[knowledge base]

Diamond Open Access

Differential Forms (Open Mathematics Knowledge Base)

Open Mathematics Collaboration^{*†}

July 21, 2020

Abstract

We present a list of mathematical results about differential forms for future implementation in a digital Open Mathematics Knowledge Base.

keywords: differential geometry, pure mathematics, knowledge base

The most updated version of this paper is available at

<https://osf.io/g7uqb/download>

Introduction

- A. [1–3]
- B. OMKB = Open Mathematics Knowledge Base (see [4])
- C. *This article is constantly being updated.*
- D. Differential Forms (OMKB) = 56 mathematical entries (30 pages)
- E. 1 entry = *notation* or *definition* or *proposition* or *theorem*

^{*}All authors with their affiliations appear at the end of this paper.

[†]Corresponding author: alvaroyucra@uft.edu.br | Join the Open Mathematics Collaboration

Overview

F. [5]

Important: guidelines for advanced search

- G. In order to maximize the efficiency while using this document, note the following guidelines for **advanced search** in *Acrobat Reader*.
- H. Acrobat Reader → Preferences → Search → Range of words for proximity searches ≈ 10
- I. Acrobat Reader → open the search window → advanced settings → show more options → select a folder where the PDF is located → check the proximity box → choose match all of the words

#alternating projection #definition #notation

1. The alternating projection is a linear map $Alt : L_r(V) \rightarrow A_r(V)$
2. $Alt(\psi)$ = alternating projection of $\psi \in L_r(V)$
3. $Alt(\psi)(u_1, \dots, u_r) = \frac{1}{r!} \sum_{\sigma \in P_r} \text{sgn} \sigma \cdot (u_{\sigma(1)}, \dots, u_{\sigma(r)})$ for any vectors $u_1, \dots, u_r \in V$

#alternating projection #properties

4. $Alt(\psi) \in A_r(V)$, for all $\psi \in L_r(V)$
5. If $\psi \in A_r(V) \Rightarrow Alt(\psi) = \psi$
6. $Alt(Alt(\psi)) = Alt(\psi)$, for all $\psi \in L_r(V)$

#anticommutativity #exterior product #linear forms

7. $\psi \wedge \xi = (-1)^{rs} \xi \wedge \psi$, where $\psi \in A_r(V)$ and $\xi \in A_s(V)$
(anticommutativity)
8. $\dim A_n(V) = 1$
9. If $r > n \Rightarrow A_r(V) = \{0\}$

#antisymmetric #definition # notation # r -linear form

10. A map $\psi \in L_r(V)$ is antisymmetric if

$$\psi(u_1, \dots, u_i, \dots, u_j, \dots, u_r) = -\psi(u_1, \dots, u_j, \dots, u_i, \dots, u_r)$$

11. $\psi \in L_r(V)$ is a r -linear form $\Leftrightarrow \psi$ é antisymmetric

12. $A_r(V)$ = set of all r -linear forms

13. $A_r(V) \subset L_r(V)$ is a vector subspace

14. $A_0(V) = \mathbb{R}$

15. $A_1(V) = V^*$

#basis #coordinate functions #definition
 #differential forms # r -linear form

16. (a) $I = \{i_1 < \dots < i_r\} = r\text{-list}$
 (b) $\{dw_I(p) = dw_{i_1}(p) \wedge \dots \wedge dw_{i_r}(p) : I \text{ is a } r\text{-list}\}$ forms a basis for $A_r(T_p M)$
17. (a) If ψ is a differential form of degree $r \Rightarrow \psi = \sum_I a_I(p) dw_I(p)$ for all $p \in X(W)$
 (b) $a_I : X(W) \rightarrow \mathbb{R} =$ coordinate functions of ψ in relation to X

#basis #definition #tensor product

18. If $\{g_1, \dots, g_n\}$ is a basis for $V^* \Rightarrow \{g_{i_1} \otimes g_{i_2} \otimes \dots \otimes g_{i_r} : 1 \leq i_1 \leq \dots i_r \leq n\}$ is a basis for $L_r(V)$

