Unified Approach of Least Squares Adjustment – an Application in 3D Geomatics

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Abstract

An application is presented in this paper to an important technique of least square (LS) adjustment using the unified approach. The unified approach is considering the contribution of weights to unknown parameters in a geomatics model similar to observations. Accordingly, the adjusted parameters are governed by their assigned a priori weights which is normally based on the instrument limitations and manufacturers specifications.

The aim of this paper, is to spot the light on such advanced LS techniques which is suitable nowadays to handle the wealth of observations collected from modern sensors and navigation systems embedded in the modern mapping systems.

Keywords: Least Squares – unified approach – photogrammetry – ellipsoid of errors

1. Introduction

The conventional LS adjustment techniques in practice and in literature is depending on categorizing different groups of variables which appear in the mathematical model. The measured variables which are normally have a priori variance – covariance are called *observations* while the unknown variables that should be determined are called *unknown parameters* (Mikhail 1976, Ghilani and Wolf 2006).

The rapid development in Geomatics, photogrammetry, computer vision, laser scanning, robotics, and satellite navigation in the last three decades necessitate the development of modified adjustment techniques. These adjustment techniques could handle the integration of different collected data in the mathematical models in a more generic and unified way which is called the *unified approach of least square adjustment* as mentioned in (Mikhail 1976).

Fig.1 shows an example of modern mobile mapping systems MMS equipped with a panoramic camera, satellite navigation system GNSS, a laser scanner and ground penetration radar GPR. This means that the system is able to collect different types of observations to determine different unknowns like:

- 3D positions and orientations of the vehicle sensors from the navigation system (GNSS+IMU) with uncertainties.
- Laser point cloud XYZ of mainly road features with uncertainties.
- Images orientations with uncertainties.

The collected information is aimed to be integrated to deliver highly absolute and relative accuracies of measurements aboveground and underground out of images and point clouds. The processing of these observed redundant data to derive the required quality needs like a unified approach of least squares where the a priori weights is crucial in the adjustment.

Fig.1.(Baker 2016)

The unified LS adjustment depends on the assumption that all the variables in the mathematical model are observations with a priori variance-covariance matrices (or a priori weights). In another word, it assumes the a priori knowledge of the variables weights in an adjustment problem.

To clarify this weighting problem, we can list two kinds of cases as:

- If an observation is given a very low weight or very a priori high error ($w = 0$), then this means that we can allow this observation to move freely during the adjustment and then it is just like an unknown parameter in the conventional adjustment techniques.
- If there is an observation with a very high weight or very low a priori error, then its weight is infinity as $w \rightarrow \infty$. It means that this observation is not allowed to change during the adjustment and behave like a constant.

Between the above two cases there are a lot of cases in adjustment computations and its worth to mention that in this paper we focus on the nonlinear mathematical problem since it is the case in most of the adjustment problems.

2. Derivation of the unified LS adjustment

The derivation is based on the following:

- l as the observations vector, which have a prior cofactor matrix Q and their residuals are v .
- The variables that represent unknowns will be treated as observations in the unified approach and therefore will be assigned as vector x with a priori cofactor matrix Q_{xx} and their corrections are Δ .
- There are n total observations in l and u unknowns in x and Δ .
- The redundancy in observations is $r = n - n_o$ where n_o is the minimum number of observations to solve a problem.

- The total number of equations is $c = r + u$. In the sense of the unified approach mentioned above, the total observations are $n + u$ and then the redundancy in observations is $(n + u - n_o) = r + u = c$
- Accordingly, the total number of observation equations will be $c + u$ where c is available from the geometric or physical condition of the adjustment problem.

The general observation equations (Ghilani and Wolf 2006) will take the matrix form in equation 1 where the weights w are assigned only to observations.

$$Av + B\Delta = F \dots\dots\dots (1)$$

Where

B : matrix of partial derivatives in the model to unknowns.

A : matrix of partial derivatives in the model to observations.

F : vector of constants evaluated using the observed values and approximate unknown values.

Δ : vector of unknown values corrections.

In the sense of the unified LS approach where unknowns are also treated as observations, we add u of new auxiliary observation equations to enable assigning the a priori weights of the unknowns w_{xx} . The added auxiliary equations which are linear are modeled by realizing that at the end of the adjustment we get the corrected unknowns $\hat{x}$ which are similarly treated as adjusted observed variables. In matrix form it means:

$$\hat{x} = x^o + \Delta = l_x + v_x \dots\dots\dots (2)$$

$$v_x - \Delta = F_x = x^o - l_x \dots\dots\dots (3)$$

Where x^o is the vector of approximate values of the unknowns and l_x is vector of the observed unknowns themselves and by merging the general observation equation 1 with equation 3 we get:

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ v_x \end{bmatrix} + \begin{bmatrix} B \\ -I \end{bmatrix} \Delta = \begin{bmatrix} F \\ F_x \end{bmatrix} \dots\dots\dots (4)$$

