1 On the Use of Discrete Fourier Transform for
2 Solving Biperiodic Boundary Value Problem of
3 Triharmonic Equation in the Unit Rectangle

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10 **Abstract**

11 This note is addressed to solving biperiodic boundary value prob-
12 lem Triharmonic equation in the unit rectangle. First, we describe
13 the necessary tools, which is discrete Fourier transform for one di-
14 mensional periodic sequence, and then extended the results to 2-
15 dimensional biperiodic sequence. Next, we use the discrete Fourier
16 transform 2-dimensional biperiodic sequence to solve discretization of
17 the biperiodic boundary value problem of Triharmonic Equation.

18 *MSC:* 15A09; 35K35; 41A15; 41A29; 94A12.

19 *Key words:* Triharmonic equation, biperiodic boundary value prob-
20 lem, discrete Fourier transform, Partial differential equations, Inter-
21 polation.

22 **1 Introduction**

23 This note is an extension of a work by Henrici [3] on fast solver for Poisson
24 equation to biperiodic boundary value problem of triharmonic equation. This

note is organized as follows, in the first part we address the tools needed to solve the problem, i.e. introducing discrete Fourier transform (DFT) follows Henrici's work [3]. The second part, we apply the tools already described in the first part to solve discretization of bi-periodic boundary value problem of triharmonic equation in the unit rectangle. The need to solve discretization of triharmonic equation is motivated by image inpainting [4] using biharmonic or harmonic equation. Note that the work of Henrici [3] on fast solver for bi-periodic Poisson equation can be seen as solving harmonic inpainting. For pedagogic reason, results from citeGarnadifft on solving biharmonic equation is reproduced in the second part.

2 Discrete Fourier Transform

2.1 One-dimensional Transform

Let $n \in \mathbb{N}$ and $(x_k)_{k=-\infty}^{\infty}$ is a bilateral infinite sequence of complex number. Let Π_n denote a set of bilateral infinite sequences with period n , i.e.:

$$\Pi_n = \{\mathbf{x} = (x_k)_{k=-\infty}^{\infty}; x_k \in \mathbb{C}, x_{k+n \cdot j} = x_k, j \in \mathbb{Z}\}.$$

Define in Π_n an addition and scalar multiplication by:

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_k + y_k)_{k=-\infty}^{\infty}, \\ c\mathbf{x} &= (c x_k)_{k=-\infty}^{\infty}, \quad \forall c \in \mathbb{C}. \end{aligned}$$

Equipped with these operations, Π_n is a finite dimensional vector space with $\dim(\Pi_n) = n$.

Denote $\omega_n := \exp(2\pi i/n)$, $i^2 = -1$. One dimensional discrete Fourier transform, denoted by $\mathcal{F}_n$, is defined as follows:

$$\begin{aligned} \mathcal{F}_n : \Pi_n &\rightarrow \Pi_n \\ \mathbf{x} &\mapsto \mathbf{y}, \end{aligned}$$

where

$$y_m = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{-mk} x_k.$$

It is clearly that $y_{m+n} = y_m$, holds for all m , as $\omega_n^n = 1$. Hence the map $\mathcal{F}_n$ is well defined. It is easily to show that $\mathcal{F}_n$ is a bijective map, hence invertible. The inverse of $\mathcal{F}_n$ is defined as:

$$\begin{aligned} \mathcal{F}_n^{-1} : \Pi_n &\rightarrow \Pi_n \\ \mathbf{y} &\mapsto \mathbf{x}, \end{aligned}$$

48 where:

$$x_r = \sum_{k=0}^{n-1} \omega_n^{mr} y_r.$$

49 Observe that the inverse has close resemblance with the DFT. Defining the
50 conjugate of DFT $\mathcal{F}_n$ by:

$$(\bar{\mathcal{F}}_n \mathbf{y})_k = \frac{1}{n} \sum_{m=0}^{n-1} \omega_n^{mk} y_m,$$

