1 An Optimal Transaction Intervals for Portfolio
2 Selection Problem
3 with Bullet Transaction Cost

4 Effendi Syahril

5 -

6 Department of Mathematics,
7 Faculty of Mathematics and Natural Sciences,
8 Bogor Agricultural University
9 Jl. Raya Pajajaran, Kampus IPB Baranangsiang, Bogor, Indonesia

10 **Abstract**

11 This paper discusses an optimal transaction interval for a con-
12 sumption and investment decision problem for an individual who has
13 available a riskless asset paying fixed interest rate and a risky asset
14 driven by Brownian motion price fluctuations. The individual ob-
15 serves current wealth when making transactions, that transactions
16 incur costs, and that decisions to transact can be made at any time
17 based on all current information. The transactions costs is fixed for
18 every transaction, regardless of amount transacted. In addition, the
19 investor is charged a fixed fraction of total wealth as management fee.
20 The investor's objective is to maximize the expected utility of con-
21 sumption over a given horizon. The problem faced by the investor is
22 formulated in a stochastic discrete-continuous-time control problem.
23 An optimal transaction interval for the inverstor is derived.

24 *Key words:* Transaction intervals, investment strategy, bullet (fixed)
25 transaction cost, continuous-discrete-time, stochastic optimal control
26 problem

27 1 Introduction

28 The publication of Merton's seminal work, see Merton(1971), has started the
29 application of stochastic optimal control and stochastic calculus techniques
30 to the area of finance. Merton (1971, 1990) studied the behaviour of a single
31 agent acting as a market price-taker who seeks to maximize expected utility
32 of consumption. The utility function of the agent was assumed to be a
33 power function, and the market was assumed to comprise a risk-free asset
34 with constant rate of return and one or more stocks, each with constant
35 mean rate of return and volatility. The only information available to the
36 agent were current prices of the assets. There were no transaction costs. It
37 was also assumed that the assets were divisible. In this idealized setting,
38 Merton was able to derive a closed-form solution to the stochastic optimal
39 control problem faced by the agent.

40 Several authors have made contributions to the stochastic optimal control
41 and stochastic calculus analyses of the Merton's model. To mention a few
42 among them are Constantinides (1979, 1986), Cox and Huang (1989), Davis
43 and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991),
44 Lelands (1985), Magill and Constantinides (1976).

45 The application of transaction costs to Merton's model was first accomplished
46 by Magill and Constantinides (1976). Several authors then have published
47 a number of works on Merton's model with transaction costs. To mention a
48 few, they are Constantinides (1979, 1986), Davis and Norman (1990), Duffie
49 and Sun (1990), Dumas and Luciano (1991), Lelands (1985). Duffie and
50 Sun (1990) treated the proportional transaction costs with different formu-
51 lation to others, which they call *discrete-continuous-time formulation*. Their
52 formulation assumes that an investor observes current wealth when making
53 transaction, and decisions to transact can be made at any time, but without
54 no costs. They treated general linear transaction costs of the form $aW_{\tau_n} + b$,
55 with W_{τ_n} denotes the amount of wealth transacted, and a and b are
56 non-negatives. Based on work of Duffie and Sun (1990), Syahril (2003a)
57 re-writes *discrete-continuous-time* formulation of Merton's model with fixed
58 transaction cost, and Syahril (2003b) derives investment strategies for a given
59 transaction intervals. This paper investigates an optimal transaction inter-
60 vals for portfolio selection problem with fixed transaction cost.

61 2 Formulation of the Model

62 Formulation of the problem as in Syahril (2003a, 2003b)

63 2.1 Securities Market and Transactions

64 It is assumed that a complete probability space $(\Omega, \mathcal{F}, P)$ is given. In
 65 addition, it is assumed that a *filtration* $\{\mathcal{F}_t : t \geq 0\}$ is also given. By a
 66 filtration is meant a family of σ -algebras $\{\mathcal{F}_t : t \geq 0\}$ which is increasing :
 67 $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. It is assumed that the one-dimensional standard
 68 Brownian motion $B = \{B_t : t \geq 0\}$ is given on a given filtered probability
 69 space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, P)$.