Or

$$A'v' + B'\Delta' = \dot{F}' \dots\dots\dots (5)$$

This observation equation system will have the cofactor matrix $\dot{Q}$ of weights as

$$\dot{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q_{xx} \end{bmatrix} \rightarrow \dot{Q}^{-1} = \dot{w} = \begin{bmatrix} w & 0 \\ 0 & w_{xx} \end{bmatrix} \dots\dots\dots (6)$$

In (Mikhail 1976), the derivation is continued to observation equation 5 as:

$$\dot{Q}_e = \dot{A}\dot{Q}\dot{A}^t = \begin{bmatrix} AQA^t & 0 \\ 0 & Q_{xx} \end{bmatrix} = \begin{bmatrix} Q_e & 0 \\ 0 & Q_{xx} \end{bmatrix} \dots\dots\dots (7)$$

$$\dot{N} = \dot{B}^t \dot{w} \dot{B} = [B^t w B + w_{xx}] = [N + w_{xx}] \dots\dots\dots (8)$$

$$\dot{t} = \dot{B}^t \dot{w} \dot{F} = B^t w F - w_{xx} F_x = [t - w_{xx} F_x] \dots\dots\dots (9)$$

Accordingly, the normal equations system with the unified approach of least square will be as:

$$[N + w_{xx}]\Delta = [t - w_{xx}F_x] \dots \dots \dots (10)$$

The variance-covariance matrix of unknowns after adjustment is computed as:

$$Q_{\Delta\Delta} = [N + w_{xx}]^{-1} \dots \dots \dots (11)$$

3. Application

To practice the application of the unified approach of least square, two numerical examples in land surveying and photogrammetry are described and solved.

3.1 application in land surveying

Given

Four stations A, B, C, and D, are intersecting to unknown point P by four measured distances, four azimuth angles and one difference in height as given below:

Station	X[m]	Y[m]	Z[m]	side	distances [m]	<i>Azimuth</i> ^o	diff. in Z[m]
A	472742.87	5191874.15	510.56	AP	3.71	267.60635 ^o	2.11
B	472751.59	5191872.49	513.76	BP	6.17	108.09892 ^o	
C	472738.08	5191886.95	518.11	CP	15.90	328.80011 ^o	
D	472715.23	5191852.36	515.39	DP	37.82	234.94600 ^o	
P ^o	472745.87	5191874.29	512.68				

The a priori standard deviations of measurements and unknowns are:

- σ_d of distances are ± 2 mm.
- σ_{Az} of azimuth angles are $\pm 5''$.
- σ_{XYZ} of ± 1 mm for points A, B, C and D.
- σ_P of ± 10 mm for point P.

Required:

Apply the unified LS adjustment to find the adjusted positions of the points A, B, C.D and P?

Solution:

The weight w matrices arranged are as follows as mentioned in equation 6:

$$\underbrace{w_D}_{4 \times 4} = \text{diagonal}(1/.002^2 \dots \dots 1/.002^2)$$

$$\underbrace{w_{AZ}}_{4 \times 4} = \text{diagonal}(1/5''^2 \dots \dots 1/5''^2)$$

$$\underbrace{w_{DH}}_{1 \times 1} = [1/.005^2]$$

$$\underbrace{w_{XYZ}}_{12 \times 12} = \text{diagonal}(1/.001^2 \dots \dots 1/.001^2)$$

$$\underbrace{w_P}_{3 \times 3} = \begin{bmatrix} 1/.01^2 & & \\ & 1/.01^2 & \\ & & 1/.01^2 \end{bmatrix}$$

$$W = \text{diagonal}(w_D \ w_{AZ} \ w_{DH} \ w_{XYZ} \ w_P)$$

It should be noted that we also add 15 linear observation equations ($3 \times \text{number of points } n$) in the sense of the unified approach as illustrated for point A:

$$\begin{aligned}
 F(X_A) &= X_A^o - X_A^a = 0 \\
 F(Y_A) &= Y_A^o - Y_A^a = 0 \\
 F(Z_A) &= Z_A^o - Z_A^a = 0
 \end{aligned}
 \dots\dots\dots (12)$$

Where ***o*** refers to observed and ***a*** for adjusted. The observation equations in matrix form as mentioned in equation 3 can be formed for the points as:

$$\begin{bmatrix} v_{xA} \\ v_{yA} \\ v_{zA} \\ \vdots \\ v_{xP} \\ v_{yP} \\ v_{zP} \end{bmatrix} = \begin{bmatrix} -1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & \ddots & & & \\ & & & & -1 & & \end{bmatrix} \begin{bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Delta_{zA} \\ \vdots \\ \Delta_{xP} \\ \Delta_{yP} \\ \Delta_{zP} \end{bmatrix} - \begin{bmatrix} F_{xA} \\ F_{yA} \\ F_{zA} \\ \vdots \\ F_{xP} \\ F_{yP} \\ F_{zP} \end{bmatrix} \dots\dots\dots (13)$$