51 where the index k denoting the k -th component of the sequence, then we
52 have

$$\mathcal{F}_n^{-1} = n \bar{\mathcal{F}}_n.$$

53 Another operator that also useful when working with DFT is the reversion
54 operator denoted by $\mathcal{R}$. This operator is defined as follow, for $\mathbf{x} \in \Pi_n$,

$$(\mathcal{R}\mathbf{x})_m := x_{-m}.$$

55 A left and right composition with $\mathcal{F}_n$, we have the following relations:

$$\begin{aligned} (\mathcal{R} \circ \mathcal{F}_n \mathbf{x})_m &= \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{km} x_k, \\ &= (\bar{\mathcal{F}}_n). \\ (\mathcal{F}_n \circ \mathcal{R}\mathbf{x})_m &= - \sum_{k=0}^{n-1} \omega_n^{-km} x_{-k} = - \sum_{k=0}^{n-1} \omega_n^{km} x_k, \\ &= (\bar{\mathcal{F}}_n). \end{aligned}$$

56 From these composition, it is clear that

$$\mathcal{F}_n^{-1} = n \mathcal{F}_n \mathcal{R} = n \mathcal{R} \mathcal{F}_n.$$

57 We introduce Hadamard product of two sequences $\mathbf{x}$ and $\mathbf{y}$ in Π_n , which
58 is a component-wise product:

$$\mathbf{x} \bullet \mathbf{y} = (x_k y_k)_{k=-\infty}^{\infty}.$$

59 Also we introduce convolution product of two sequences $\mathbf{x}$ and $\mathbf{y}$ in Π_n by:

$$\mathbf{x} * \mathbf{y} = \mathbf{z} = (z_k)_{k=-\infty}^{\infty},$$

60 where:

$$z_k = \sum_{k=0}^{n-1} x_k y_{k-m} = \sum_{k=0}^{n-1} x_{k-m} y_k.$$

Both product relate to each other of as follows. Let $\mathbf{x}$ and $\mathbf{y}$ be two sequences in Π_n , and

$$\mathbf{u} = \mathcal{F}_n \mathbf{x}, \quad \mathbf{v} = \mathcal{F}_n \mathbf{y}.$$

By definition,

$$(\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}))_m = \frac{1}{n} \sum_{k=0}^{n-1} x_k y_k \omega_n^{-km},$$

as $y_k = (\mathcal{F}_n^{-1} \mathbf{y})_k$, then

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} x_k y_k \omega_n^{-km} &= \frac{1}{n} \sum_{k=0}^{n-1} x_k \left(\sum_{l=0}^{n-1} \omega_n^{lk} v_l \right) \omega_n^{-km} \\ &= \sum_{l=0}^{n-1} v_l \left(\frac{1}{n} \sum_{k=0}^{n-1} x_k \omega_n^{-k(m-l)} \right). \end{aligned}$$

The last inner sum is obviously equal to $(\mathcal{F}_n \mathbf{x})_{(m-l)} = u_{m-l}$, hence

$$\begin{aligned} (\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}))_m &= \sum_{l=0}^{n-1} v_l u_{m-l} = (\mathbf{v} * \mathbf{u})_m = (\mathbf{u} * \mathbf{v})_m \\ &= (\mathcal{F}_n(\mathbf{x}) * \mathcal{F}_n(\mathbf{y}))_m \end{aligned}$$

As it holds for all m , then

$$\mathcal{F}_n(\mathbf{x} \bullet \mathbf{y}) = \mathcal{F}_n(\mathbf{x}) * \mathcal{F}_n(\mathbf{y}).$$

Working on $\mathbf{u} = \mathcal{F}_n(\mathbf{x})$ and $\mathbf{v} = \mathcal{F}_n(\mathbf{y})$, for convolution form, then

$$\begin{aligned} \mathbf{u} * \mathbf{v} &= n^2 \mathcal{F}_n(\mathcal{F}_n^{-1}(\mathbf{x}) \bullet \mathcal{F}_n^{-1}(\mathbf{y})) \\ &= n^2 \mathcal{F}_n(R \mathcal{F}_n(\mathbf{x}) \bullet R \mathcal{F}_n(\mathbf{y})) \\ &= n^2 \mathcal{F}_n(R(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y}))) \\ &= n(nR \mathcal{F}_n)(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \\ &= n(\mathcal{F}_n^{-1})(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \\ \mathcal{F}_n(\mathbf{u} * \mathbf{v}) &= n(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})) \end{aligned}$$