#basis #exterior product #linear forms
#sequence of indices

19. If $\{f_1, \dots, f_n\}$ is any basis for V^* and $I = (i_1, \dots, i_r)$ a sequence of indices of the set $\{1, 2, \dots, n\} \Rightarrow f_I = f_{i_1} \wedge \dots \wedge f_{i_r} \in A_r(V)$
20. $I = \{i_1 < \dots < i_r\}$ is called a r -list
21. If $\{f_1, \dots, f_n\}$ is any basis for $V^* \Rightarrow \{f_I : I \text{ is a } r\text{-list}\}$ forms a basis for $A_r(V)$
22. $\dim A_r(V) = \frac{n!}{r!(n-r)!}$

#coordinate functions #definition #differentiable
#differential forms #notation

- 23. If a_I are differentiable in each coordinate neighborhood of M for all r -list $I \Rightarrow$ the differential form of degree r , ψ , is differentiable
- 24. $\Omega^r(M)$ = set of differential forms of degree r that are differentiable in M

~~#~~definition ~~#~~differentiable ~~#~~differential forms
~~#~~exterior product ~~#~~wedge product

25. (a) Let $\beta \in \Omega^r(M)$ and let $\psi \in \Omega^s(M)$
(b) $\beta \wedge \psi$ = wedge product of β and ψ . This is a differential form of degree $r + s$, defined by

$$\beta \wedge \psi(p) = \beta(p) \wedge \psi(p)$$

for all $p \in M$

- (c) $\beta(p) \wedge \psi(p)$ = exterior product of the linear forms $\beta(p)$ and $\psi(p)$

~~#definition~~ ~~#differentiable~~ ~~#differential forms~~
~~#pullback~~ ~~#smooth map~~

26. (a) Let $f : M \rightarrow N$ be a smooth map between two surfaces, and $\psi \in \Omega^r(N)$

(b) $f^*\psi \in \Omega^r(M)$ = pullback of ψ through f , defined by

$$f^*\psi(p)(u_1, \dots, u_r) = \psi(f(p))(df(p) \cdot u_1, \dots, df(p) \cdot u_r)$$

for any $p \in M$ and $u_1, \dots, u_r \in T_p M$

27. If $g \in \Omega^0(N) \Rightarrow g \in C^\infty(N)$ and $f^*g = g \circ f : M \rightarrow \mathbb{R}$

#definition #differential forms #dual basis
 #parametrization #tangent space

28. (a) $X : W \rightarrow X(W)$ = parametrization of M , $W \subset \mathbb{R}^n$ is an open set
 (b) For each $p = X(x) \in X(W) \subset M$, $\{dX(x) \cdot e_1, \dots, dX(x) \cdot e_n\}$ is a basis for $T_p M$
 (c) X determines in $X(W) \subset M$, n differential forms of degree 1, $dw_1, \dots, dw_n$
29. (a) $(T_p M)^*$ = dual space to $T_p M$
 (b) For each $p \in X(W)$, $\{dw_1(p), \dots, dw_n(p)\}$ forms a basis for $(T_p M)^*$
 (c) $\{dw_1(p), \dots, dw_n(p)\}$ = dual basis to $\{dX(x) \cdot e_1, \dots, dX(x) \cdot e_n\}$

~~#~~definition ~~#~~differential forms ~~#~~ r -linear form

30. (a) M^n = a surface of dimension n
- (b) $T_p M$ = tangent space to M at p
- (b) For each $p \in M$, the map $\psi : M \rightarrow A_r(T_p M)$ associates the r -linear form $\psi(p) \in A_r(T_p M)$
- (c) ψ = differential form of degree r on M

#definition #dual map #linear map #transpose

31. (a) Let V and W be two vector spaces
- (b) For each linear map $\psi : V \rightarrow W$ we define $\psi^* : V^* \rightarrow W^*$, by
 $\psi^*(f)(u) = f(\psi(u))$ for $f \in V^*$, $u \in W^*$
- (c) ψ^* = the dual map or transpose of ψ