For clarity, we list the first two iterations of adjustment as follows:

LS adjustment - Iteration 1

	dxA	dyA	dzA	dxB	dyB	dzB	dxC	dyC	dzC	dxD	dyD	dzD	dxP	dyP	dzP	F
D-AP	-0.81622	-0.03915	-0.57641	0	0	0	0	0	0	0	0	0	0.816221	0.039152	0.576412	0.076
D-BP	0	0	0	0.938559	-0.29611	0.177272	0	0	0	0	0	0	-0.93856	0.296111	-0.17727	0.097
D-CP	0	0	0	0	0	0	-0.49244	0.799837	0.343166	0	0	0	0.492441	-0.79984	-0.34317	0.110
D-DP	0	0	0	0	0	0	0	0	0	-0.81105	-0.58056	0.07173	0.81105	0.580562	-0.07173	0.076
Az-AP	-0.01065	0.221924	0	0	0	0	0	0	0	0	0	0	0.010645	-0.22192	0	0.006
Az-BP	0	0	0	-0.0486	-0.15406	0	0	0	0	0	0	0	0.048604	0.154056	0	0.010
Az-CP	0	0	0	0	0	0	0.050548	0.031121	0	0	0	0	-0.05055	-0.03112	0	0.007
Az-DP	0	0	0	0	0	0	0	0	0	-0.01537	0.021467	0	0.015367	-0.02147	0	0.009
DZ-AP	0	0	1	0	0	0	0	0	0	0	0	0	0	0	-1	-0.009

The normal equation matrices ***N*** and ***t*** are as follows:

								N-1								T	Δ
9.31E-07	7.59E-09	-2.31E-08	3.17E-08	-2.01E-08	6.53E-09	1.65E-08	4.32E-08	1.03E-08	2.03E-08	-3.04E-08	8.51E-11	7.02E-08	-3.24E-08	6.14E-07	-9.79E+04	-0.004	
7.59E-09	3.93E-07	-1.46E-08	5.42E-08	3.19E-07	-7.95E-09	3.80E-08	1.02E-07	2.45E-08	-9.86E-08	1.81E-07	-1.83E-09	-1.16E-07	3.80E-07	-1.39E-08	1.84E+06	-0.052	
-2.31E-08	-1.46E-08	9.72E-07	2.80E-08	-1.21E-08	5.46E-09	-8.15E-09	4.25E-08	1.48E-08	2.91E-09	-1.57E-08	4.88E-10	2.86E-08	-1.23E-08	6.98E-07	-8.90E+03	-0.007	
3.17E-08	5.42E-08	2.80E-08	7.98E-07	-1.33E-07	-2.75E-08	1.27E-07	8.02E-08	6.21E-10	4.16E-08	-2.34E-09	-2.34E-09	1.68E-07	6.12E-08	1.25E-07	-8.88E+05	-0.002	
-2.01E-08	3.19E-07	-1.21E-08	-1.33E-07	4.15E-07	8.86E-09	1.72E-07	1.37E-07	9.53E-09	-2.04E-08	1.26E-07	-4.09E-09	9.67E-08	3.28E-07	-2.22E-07	-2.90E+06	-0.119	
6.53E-09	-7.95E-09	5.46E-09	-2.75E-08	8.86E-09	9.95E-07	1.25E-08	6.39E-09	-4.09E-10	8.26E-09	-7.23E-09	-1.80E-10	2.36E-08	-7.25E-09	3.35E-08	4.37E+03	0.006	
1.65E-08	3.80E-08	-8.15E-09	1.27E-07	1.72E-07	1.25E-08	6.99E-07	-1.60E-07	7.80E-09	1.53E-07	-5.05E-08	-6.83E-09	4.96E-07	6.08E-08	-5.34E-07	5.95E+05	0.099	
4.32E-08	1.02E-07	4.25E-08	8.02E-08	1.37E-07	6.39E-09	-1.60E-07	7.45E-07	-4.88E-08	3.52E-08	1.57E-08	-2.71E-09	1.78E-07	1.10E-07	2.59E-07	4.00E+05	0.064	
1.03E-08	2.45E-08	1.48E-08	6.21E-10	9.53E-09	-4.09E-10	7.80E-09	-4.88E-08	9.83E-07	-1.83E-08	1.45E-08	4.63E-10	-3.97E-08	2.24E-08	1.83E-07	1.07E+04	0.001	
2.03E-08	-9.86E-08	2.91E-09	4.16E-08	-2.04E-08	8.26E-09	1.53E-07	3.52E-08	-1.83E-08	7.83E-07	8.48E-08	9.17E-09	2.58E-07	-8.87E-08	-2.01E-07	-2.64E+05	-0.093	
-3.04E-08	1.81E-07	-1.57E-08	-2.34E-09	1.26E-07	-7.23E-09	-5.05E-08	1.57E-08	1.45E-08	8.48E-08	6.75E-07	8.65E-09	-1.57E-07	1.78E-07	-2.24E-08	3.32E+05	0.107	
8.51E-11	-1.83E-09	4.88E-10	-2.34E-09	-4.09E-09	-1.80E-10	-6.83E-09	-2.71E-09	4.63E-10	9.17E-09	8.65E-09	9.99E-07	-8.52E-09	-2.25E-09	1.27E-08	1.50E+03	0.001	
7.02E-08	-1.16E-07	2.86E-08	1.68E-07	9.67E-08	2.36E-08	4.96E-07	1.78E-07	-3.97E-08	2.58E-07	-1.57E-07	-8.52E-09	7.59E-07	-8.58E-08	-3.98E-07	6.55E+05	0.066	
-3.24E-08	3.80E-07	-1.23E-08	6.12E-08	3.28E-07	-7.25E-09	6.08E-08	1.10E-07	2.24E-08	-8.87E-08	1.78E-07	-2.25E-09	-8.58E-08	3.82E-07	-6.16E-08	3.31E+05	-0.071	
6.14E-07	-1.39E-08	6.98E-07	1.25E-07	-2.22E-07	3.35E-08	-5.34E-07	2.59E-07	1.83E-07	-2.01E-07	-2.24E-08	1.27E-08	-3.98E-07	-6.16E-08	7.33E-06	-7.65E+03	-0.061	