Hence for $\mathbf{x}, \mathbf{y} \in \Pi_n$

$$\mathcal{F}_n(\mathbf{u} * \mathbf{v}) = n(\mathcal{F}_n(\mathbf{x}) \bullet \mathcal{F}_n(\mathbf{y})).$$

69 2.2 Two-dimensional Transform

70 Let $n \in \mathbb{N}$ and $\mathbf{x}^{(2)} = (x_{k_1 k_2})_{k_1, k_2=-\infty}^{\infty}$ is a two dimensional bilateral infi-
 71 nite sequence of complex number. Let $\Pi_n^{(2)}$ denote a set of bilateral infinite
 72 sequences with period n on each direction, i.e.

$$\begin{aligned} \Pi_n^{(2)} = \{ \mathbf{x} = (x_{k_1 k_2})_{k_1, k_2=-\infty}^{\infty}; x_{k_1 k_2} \in \mathbb{C}, \\ x_{k_1+n, j, k_2} = x_{k_1 k_2}, x_{k_1, k_2+n, j} = x_{k_1 k_2}, j \in \mathbb{Z} \}. \end{aligned}$$

73 Observe that this set of two-dimensional array is isomorph with a set of bilat-
 74 eral infinite sequence of element in Π_n , i.e. $\{(\mathbf{x}_{k_2})_{k_2=-\infty}^{\infty}; \mathbf{x}_{k_2} \in \Pi_n, \mathbf{x}_{k_2+n, j} =$
 75 $\mathbf{x}_{k_2}, j \in \mathbb{Z}\}$.

76 Addition, scalar multiplication, Hadamard product, and convolution in
 77 Π_n easily extended to $\Pi_n^{(2)}$ by using the fact that $\Pi_n^{(2)}$ is a set of bilateral
 78 infinite array of element in Π_n .

79 The 2-dimensional DFT is defined by

$$\mathbf{y}^{(2)} = \mathcal{F}_n^{(2)} \mathbf{x}^{(2)},$$

80 where

$$y_{m_1 m_2} = \frac{1}{n^2} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \omega_n^{-(m_1 k_1 + m_2 k_2)} x_{k_1 k_2}.$$

81 Rearrangement of the right hand side, we have the following form:

$$y_{m_1 m_2} = \frac{1}{n} \sum_{k_1=0}^{n-1} \omega_n^{-m_1 k_1} \left(\frac{1}{n} \sum_{k_2=0}^{n-1} \omega_n^{-m_2 k_2} x_{k_1 k_2} \right),$$

82 Clearly that the 2-dimensional DFT can be seen as a successive of two one-
 83 dimensional DFT. Hence, the 2-dimensional DFT is a bijective map, so it is
 84 invertible. The inverse is given by:

$$(\mathcal{F}_n^{(2)})^{-1} = n^2 \mathcal{F}_n^{(2)*}.$$

85 As 2-dimensional DFT on Hadamard product and convolutions of two ele-
 86 ments in $\Pi_n^{(2)}$, the following are satisfied:

$$\begin{aligned} \mathcal{F}_n^{(2)}(\mathbf{x} \bullet \mathbf{y}) &= \mathcal{F}_n^{(2)}(\mathbf{x}) * \mathcal{F}_n^{(2)}(\mathbf{y}), \\ \mathcal{F}_n^{(2)}(\mathbf{u} * \mathbf{v}) &= n^2 (\mathcal{F}_n^{(2)}(\mathbf{x}) \bullet \mathcal{F}_n^{(2)}(\mathbf{y})). \end{aligned}$$

87 2.3 Applying DFT for solving Triharmonic equations

88 Suppose that S is a unit square,

$$S = \{(\xi, \eta); 0 \leq \xi, \eta \leq 1\}.$$

89 We are going to solve triharmonic equation:

$$\Delta^3 u(x, y) = f(x, y),$$

90 in the rectangle S , where f is bi-periodic on S . Furthermore, we restricts u
91 to be bi-periodic, i.e. bi-periodic boundary value problem.