70 There are two securities available in the economy to an investor. One is a
 71 riskless security with fixed interest rate r , and the other is a risky security
 72 whose price is a geometric Brownian motion with expected rate of return α
 73 and rate of return variation σ^2 . At time $t \geq 0$, the price processes $\{P_0(t)\}$ of
 74 the riskless security satisfy a deterministic differential equation

$$dP_0(t) = rP_0(t)dt, \quad (1)$$

75 while the price processes $\{P_1(t)\}$ of the risky security satisfy a stochastic
 76 differential equation

$$dP_1(t) = \alpha P_1(t)dt + \sigma P_1(t)dB_t. \quad (2)$$

77 Money is available for the investor in the economy as a medium of exchange
 78 and numeraire. Only money is exchangable for consumption. Let M_t de-
 79 notes money holdings at time t . The investor is assumed to receive no fur-
 80 ther income from noncapital sources, and starts with the initial stock of
 81 money $M_0 = 0$. Trading opportunities are available continuously in time,
 82 but with costs. Transaction costs are incurred when information is processed
 83 and a portfolio transaction is made. There are two forms of transaction
 84 costs: portfolio management fees and withdrawal costs. The investor pays
 85 a fraction $\varepsilon > 0$ of the total wealth in the securities at the beginning
 86 of each interval as a portfolio management fee. The portfolio management
 87 fee is meant to include the cost of adjusting the portfolio and the cost of
 88 processing information. In this paper, transaction costs is the costs which
 89 incurs during withdrawing wealth from the portfolio. The transaction costs
 90 is a fixed for every transaction, regardless of amount of wealth transacted.
 91 Then the total transaction costs function is of the form $b + \varepsilon(X_{\tau_n} - W_{\tau_n})$,
 92 where X_{τ_n} is the total wealth at time τ_n before transaction. Filtration $(\mathcal{F}_t)$
 93 defined by $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$, will be interpreted as information available
 94 up to time t . Given the structure of transaction costs, consumption and
 95 investment decisions are made at intervals. During each interval there is no
 96 transaction. All dividends of risky security are re-invested continually in the
 97 risky security, and all interest income is re-invested continually in the riskless
 98 security.

99 The investor chooses instants of time at which to process information and
 100 make consumption and investment decisions. In other words, information
 101 is available continuously through the filtration $\{\mathcal{F}_t : t \geq 0\}$. The investor
 102 receives information via controllable filtration

$$\mathbf{H} = \{\mathcal{H}_t : t \geq 0\} \text{ with } \mathcal{H}_t = \mathcal{F}_t, t \in [\tau_n, \tau_{n+1}),$$

103 where τ_n is a $\mathcal{H}_{\tau_{n-1}}$ -measurable stopping time at which the n -th transaction
 104 occurs. The filtration $\mathbf{H}$ is controllable in the sense that the investor is al-
 105 lowed to choose any sequence $\tau = \{\tau_n : n = 1, 2, 3, \dots\}$ of such transaction
 106 times with $\tau_1 \equiv 0$. Let $T = \{T_n = \tau_{n+1} - \tau_n : n = 1, 2, 3, \dots\}$ denotes the
 107 corresponding sequence of transaction intervals. Finding an optimal stop-
 108 ping policy τ is clearly equivalent to finding an optimal transaction interval
 109 policy T .

110 2.2 Formulation of The Model

111 Let the consumption space $\mathcal{C}$ for the investor consists of positive $\mathbf{H}$ -adapted
 112 consumption processes $\mathbf{C} = \{C_t : t \geq 0\}$ satisfying $\int_0^t C_s ds < \infty$ almost
 113 surely for all $t \geq 0$, and

$$E\left[\int_0^\infty e^{-\delta t} u(C_t) dt\right] < \infty, \quad (3)$$

114 where E denotes the *expected value function*, with respect to P , δ is a
 115 strictly positive scalar discount factor and the utility function u , is one of
 116 the HARA (hyperbolic absolute risk-aversion) type function, as defined in
 117 Merton (1971). We take u as given by

$$u(C) = \frac{1}{\gamma} C^\gamma, \quad 0 < \gamma < 1. \quad (4)$$

118 Let $\tau = \{\tau_n : n = 1, 2, 3, \dots\}$ be sequence of transaction times with $\tau_1 \equiv 0$.
 119 Let $T = \{T_n = \tau_{n+1} - \tau_n, n = 1, 2, 3, \dots\}$ be the sequence of corresponding
 120 transaction intervals. Let $W = \{W_{\tau_n} : n = 1, 2, 3, \dots\}$ be the sequence of
 121 money withdrawal processes, and $V = \{V_{\tau_n} : n = 1, 2, 3, \dots\}$ be the sequence
 122 of investment for the risky security.

123 Let $\mathcal{T}$ denote the space of sequences of strictly positive transaction intervals,
 124 $\mathcal{W}$ the space of positive $\mathbf{H}$ -adapted money withdrawal processes, and $\mathcal{V}$ the
 125 space of $\mathbf{H}$ -adapted investment processes for the risky security. Let $\mathcal{U} =$
 126 $\mathcal{T} \times \mathcal{W} \times \mathcal{V} \times \mathcal{C}$.