LS adjustment - Iteration 2

								B								F	
-0.82915168	-0.03262053	-0.55807	0	0	0	0	0	0	0	0	0	0	0.82915	0.03262	0.55807	0.0465	
0	0	0	0.93331	-0.30506	0.18941	0	0	0	0	0	0	0	-0.93331	0.30506	-0.18941	0.1474	
0	0	0	0	0	0	0	-0.48664	0.80278	0.34457	0	0	0	0.48664	-0.80278	-0.34457	0.0108	
0	0	0	0	0	0	0	0	0	0	-0.81462	-0.57533	0.07332	0.81462	0.57533	-0.07332	0.0410	
-0.00881024	0.223939462	0	0	0	0	0	0	0	0	0	0	0	0.00881	-0.22394	0	-0.0025	
0	0	0	-0.05037	-0.15411	0	0	0	0	0	0	0	0	0.05037	0.15411	0	0.0000	
0	0	0	0	0	0	0.05037	0.03053	0	0	0	0	0	-0.05037	-0.03053	0	0.0004	
0	0	0	0	0	0	0	0	0	0	-0.01522	0.02155	0	0.01522	-0.02155	0	0.0031	
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	-1	-0.0255	
								N-1								T	Δ
9.26E-07	3.20E-09	-2.39E-08	3.40E-08	-1.78E-08	7.29E-09	1.88E-08	4.47E-08	1.05E-08	2.02E-08	-2.98E-08	5.46E-11	7.20E-08	-3.03E-08	6.13E-07	3.46E+04	0.000	
3.20E-09	3.94E-07	-1.40E-08	5.72E-08	3.18E-07	-8.59E-09	3.84E-08	1.02E-07	2.46E-08	-9.77E-08	1.82E-07	-1.87E-09	-1.15E-07	3.82E-07	-1.64E-08	-8.87E+05	-0.006	
-2.39E-08	-1.40E-08	9.72E-07	2.89E-08	-1.23E-08	6.03E-09	-7.76E-09	4.19E-08	1.46E-08	2.55E-09	-1.55E-08	5.03E-10	2.79E-08	-1.21E-08	6.83E-07	4.59E+03	0.000	
3.40E-08	5.72E-08	2.89E-08	7.96E-07	-1.38E-07	-2.91E-08	1.27E-07	8.11E-08	1.23E-09	4.08E-08	-1.39E-09	-2.39E-09	1.68E-07	6.32E-08	1.42E-07	3.58E+04	0.002	
-1.78E-08	3.18E-07	-1.23E-08	-1.38E-07	4.18E-07	9.64E-09	1.73E-07	1.34E-07	9.21E-09	-1.89E-08	1.26E-07	-4.20E-09	9.97E-08	3.27E-07	-2.33E-07	1.14E+05	0.008	
7.29E-09	-8.59E-09	6.03E-09	-2.91E-08	9.64E-09	9.94E-07	1.30E-08	6.82E-09	-3.27E-10	8.61E-09	-7.79E-09	-1.86E-10	2.48E-08	-8.00E-09	4.00E-08	3.75E+02	0.000	
1.88E-08	3.84E-08	-7.76E-09	1.27E-07	1.73E-07	1.30E-08	6.96E-07	-1.61E-07	7.21E-09	1.53E-07	-5.11E-08	-6.99E-09	4.96E-07	5.76E-08	-5.43E-07	-6.43E+04	-0.004	
4.47E-08	1.02E-07	4.19E-08	8.11E-08	1.34E-07	6.82E-09	-1.61E-07	7.47E-07	-4.87E-08	3.36E-08	1.58E-08	-2.69E-09	1.73E-07	1.08E-07	2.71E-07	-4.00E+04	-0.004	
1.05E-08	2.46E-08	1.46E-08	1.23E-09	9.21E-09	-3.27E-10	7.21E-09	-4.87E-08	9.83E-07	-1.85E-08	1.47E-08	4.87E-10	-4.00E-08	2.29E-08	1.88E-07	-2.42E+01	0.000	
2.02E-08	-9.77E-08	2.55E-09	4.08E-08	-1.89E-08	8.61E-09	1.53E-07	3.36E-08	-1.85E-08	7.84E-07	8.39E-08	9.41E-09	2.56E-07	-8.95E-08	-2.08E-07	1.64E+03	0.002	
-2.98E-08	1.82E-07	-1.55E-08	-1.39E-09	1.26E-07	-7.79E-09	-5.11E-08	1.58E-08	1.47E-08	8.39E-08	6.74E-07	8.78E-09	-1.57E-07	1.80E-07	-2.15E-08	-4.41E+02	0.001	
5.46E-11	-1.87E-09	5.03E-10	-2.39E-09	-4.20E-09	-1.86E-10	-6.99E-09	-2.69E-09	4.87E-10	9.41E-09	8.78E-09	9.99E-07	-8.72E-09	-2.24E-09	1.34E-08	6.89E+00	0.000	
7.20E-08	-1.15E-07	2.79E-08	1.68E-07	9.97E-08	2.48E-08	4.96E-07	1.73E-07	-4.00E-08	2.56E-07	-1.57E-07	-8.72E-09	7.56E-07	-8.91E-08	-4.08E-07	-7.76E+03	0.005	
-3.03E-08	3.82E-07	-1.21E-08	6.32E-08	3.27E-07	-8.00E-09	5.76E-08	1.08E-07	2.29E-08	-8.95E-08	1.80E-07	-2.24E-09	-8.91E-08	3.84E-07	-5.70E-08	8.13E+05	0.005	
6.13E-07	-1.64E-08	6.83E-07	1.42E-07	-2.33E-07	4.00E-08	-5.43E-07	2.71E-07	1.88E-07	-2.08E-07	-2.15E-08	1.34E-08	-4.08E-07	-5.70E-08	7.49E-06	-564.60411	-0.009	