#definition #exterior product #linear forms

32. The exterior product of $\psi \in A_r(V)$ and $\xi \in A_s(V)$, is the map $\psi \wedge \xi \in A_{r+s}(V)$ defined by

$$\psi \wedge \xi = \frac{(r+s)!}{r!s!} \text{Alt}(\psi \otimes \xi)$$

#definition # n -linear form #volume element

33. (a) V = vector space with inner product $\langle, \rangle$
(b) If $\{e_1, \dots, e_n\}$ is an positive orthonormal basis for V
(c) For all $u_1, \dots, u_n \in V \Rightarrow u_j = \sum_{i=1}^n a_{ij} e_i$ and $A = (a_{ij})$
(d) Let us define $\psi(u_1, \dots, u_n) = \det A$, called the volume element of V
34. (a) $\psi \in A_n(V)$
(b) $g = (\langle v_i, v_j \rangle)$ (The Gram matrix)
(c) $g = A^t \cdot A \Rightarrow \det g = (\det A)^2$
(d) $\psi(u_1, \dots, u_n) = \pm \sqrt{\det(\langle v_i, v_j \rangle)} = (\text{sign of } \det A) \sqrt{\det(\langle v_i, v_j \rangle)}$

#definition #permutation #sign of a permutation #transposition

35. P_r = set of all permutations of the set $\{1, 2, \dots, r\}$
36. $\sigma \in P_r$ is a transposition if there is (i, j) in $\{1, 2, \dots, r\}$ that such $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(l) = l \ \forall \ l \neq i, j$
37. If $\sigma \in P_r \rightarrow \sigma = \beta_1 \cdot \dots \cdot \beta_b$, where $\beta_1, \dots, \beta_b$ are transpositions
38. For all $\sigma \in P_r$, $\text{sgn } \sigma = \text{sign of } \sigma$
39. $\text{sgn } \sigma = 1$ if b is even, and $\text{sgn } \sigma = -1$ if b is odd

#definition #pullback # r -linear form

40. (a) Let V and W be two vector spaces
- (b) For each r , the linear map $\psi : V \rightarrow W$ determines $\psi^* : A_r(W) \rightarrow A_r(V)$ defined by

$$\psi^*(\xi)(u_1, \dots, u_r) = \xi(\psi(u_1), \dots, \psi(u_r))$$

for $\xi \in A_r(W)$, $u_1, \dots, u_r \in V$

- (c) $\psi^*(\xi)$ = the pullback of ξ for the space V through ψ

#definition # r -form # r -linear form

41. A map $\psi \in L_r(V)$ is a r -linear form ou r -form if $\psi(u_1, \dots, u_r) = 0$, as long as there are $i \neq j$ such that $u_i = u_j$

#definition # r -linear map

- 42. (a) V vector space of dimension n
 - (b) A map $\psi : V \times \cdots \times V \rightarrow \mathbb{R}$ is said to be r -linear if ψ is linear in each of its variables
- 43. $L_r(V)$ = real vector space of all r -linear maps
- 44. $L_1(V) = E^*$, the dual space to E

#definition #tensor product

45. The tensor product of $\psi \in L_r(V)$ and $\xi \in L_s(V)$ is the map $\psi \otimes \xi \in L_{r+s}(V)$ defined by

$$\psi \otimes \xi(u_1, \dots, u_r, v_1, \dots, v_s) = \psi(u_1, \dots, u_r) \cdot \xi(v_1, \dots, v_s)$$

for any $u_1, \dots, u_r, v_1, \dots, v_s \in V$

46. The tensor product is associative, but it is not commutative

~~#determinant~~ ~~#linear forms~~

47. (a) If $\psi \in A_n(V)$ and $\{u_1, \dots, u_n\}$ is any basis for V
- (b) $\psi(w_1, \dots, w_n) = \det(a_{ij}) \cdot \psi(u_1, \dots, u_n) \quad \forall u_1, \dots, u_n \in V$, where
 $w_i = \sum a_{ij} u_j$