127 **Definition 2.1** A budget policy is a quadruplet $(T, W, V, C) \in \mathcal{U}$.

128 We characterize budget feasible policies as follows. Let $\mathcal{U}$ denotes a class of
 129 budget policies. Given a policy $(T, W, V, C) \in \mathcal{U}$, then the money holding at
 130 any time t is defined by

$$M_t = \sum_{\{n: \tau_n \leq t\}} [W_{\tau_n} - b] - \int_0^t C_s ds, \quad (5)$$

131 Let X_{τ_n} denotes the total wealth invested in the securities at time τ_n , before
 132 the n th transaction. Let W_{τ_n} denotes the amount of money withdrawn at
 133 time τ_n from the total wealth X_{τ_n} , and V_{τ_n} denotes the market value
 134 of the investment in the risky security chosen at time τ_n . After an amount
 135 W_{τ_n} is withdrawn from the total wealth X_{τ_n} , and a fraction ε of the
 136 remainder, is paid as management fees, then the wealth left for re-investment
 137 is $Z_{\tau_n} = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}]$. Of this amount, V_{τ_n} is invested in the risky
 138 security with a per-dollar payback of Γ_{n+1} at the next transaction date,
 139 including continually re-invested dividends. And the remainder, $Z_{\tau_n} - V_{\tau_n}$,
 140 is invested in the riskless security at the continuously compounding interest
 141 rate $r > 0$.

142 The investor's total wealth invested at the time of the $(n + 1)$ th transaction
 143 is therefore

$$X_{\tau_{n+1}} = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}] e^{r T_n} + V_{\tau_n} [\Gamma_{n+1} - e^{r T_n}]. \quad (6)$$

144 for $n = 1, 2, 3, \dots$,

145 According to the equation (2) and the Itô's formula,¹ the return of the risky
 146 investment Γ satisfies

$$\Gamma_{n+1} = \exp \left[\left(\alpha - \frac{1}{2} \sigma^2 \right) T_n + \sigma (B_{\tau_{n+1}} - B_{\tau_n}) \right]. \quad (7)$$

147 Since $M_0 = 0$, then X_0 is considered as the initial wealth endowment for
 148 the investor.

149 **Definition 2.2** *The budget policy $(T, W, V, C) \in \mathcal{U}$ is budget feasible policy*
 150 *if the associated money process M of (5) and invested wealth process X of (6)*
 151 *are non-negative.*

152 2.3 Optimal Control Statement of the Problem

153 **Definition 2.3** *Let $\mathcal{U}$ be the set of all budget feasible policies as defined*
 154 *previously. The optimal control problem for the investor is to maximize*

$$U(X_0) \equiv \max_{(T, W, V, C) \in \mathcal{U}} E \left[\int_0^\infty e^{-\delta t} u(C_t) dt \right], \quad (8)$$

¹Details may be found in Karatzas and Shreve (1988), or Protter (1990)

155 subject to, for $n = 1, 2, 3, \dots$,

$$X_{\tau_{n+1}} = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}] e^{r T_n} + V_{\tau_n} [\Gamma_{n+1} - e^{r T_n}], \quad (9)$$

156 with $M_t \geq 0$, and $X_{\tau_{n+1}} \geq 0$.

157 We assume that only money is available to the investor as a medium of
 158 exchange and numeraire in the economy. Only money is exchangeable for
 159 consumption. It is also assumed that money cannot be borrowed, it can only
 160 be acquired by selling the securities, and it is put in the purse M . Because
 161 there exists a riskless security with a positive interest rate in the economy,
 162 there is no investment demand for money. Duffie and Sun (1990) argued
 163 that it will not be optimal for the investor to withdraw more money than
 164 the amount needed for financing consumption before the next transaction.

165 The following result is similar to those in Duffie and Sun (1990), the proof
 166 can be found in Duffie and Sun (1990) or in Syahril (2003).

167 **Theorem 2.4** *Let the value function U be defined as in (8), and the*
 168 *transaction costs function $\Psi(W_{\tau_n}) = b$, $b \geq 0$. Then the optimal policy*
 169 *(T, W, V, C) must satisfy for all $n = 1, 2, 3, \dots$*

$$\int_{\tau_n}^{\tau_{n+1}} C_t dt = W_{\tau_n} - b. \quad (10)$$

170 **Corollary 1** *By the definition of money holding M_t of equation (5), then*

$$M_{\tau_n} = W_{\tau_n} - b, \quad n = 1, 2, 3, \dots$$

171 Therefore, the optimal control problem (8)-(9) is equivalent to the optimal
 172 control problem :

$$U(X_0) = \max_{(T, W, V, C) \in \mathcal{U}} E \left[\int_0^\infty e^{-\delta t} u(C_t) dt \right] \quad (11)$$

173 subject to

$$\int_{\tau_n}^{\tau_{n+1}} C_t dt = W_{\tau_n} - b, \quad (12)$$

$$X_{\tau_{n+1}} = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}] e^{r T_n} + V_{\tau_n} [\Gamma_{n+1} - e^{r T_n}] \geq 0, \quad (13)$$

175 for $n = 1, 2, 3, \dots$

176 We summarize the problem faced by the investor in the following definition.
 177 For the complete formulation, one can consult ².