The adjustment iterations continue until reach negligible values ($<.0001$) as shown in Fig. 2.

Fig.2. the adjustment iterations of the five points A, B, C, D and P.

The final adjusted points with their coordinates are as follows:

	X[m]	Y[m]	Z[m]	σ_X	σ_Y	σ_Z
A	472,742.87	5,191,874.10	510.55	0.09	0.06	0.09
B	472,751.59	5,191,872.38	513.77	0.08	0.06	0.09
C	472,738.18	5,191,887.00	518.11	0.08	0.08	0.09
D	472,715.14	5,191,852.47	515.39	0.08	0.07	0.09
P	472,745.94	5,191,874.22	512.62	0.08	0.06	0.25

Fig. 3 illustrates the ellipsoid of errors of the adjusted points.

Fig.3. Ellipsoid of errors of the adjusted points with σ_P of 10mm.

To illustrates the effect of weighting on the final adjusted points, we take a high weight of P as:

- σ_{XYZ} of $\pm 1\text{mm}$ for points A, B, C and D.
- σ_{XYZ} of $\pm 1\text{mm}$ for point P.

After running the unified LS adjustment with these weights, we get the following results as illustrated in Fig.4:

	X[m]	Y[m]	Z[m]	σ_X	σ_Y	σ_Z
A	472,742.87	5,191,874.12	510.56	0.09	0.05	0.09
B	472,751.59	5,191,872.39	513.77	0.08	0.05	0.09
C	472,738.16	5,191,887.00	518.11	0.07	0.08	0.09
D	472,715.13	5,191,852.49	515.39	0.08	0.07	0.09
P	472,745.90	5,191,874.24	512.68	0.06	0.05	0.09

Fig.4. Ellipsoid of errors of the adjusted points with σ_P of 1mm.

It is obvious how the high weight assigned to P is reflected in the unified LS adjustment where the confidence limits are much improved (smaller ellipsoid of errors) in its adjusted position compared to the previous run shown in Fig.3.