92 To solve the problem, rewrite the triharmonic problem in mixed form as
93 follows :

$$\Delta v = f(x, y), \quad \Delta^2 u(x, y) = v.$$

94 Let we discretize the equation over a discrete grid on S . Suppose that on
95 S we construct a uniform grid with constant spacing $h = 1/(2n)$ on each
96 direction. Let

$$\xi_i = ih; \quad \eta_j = jh \quad i, j = 0, 1, \dots, 2n,$$

97 and obtains approximate values $v_{i,j}$ of $v(\xi_i, \eta_j)$ by solving linear equation
98 system from five-points finite difference discretization of harmonic equations,
99 hence the algorithms of Henrici [3] on fast solution of Poisson problem on
100 bi-periodic domain is applicable to get the solution of $v_{i,j}$.

101 While approximate values $u_{i,j}$ of $u(\xi_i, \eta_j)$ by solving linear equation sys-
102 tem from thirteen-points finite difference discretization of biharmonic equa-
103 tions, we have the following discrete equation for biharmonic equation ([6])

$$\begin{aligned} [20 u_{i,j} &- 8(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) + \\ &+ 2(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}) + \\ &+ (u_{i+2,j} + u_{i-2,j} + u_{i,j+1} + u_{i,j-2})] \\ &= h^4 f_{i,j}, \end{aligned}$$

104 where $f_{i,j} =: \mathbf{v} \in \Pi_{2n}^{(2)}$. Observe that the left hand side is a convolution of
105 $\mathbf{u} \in \Pi_{2n}^{(2)}$ with a bilateral sequence $\mathbf{d} = (d_{km})$ with vanishes element except
106 that

$$\begin{aligned} d_{0,0} &= 20, \\ d_{1,0} = d_{-1,0} = d_{0,1} = d_{0,-1} &= -8, \\ d_{1,1} = d_{1,-1} = d_{-1,1} = d_{-1,-1} &= 2, \\ d_{2,0} = d_{-2,0} = d_{0,2} = d_{0,-2} &= 1. \end{aligned}$$

107 Then the discrete equation can be written as:

$$\mathbf{d} * \mathbf{u} = h^4 \mathbf{v}.$$

108 Applying $\mathcal{F}_{2n}^{(2)}$ to both side, we obtain

$$(2n)^2 \hat{\mathbf{d}} \bullet \hat{\mathbf{u}} = h^4 \hat{\mathbf{v}},$$

109 where $\hat{\mathbf{d}}$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are images under $\mathcal{F}_{2n}^{(2)}$ of $\mathbf{d}$, $\mathbf{u}$, and, $\mathbf{v}$ respectively, and the
110 images of $\mathbf{d}$ is

$$\begin{aligned} \hat{\mathbf{d}} = & [20 - (\cos(\frac{k_1 + k_2}{2n}\pi) \cos(\frac{k_1 - k_2}{2n}\pi)) + \\ & + 4(\cos(\frac{k_1 + k_2}{n}\pi) \cos(\frac{k_1 - k_2}{n}\pi)) + \\ & + 8(\cos(\frac{k_1}{n}\pi) \cos(\frac{k_2}{n}\pi))]_{k_1, k_2 = -\infty}^{\infty}. \end{aligned}$$

111 Multiplying using Hadamard product of $\hat{\mathbf{d}}^{-1}$, we have

$$\hat{\mathbf{u}} = \frac{h^4}{(2n)^2} \hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}},$$

112 inverting using 2-dimensional inverse DFT, we get

$$\mathbf{u} = \frac{h^4}{(2n)^2} \mathcal{F}_{2n}^{(2)-1} (\hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}}),$$

113 and we get the solutions of discrete biharmonic problem

$$\mathbf{u} = h^4 \bar{\mathcal{F}}_{2n}^{(2)} (\hat{\mathbf{d}}^{-1} \bullet \hat{\mathbf{v}}),$$

114 using the fact that

$$\frac{1}{(2n)^2} \mathcal{F}_{2n}^{(2)-1} = \bar{\mathcal{F}}_{2n}^{(2)}.$$

115 3 Epilogue

116 We have demonstrated an algorithmic development of solving discretization
117 of bi-periodic triharmonic equations in unit rectangle. The availability of
118 DFT in several computational platforms such as MATLAB or Scilab, means
119 that the algorithms is easily benchmarked with several necessary additional
120 tools such as Hadamard product, which is unfortunately is not widely avail-
121 able in most computational platforms, hence opening for further work for
122 implementation to be pursued.

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