g_n, \{g_1, b_1\}, \dots, \{g_n, b_n\}\}$ subject to the condition that at most one of g_i or $\{g_i, b_i\}$ belong to T . Here the primitive options are not the girls themselves, but paired and non-paired girls.

Then there is a clear ordering $\{g_1, b_1\} > \dots > \{g_n, b_n\} > g_1 > \dots > g_n$ of the alternatives and b ’s decision can be seen as choosing the maximal one.

The same choice situation can be represented by different menus, leading to different choice functions. Given a choice function, can we measure the amount of background information that is not captured by it?

3. BACKGROUND INFORMATION

The value/meaning of an alternative can change with the context. We split an alternative into two alternatives (or *alleles*¹) whenever the decision maker *seems* to evaluate it differently over different submenus. The number of additional alleles (compared to the size of the original menu) then gives a measure of the background information not visible on the menu. This might be a crude measure for the information but it is easy to calculate from the choice data. In the *Eugene Onegin* example in the last section, paired and non-paired girls can be thought of as two mutant forms of the same alternatives.

In Theorem 3.2, we show that two mutant forms are sufficient to view the choice behavior as consistent. We capture the information without reducing our problem to binary preferences. No assumption is made about the choice function, except that the set of options is finite. In particular, the decision maker may choose multiple items from a menu.

We start with a choice function C defined over a domain consisting of some subsets of a finite set X . (A domain simply means a collection of menus.) Recall that for any submenu $S \subseteq X$, we have $C(S) \subseteq S$. No other restriction is placed on the choice function.

Definition 3.1. An element $g \in X$ is said to be **incomplete** if there is a pair of submenus (S, T) such that $g \in S \subset T$ and $g \in C(T)$ but $g \notin C(S)$. Otherwise g is said to be **complete**.

Notice that a domain has no incomplete elements if and only if it satisfies Property α : Let $x \in S \subseteq T$. Then $x \in C(T) \implies x \in C(S)$.

Theorem 3.2. Two values (for each alternative) are sufficient for menu consistency (with respect to α).

Proof. Let S be a submenu. Whenever there is a pair $\langle g, T \rangle$ such that $g \in S \subset T$ and $g \in C(T)$ but $g \notin C(S)$, the idea is to replace g in S with a new element $(g, 1)$ in S .

- Replace all *incomplete* elements g with $(g, 1)$ in S , whenever applicable as above. Do this simultaneously for all submenus S in the choice domain. Then g is complete in the transformed domain.

¹An *allele* is an alternative form of a gene. For each gene (for eye color, hair color etc) there are two (or more) alleles. When the two alleles are of different type, the trait (blue or brown eyes) that is revealed corresponds to the dominant allele.

Notice that $(g, 1) \notin C(S)$ for any S in the transformed domain. In particular $(g, 1)$ is also *complete*. Therefore the new transformed domain [which differentiates between $(g, 1)$ and g] has no incomplete elements and satisfies α . $\square$

The two alleles g and $(g, 1)$ can be interpreted as the dominant and recessive variants respectively. We can think of number of additional alleles as a measure of the background information. In other words,

$$\text{Background Information} = \frac{\text{no. of extra alleles}}{\text{total no. of alternatives (including extra alleles)}}$$

Since each alternative requires at most one extra allele for menu consistency (Theorem 3.2), this motivates the following definition for menu and background information.

Definition 3.3. Let $n = \text{Card}(X)$ and i be the number of incomplete elements in X . Then the menu information of C is defined as $I(C) = \frac{n}{n+i}$ and

$$\text{Background Information} = \frac{i}{n+i} = 1 - I(C).$$

Menu Information gives a measure of how close the choice function is to satisfying α .

Theorem 3.4. For any finite menu X with a choice function C ,

$$\text{Menu Information} \geq \text{Background Information}.$$

Proof. Since $i \leq n$, we have $I(C) = \frac{n}{n+i} \geq \frac{1}{2}$. $\square$

Notice that if the choice function C already satisfies α , then all elements are complete and we have $I(C) = 1$. Let us return to the example of the dancing girls, where $X_{bg} = \{b_1, \dots, b_n, g_1, \dots, g_n\}$ and the choice-menu consists of people present at the table. The two alleles of g_i can be interpreted as paired and non-paired. The amount of menu information is then $2/3$ and the background information is $1/3$.

Remark 3.5. It would be interesting to find upper bounds for the amount of background information allowed by different models of rationality in Economics such as those introduced in Manzini and Mariotti (2007), Salant and Rubinstein (2008), etc. In this way, one can classify different models of rationality based on the amount of *menu information* required for the choice function.

The following result says that menu information doesn't increase when we combine two menus.

Theorem 3.6. Let X_1, X_2 be two menus with choice functions C_1, C_2 . Let $X = X_1 \cup X_2$ and C be the choice function on X such that $C = C_k$ when restricted to X_k . Then $I(C) \leq \max\{I(C_1), I(C_2)\}$.

Proof. Let $n_i = \text{Card}(X_k)$ and $N_k = n_k + i_k$. By definition, we have $I(C_k) = \frac{n_k}{N_k}$.

Suppose $I(C_1) \leq I(C_2)$. Then $\frac{n_1}{n_2} \leq \frac{N_1}{N_2}$. Adding 1 on both sides of the inequality, we get $\frac{n_1+n_2}{n_2} \leq \frac{N_1+N_2}{N_2}$, which is equivalent to

$$\frac{n_1 + n_2}{N_1 + N_2} \leq \frac{n_2}{N_2}.$$

On the other hand, since C restricts to C_k on X_k , it follows that $N \geq N_1 + N_2$. So we clearly have

$$I(C) \leq \frac{n_1 + n_2}{N_1 + N_2}.$$

Therefore $I(C) \leq I(C_2) = \frac{n_2}{N_2}$. □

The following generalization is straightforward.

Theorem 3.7. *Let $X_1, \dots, X_t$ be menus with choice functions $C_1, \dots, C_t$. Let $X = X_1 \cup \dots \cup X_t$ and C be the choice function on X such that $C = C_k$ when restricted to X_k . Then $I(C) \leq \max\{I(C_1), \dots, I(C_t)\}$.*

REFERENCES

- [1] Apesteguia, J., and M. Ballester, “A Measure of Rationality and Welfare”, *Journal of Political Economy*, 123(6):1278–1310, 2015.
- [2] Kanger, S., “Choice based on Preference”, *Collected Papers of Stig Kanger with Essays on his Life and Work*, Kluwer Academic Publishers, 214–230, 2001
- [3] Manzini, P., Mariotti, M., “Sequentially Rationalizable Choice”, *American Economic Review*, 97 (5), 1824–1839. [1154], 2007.
- [4] Pushkin, A., “Eugene Onegin”, 1837.
- [5] Rabin, M., and B. Köszegi “Revealed Mistakes and Revealed Preferences”, in *Methodologies of Modern Economics*, Andrew Caplin and Andrew Schotter, eds., 2008.
- [6] Salant, Y., and A. Rubinstein “(A, f): Choice with Frames.” *Rev. Econ. Studies* 75:1287–96, 2008.
- [7] Sen, Amartya, “Internal Consistency of Choice”, *Econometrica*, 61, 495–521, 1993.
- [8] ——— “Maximization and the act of choice”, *Econometrica*, Vol. 65, No. 4 (Jul., 1997), pp. 745–779
- [9] ——— “Non-binary choice and preference: A tribute to Stig Kanger.” *Studies in Logic and the Foundations of Mathematics* 134 (1995): 913–924.

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