3.2 application in photogrammetry

A smartphone image is taken with the following settings:

- focal length = 4.15 mm.
- pixel size = 0.0015mm
- image format = 2448*3264 pixels.
- neglected lens distortion.
- Five control points are imaged and their XYZ coordinates are given with their pixel and transformed coordinates as follows:

	X[m]	Y[m]	Z[m]	Ln_{pixel}	Sn_{pixel}	x [mm]	y [mm]
A	145899.21	424106.18	7.15	303	416	-0.433	0.442
B	145904.39	424114.19	7.16	2709	490	-1.398	1.845
C	145899.71	424106.94	4.65	540	1073	2.345	-0.280
D	145904.55	424114.43	5.31	2770	1816	2.253	1.732
E	145902.38	424111.14	3.63	939	1340	-1.039	0.848

- Initial exterior orientation parameters are given as follows where angles are in radian:

Omega	phi	kappa	Xo[m]	Yo[m]	Zo[m]
1.57	1.57	0	145902.05	424110.58	5.58

Required:

Apply the unified LS adjustment to find the adjusted image orientation if you consider the following a priori standard deviations of measurements and unknowns:

- σ_{xy} of image coordinates are $\pm 0.003\text{mm}$.
- σ_{angle} of orientation angles are $\pm 5'$.
- $\sigma_{X_oY_oZ_o}$ of $\pm 5\text{cm}$ for camera coordinates.
- σ_{XYZ} of $\pm 1\text{mm}$ for control points A, B, C, D and E .

Solution

The observation equations necessary to solve this problem is divided into three types as:

- Collinearity equations ($2 \times \text{no. of points} = 10$) with weight w_{xy} (Luhmann T., Robson S. et al. 2006). For point A, the collinearity equations is:

$$\begin{aligned} F_{xA} &= -f \frac{m_{11}(X_A - X_o) + m_{12}(Y_A - Y_o) + m_{13}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} - x_A \\ F_{yA} &= -f \frac{m_{21}(X_A - X_o) + m_{22}(Y_A - Y_o) + m_{23}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} - y_A \end{aligned} \quad (14)$$

where

f : focal length in mm.

m 's: rotation matrix elements.

X_o, Y_o, Z_o are the camera coordinates in object coordinate system.

X_A, Y_A, Z_A : coordinates of point A in object system.

x_a, y_a : observed image coordinates in mm.

$$w_{xy} = \text{diagonal} \left(\frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \quad \frac{1}{.003^2} \right)$$

- Six observation equations of the exterior orientation parameters with weight

$$w_{exterior} = \begin{bmatrix} w_{angle} & \\ & w_{X_oY_oZ_o} \end{bmatrix} = \text{diagonal} \left(\frac{1}{5'^2} \quad \frac{1}{5'^2} \quad \frac{1}{5'^2} \quad \frac{1}{.05^2} \quad \frac{1}{.05^2} \quad \frac{1}{.05^2} \right)$$

$$\begin{bmatrix} v_w \\ v_\phi \\ v_k \\ \vdots \\ v_{X_o} \\ v_{Y_o} \\ v_{Z_o} \end{bmatrix} = \begin{bmatrix} -1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & -1 \end{bmatrix} \begin{bmatrix} \Delta_w \\ \Delta_\phi \\ \Delta_k \\ \vdots \\ \Delta_{X_o} \\ \Delta_{Y_o} \\ \Delta_{Z_o} \end{bmatrix} - \begin{bmatrix} F_w \\ F_\phi \\ F_k \\ \vdots \\ F_{X_o} \\ F_{Y_o} \\ F_{Z_o} \end{bmatrix}$$

Like equation 13, the observation equations of the control points ($3 \times \text{no. of points} = 15$ equations) with weight w_{XYZ} are:

$$\begin{bmatrix} v_{xA} \\ v_{yA} \\ v_{zA} \\ \vdots \\ v_{xE} \\ v_{yE} \\ v_{zE} \end{bmatrix} = \begin{bmatrix} -1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & -1 \end{bmatrix} \begin{bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Delta_{zA} \\ \vdots \\ \Delta_{xE} \\ \Delta_{yE} \\ \Delta_{zE} \end{bmatrix} - \begin{bmatrix} F_{xA} \\ F_{yA} \\ F_{zA} \\ \vdots \\ F_{xE} \\ F_{yE} \\ F_{zE} \end{bmatrix}$$

$$w_{XYZ} = \text{diagonal} \left(\frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \quad \frac{1}{.001^2} \right)$$

The total weight matrix will be:

$$w_{31 \times 31} = \begin{bmatrix} w_{xy} & & \\ & w_{exterior} & \\ & & w_{XYZ} \end{bmatrix}$$

The least square adjustment matrices are:

[illegible]

The normal equation matrices are:

N-1																			T		Δ	
1.07E-06	-4.92E-11	1.04E-06	8.46E-08	-2.41E-08	-2.42E-08	-2.64E-09	7.14E-09	-2.06E-08	3.64E-09	1.95E-08	2.30E-08	8.40E-09	8.05E-09	-3.06E-08	2.39E-09	2.51E-08	-2.17E-09	5.05E-09	4.64E-09	3.04E-08	1.382E+06	0.194
-4.92E-11	1.60E-07	-4.92E-11	3.83E-07	9.95E-08	5.13E-08	9.77E-10	4.10E-08	-2.76E-08	3.48E-08	8.58E-08	-3.35E-08	3.97E-08	6.73E-09	8.63E-08	1.94E-08	1.89E-07	-2.70E-09	-2.52E-08	8.24E-08	2.26E-08	1.216E+07	0.281
1.04E-06	1.92E-11	1.07E-06	8.46E-08	2.41E-08	-2.42E-08	-2.64E-09	7.14E-09	-2.06E-08	3.64E-09	1.95E-08	2.30E-08	8.40E-09	8.05E-09	-3.06E-08	2.39E-09	2.51E-08	-2.17E-09	5.05E-09	4.64E-09	3.04E-08	1.382E+06	0.194
8.46E-08	-9.87E-07	8.46E-08	5.74E-06	6.43E-07	-9.88E-07	-3.48E-07	5.94E-07	1.87E-07	6.14E-08	1.84E-07	8.08E-09	5.24E-07	-5.12E-07	6.14E-07	5.70E-08	8.00E-07	2.50E-07	7.30E-09	1.22E-07	1.85E-07	1.811E+06	8.603
-2.41E-08	1.93E-08	-2.41E-08	4.33E-07	4.35E-07	7.84E-08	-5.23E-08	9.12E-08	1.14E-08	3.52E-08	8.36E-08	5.63E-08	5.45E-08	-1.32E-07	1.13E-07	6.15E-08	5.94E-07	3.56E-08	1.20E-08	-9.90E-08	6.66E-08	1.062E+07	0.865
-2.42E-08	5.13E-08	-2.42E-08	9.88E-07	7.84E-08	7.15E-07	6.75E-08	1.03E-07	1.41E-07	5.12E-08	-5.31E-08	1.11E-07	4.05E-08	8.79E-08	1.74E-08	4.69E-08	1.06E-07	-5.43E-08	-3.03E-08	-3.15E-08	1.87E-07	-3.290E+07	-2.480
2.64E-09	9.77E-09	-2.64E-09	3.48E-07	8.24E-08	6.75E-08	9.61E-07	6.92E-08	1.96E-09	5.32E-09	-5.62E-09	8.15E-09	3.95E-08	5.38E-08	2.46E-08	4.82E-08	2.07E-08	4.59E-08	-1.29E-07	3.74E-10	5.86E-09	9.192E+09	-0.490
-7.14E-09	1.40E-08	-7.14E-09	5.94E-07	9.12E-08	1.03E-07	-6.92E-08	9.75E-07	2.25E-08	1.01E-08	-1.30E-08	1.28E-08	6.93E-09	9.00E-08	4.89E-08	7.99E-09	4.79E-08	-2.80E-08	-2.33E-09	1.77E-10	1.12E-08	1.610E+06	-0.058
-2.06E-08	2.76E-08	-2.06E-08	1.87E-07	1.14E-08	-1.41E-07	-1.96E-09	2.25E-08	8.41E-07	2.47E-09	-2.52E-08	1.36E-08	4.55E-08	-4.28E-08	4.47E-08	5.15E-09	9.45E-08	1.35E-07	-3.83E-10	4.07E-09	6.68E-09	-1.341E+07	0.247
3.64E-09	8.46E-08	3.64E-09	6.54E-08	3.12E-09	-3.12E-08	5.52E-09	1.01E-08	2.47E-09	2.70E-07	2.38E-08	4.87E-08	4.73E-09	1.51E-08	9.21E-08	3.78E-09	6.26E-09	2.55E-08	1.60E-08	-4.27E-08	2.60E-08	3.080E+05	-0.039
1.95E-08	8.58E-08	1.95E-08	1.84E-07	8.36E-08	5.31E-08	-5.62E-09	1.30E-08	2.52E-08	9.38E-08	9.20E-07	-6.76E-09	1.73E-09	3.60E-09	-8.89E-09	2.32E-09	6.33E-09	1.94E-08	-1.87E-08	9.52E-08	2.53E-08	8.172E+05	0.150
2.30E-08	-3.57E-08	2.30E-08	8.08E-09	-5.63E-08	-1.11E-07	8.15E-09	1.28E-08	-1.36E-08	4.87E-08	-6.76E-08	8.85E-07	8.40E-08	3.66E-08	-2.75E-08	1.03E-08	1.89E-08	7.79E-08	2.18E-08	-2.38E-08	7.77E-08	7.331E+04	0.026
-8.40E-09	3.97E-08	-8.40E-09	5.34E-07	-6.45E-08	4.05E-08	3.95E-09	6.93E-08	4.55E-09	4.33E-09	-1.76E-08	8.40E-09	9.55E-07	-6.35E-08	-2.47E-08	1.03E-08	1.29E-08	6.64E-08	-3.35E-09	8.91E-05	5.50E-05	1.298E+06	0.075
8.05E-09	6.73E-09	8.05E-09	5.12E-07	1.32E-07	8.79E-08	5.38E-08	9.00E-08	-4.28E-08	4.73E-08	1.51E-08	3.60E-09	3.66E-08	6.35E-08	8.62E-07	2.70E-08	-8.44E-09	4.12E-08	-3.89E-08	2.95E-09	2.94E-08	1.704E+06	0.050
-3.06E-08	6.73E-08	-3.06E-08	6.14E-07	1.13E-07	-1.74E-08	2.46E-08	8.90E-08	4.47E-08	-9.21E-08	-8.89E-09	-2.75E-08	4.47E-08	2.70E-08	2.05E-07	2.09E-08	8.45E-08	1.14E-09	1.15E-08	1.75E-08	3.66E-08	8.923E+05	0.114
2.39E-09	1.94E-08	2.39E-09	5.70E-08	6.15E-08	-4.69E-08	-4.82E-09	-7.99E-09	4.51E-09	3.78E-08	2.32E-09	1.03E-08	4.16E-08	-8.44E-09	2.89E-09	9.93E-07	1.83E-08	5.14E-08	4.24E-09	-4.14E-09	1.57E-08	-4.503E+06	-0.062
-2.51E-08	-1.89E-07	-2.51E-08	8.00E-07	6.94E-07	-1.06E-07	2.07E-08	4.79E-08	9.45E-08	6.26E-09	-6.33E-09	-1.89E-08	1.29E-08	4.12E-08	8.05E-08	1.83E-08	8.59E-07	3.36E-08	-1.95E-08	5.77E-08	2.48E-08	9.90E-08	0.068
-2.17E-09	-2.70E-09	-2.17E-09	2.50E-07	5.26E-09	-5.43E-07	-4.29E-08	2.80E-08	1.35E-07	4.29E-08	1.55E-07	7.79E-08	2.64E-08	-3.89E-08	1.14E-07	-5.14E-08	3.36E-08	5.50E-07	2.07E-08	1.77E-08	1.23E-07	-3.324E+07	-0.517
5.05E-09	-2.52E-08	5.05E-09	7.30E-09	1.30E-08	-3.03E-08	-1.29E-09	-2.33E-08	-3.83E-10	1.60E-08	-1.87E-08	2.18E-08	3.35E-09	2.95E-09	-1.19E-08	4.54E-08	1.95E-08	2.07E-08	9.84E-07	3.76E-08	3.03E-08	1.667E+05	0.059
4.64E-09	8.24E-08	4.64E-09	1.22E-07	-9.20E-08	-3.15E-08	5.74E-10	1.77E-07	-1.07E-09	-4.27E-08	9.52E-08	-2.38E-08	8.91E-09	2.94E-07	1.75E-08	4.14E-08	5.77E-07	1.77E-08	3.76E-08	8.44E-07	8.33E-08	-1.068E+06	-0.210
3.04E-08	-2.66E-08	3.04E-08	1.85E-07	6.26E-08	-1.87E-07	-3.86E-09	-1.12E-08	-6.86E-09	2.60E-08	2.53E-08	7.77E-05	5.00E-09	1.80E-08	-3.66E-08	1.57E-08	-2.48E-08	1.23E-07	-3.03E-08	-7.38E-08	7.43E-07	-4.927E+05	0.122