#differentiable #differential forms #exterior product
#properties #wedge product

48. (a) $(\beta \wedge \psi) \wedge \gamma = \beta \wedge (\psi \wedge \gamma)$
(b) $\beta \wedge \psi = (-1)^{rs} \psi \wedge \beta$
(c) $\beta \wedge (\psi + \gamma) = \beta \wedge \psi + \beta \wedge \gamma$
(d) $(\beta + \psi) \wedge \gamma = \beta \wedge \gamma + \psi \wedge \gamma$
(e) $(g \cdot \beta) \wedge (h \cdot \psi) = gh(\beta \wedge \psi)$, for any $g, h \in C^\infty(M)$

#differentiable #differential forms #properties
#pullback #smooth map

49. Let $f : M \rightarrow N$ and $h : N \rightarrow S$ be two smooth maps

(a) $f^*(\beta + \psi) = f^*\beta + f^*\psi$

(b) $f^*(\beta \wedge \psi) = f^*\beta \wedge f^*\psi$

(c) $f^*(h \cdot \psi) = f^*h \cdot f^*\psi$

(d) $(h \circ f)^*\psi = f^*h^*\psi$

#differential forms # r -linear form #vector space

- 50. The set of differential forms of degree r on M is a real vector space.
- 51. A differential form of degree zero is a real function $\psi : M \rightarrow \mathbb{R}$

#exterior product #linear forms #properties

52. $\psi \wedge (\xi + \eta) = \psi \wedge \xi + \psi \wedge \eta$ (distributive over addition)

53. $\psi \wedge (\xi \wedge \eta) = (\psi \wedge \xi) \wedge \eta$ (associativity)

54. If $\psi, \xi \in V^* \Rightarrow \psi \wedge \xi = \psi \otimes \xi - \xi \otimes \psi$

55. $\psi \wedge \xi = -\xi \wedge \psi$

56. $\psi \wedge \psi = 0$

Open Invitation

Please *review* this article, *add* content, co-author, and *join* the **Open Mathematics Collaboration**. Contact alvaroyucra@uft.edu.br.

Open Science

The **latex file** for this paper together with other *supplementary files* are available [5].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [6] following the structure from [4], please cite it accordingly [7]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

Acknowledgement

+ **Center for Open Science**

<https://www.cos.io>

+ **Open Science Framework**

<https://osf.io>

References

- [1] Lee, John M. *Introduction to Smooth Manifolds*. New York: Springer, 2001.
- [2] Lee, Jeffrey M. *Manifolds and Differential Geometry*. Graduate Studies in Mathematics. Vol. 107, 2009.

- [3] Hirsch, Morris W. *Differential Topology*. New York: Springer. Vol. 33, 1976.
- [4] Lobo, Matheus P. “Open Mathematics Knowledge Base.” *OSF Preprints*, 28 Jun. 2020. <https://doi.org/10.31219/osf.io/evq8a>
- [5] Lobo, Matheus P. “Open Journal of Mathematics and Physics (OJMP).” *OSF*, 21 Apr. 2020. <https://doi.org/10.17605/OSF.IO/6HZYP>
- [6] COS. *Open Science Framework*. <https://osf.io>
- [7] Hanco, Alvaro J. Y. “Differential Forms (Open Mathematics Knowledge Base).” *OSF Preprints*, 19 July 2020. <https://doi.org/10.31219/osf.io/g7uqb>

The Open Mathematics Collaboration

Alvaro Julio Yucra Hanco (lead author, alvaroyucra@uft.edu.br)¹
<https://orcid.org/0000-0001-6280-8775>

Matheus Pereira Lobo (mplobo@uft.edu.br)^{1,2}
<https://orcid.org/0000-0003-4554-1372>

¹Federal University of Northern Tocantins (Brazil)

²Universidade Aberta (UAb, Portugal)