There is no uniformly unbiased independence test for densities

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Abstract

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1. Introduction

Is there an uniformly unbiased independence test for densities? This question can be traced back to Hoeffding (1948), who proved the non-existence of uniformly unbiased independence tests based on ranks. As a test that fails to be uniformly unbiased has no power against at least one alternative, uniform unbiasedness is a property any sensible test should satisfy.

In the tradition of Bahadur and Savage (1956) and Romano (2004), Shah and Peters (2018) recently showed a strong impossibility result for conditional independence testing. In this case there is no test that is unbiased against any alternative, a property Shah and Peters (2018) and termed *untestable*. This is not true for unconditional independence testing as there are tests such as Hoeffding's rank test (Hoeffding, 1948) with pointwise asymptotic power equal to 1.

2. The Result

Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of probabilities. Consider the sequence of testing problems

$$H_0 : P^n \in \mathcal{P}^n$$

$$H_a : Q^n \in \mathcal{Q}^n$$

Here P^n is the product of n probability measures P . This corresponds to independent and identical sampling of n points from P . A test with level α is a sequence of sets $(A_1, A_2, \dots)$

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satisfying $P^n(A_n) \geq 1 - \alpha$ for all n . The sets A_n are the *acceptance sets* for H_0 . A test is biased if there for every n is a member $Q \in \mathcal{Q}$ satisfying $Q^n(A_n) \geq 1 - \alpha$. A test is uniformly unbiased if it is not biased. These definitions are straightforward adaptations of the classical definitions found in e.g. Lehmann and Romano (2008).

All mentions of "almost everywhere" are with respect to the Lebesgue measure.

Theorem 1. *There is no uniformly unbiased test of independence of any level for densities on $\mathbb{R}^k$, $k \geq 2$.*

Let $\mathcal{P}$ be the family of independent densities, $\mathcal{Q}$ be the family of dependent densities, and A_n a set satisfying $P^n(A_n) \geq 1 - \alpha$ for all $P \in \mathcal{P}$. Let $I \in \Sigma$ be a hypercube in $\mathbb{R}^k$. Then there is a $Q \in \mathcal{Q}$ that fails to be independent on I , for instance when Q equals the diagonal of I . Choose an $n \in \mathbb{N}$ and take a look at $I^n \cap A_n$, where one of three things can happen.

- (i) The intersection is empty almost everywhere. Now choose a $Q \in \mathcal{Q}$ which equals a $P \in \mathcal{P}$ everywhere except I . Then $Q^n(A_n) \geq 1 - \alpha$, and the test is biased.
- (ii) The intersection equals I^n almost everywhere. Again choose a $Q \in \mathcal{Q}$ which equals some $P \in \mathcal{P}$ everywhere except I . Then $Q^n(A_n) \geq 1 - \alpha$, and the test is biased.
- (iii) The intersection is something else entirely. That is, both $I^n \cap A_n \neq \emptyset$ and $I^n \cap A_n \neq I^n$ are true almost everywhere.

If (i) or (ii) holds the test A_n is biased, so we will have to take a look at case (iii). For the test to have a chance of being unbiased, this condition has to hold for all hypercubes I .

The family $\mathcal{P}$ includes the uniform distribution on I for every hypercube I . A uniform distribution has probability measure $P(B) = \lambda(B \cap I) / \lambda(I)$. Since $P(A_n) \geq 1 - \alpha$ for all $P \in \mathcal{P}$, $\lambda(A_n \cap I) / \lambda(I) \geq 1 - \alpha$. But then $\lambda(A^c) = 0$ by the following lemma.

Lemma 2. *Let A be Lebesgue measurable. If there is a real number $c > 0$ such that $\lambda(A \cap U) \geq c\lambda(U)$ for every hypercube U then A^c has measure 0.*

Proof. By assumption

$$\lambda(A^c \cap U) \leq (1 - c) \lambda(U)$$

for each U . By Lebesgue's density theorem (Rudin, 1986, Theorem 7.10), when U_x is a hypercube centered on x , $\lambda(A^c \cap U_x) / \lambda(U_x)$ converges to 1 as $\lambda(U_x) \rightarrow 0$ for almost all $x \in A^c$. But

$\lambda(A^c \cap U_x) / \lambda(U_x) \leq 1 - c$, so $\lambda(A^c \cap U_x) / \lambda(U_x)$ fails to converge to 1 for all $x \in A^c$. Hence A^c has measure 0. $\square$

If $\lambda(A^c) = 0$ then condition (ii) holds for every I . This implies the test is biased and Theorem 1 follows.

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