LS adjustment - iteration 2

[illegible]

The iterations continue until reach negligible values and becoming stable after five iterations as illustrated in Fig.5

Fig.5. the corrected exterior orientation parameters of the image.

The final adjusted exterior orientation parameters of the image are:

$X_o[m]$	$Y_o[m]$	$Z_o[m]$	$\omega[deg.]$	$\phi[deg.]$	$\kappa[deg.]$
145909.624	424112.864	4.347	101.8807	98.5428	10.3834

To clarify the weight effect on the adjusted camera coordinates and its accuracy, we applied two cases when the a priori standard deviation of control points is 1mm and 5mm as follows:

$\sigma_{XYZ} = 1mm$			$\sigma_{XYZ} = 5mm$		
$\sigma_{X_o}[m]$	$\sigma_{Y_o}[m]$	$\sigma_{Z_o}[m]$	$\sigma_{X_o}[m]$	$\sigma_{Y_o}[m]$	$\sigma_{Z_o}[m]$
0.078	0.081	0.041	0.128	0.115	0.063

Fig.6. Ellipsoid of errors of the adjusted camera position and the control points.

Left) adjustment with σ_{XYZ} of 1mm. right) adjustment with σ_{XYZ} of 5mm.

These two weight settings illustrate the effect of the a priori weights of control points on the adjusted camera coordinates. When using highly trusted control points (smaller σ_{XYZ}) we get higher trusted camera coordinates (Fig.6a) while using less trusted control points (larger σ_{XYZ}) results in less trusted camera coordinates (Fig.6b).

4. Conclusions

In this review paper, the unified approach of least square is presented and evaluated through the application of two tests. The two computational tests showed the effectiveness of such adjustment approach to handle the multi-weighted geomatical problems where all the variables in the model are treated as observations. The effect of the different weights on the final adjusted unknowns are also tested and visualized in Figures 4 and 6.

5. References

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