²Syahril (2003a)

178 **Definition 2.5** Let $\mathcal{U}$ be the set of all budget feasible policies as defined
 179 previously. The optimal control problem for the investor is to maximize

$$U(X_{\tau_n}) = \max_{\{T_n, W_{\tau_n}, V_{\tau_n}\}} \left\{ Q_n^\nu \frac{1}{\gamma} (W_{\tau_n} - b)^\gamma + e^{-\delta T_n} E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] \right\}, \quad (14)$$

180 subject to

$$X_{\tau_{n+1}} = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}] e^{r T_n} + V_{\tau_n} [\Gamma_{n+1} - e^{r T_n}], \quad (15)$$

181 for $n = 1, 2, 3, \dots$, with $M_t \geq 0$, and $X_{\tau_{n+1}} \geq 0$.

182 The following result about optimal investment strategies for the investor
 183 was derived in Syahril (2003b), but is presented here for completeness.

184 **Theorem 2.6** Let T_n be fixed for $n = 1, 2, 3, \dots$. Then the optimal value
 185 function and unique solution to problem (14)- (15), is given by

$$U(X_{\tau_n}) = Q_n^\nu A_n^{-\nu} \frac{1}{\gamma} (X_{\tau_n} - Y_n)^\gamma \quad (16)$$

186 with the optimal withdrawal and investment strategies are given by

$$W_{\tau_n} = A_n (X_{\tau_n} - Y_n) + b \quad (17)$$

$$V_{\tau_n} = (1 - \varepsilon) (1 - A_n) (X_{\tau_n} - Y_n) \pi_n, \quad (18)$$

187 respectively, and where A_n , and Y_n are given by

$$A_n = \frac{A_{n+1} Q_n}{A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}} \quad (19)$$

$$Y_n = b + (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1} \quad (20)$$

188 respectively, with $R_n = (1 - \varepsilon)^\gamma e^{-\delta T_n} \Omega_n$, and where Ω_n and π_n are
 189 defined by the optimization problem

$$\begin{aligned} \Omega_n &\equiv E([e^{r T_n} + \pi_n (\Gamma_{n+1} - e^{r T_n})]^\gamma) \\ &\equiv \sup_{\{0 \leq \pi \leq 1\}} E([e^{r T_n} + \pi (\Gamma_{n+1} - e^{r T_n})]^\gamma). \end{aligned} \quad (21)$$

190 **Remark 2.7** If at any τ_n , $W_{\tau_n} = b$, then the implications are as follows :

191 1. $Y_n = X_{\tau_n}$, by (17).

192 2. T_n can be computed from (20) as

$$0 < T_n = \frac{1}{r} \ln \left[\frac{X_{\tau_{n+1}}}{(X_{\tau_n} - b)(1 - \varepsilon)} \right] < \infty.$$

- 193 3. $U(X_{\tau_n}) = 0$, implying by (8) that $C_t = 0$, almost every where for
 194 $t \geq \tau_n$. This can occur only for utility function which satisfy $u(0) = 0$.
- 195 4. Case 2 and 3 imply that $W_{\tau_n} > b$ always in the case of infinite-time
 196 horizon

197 **Corollary 2** It is possible that $W_{\tau_n} = b$, in which case

$$T_n = \frac{1}{r} \ln \left[\frac{X_{\tau_{n+1}}}{(X_{\tau_n} - b)(1 - \varepsilon)} \right]. \quad (22)$$

198 This implies that $b < W_{\tau_n}$ if the specified T_n are not given by (22).

199 3 Optimal Transaction Intervals

200 **Theorem 3.1** Let the problem faced by an investor satisfy relation (14) sub-
 201 ject to (15). Then the optimal transaction intervals T_n , $n = 1, 2, 3, \dots$ satisfy

$$g(T_n) + (X_{\tau_n} - Y_n) h(T_n) = 0, \quad (23)$$

202 where g and h are defined by

$$g(T_n) = \gamma r (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1} [A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}], \quad (24)$$

203

$$\begin{aligned} h(T_n) &= \delta e^{-\frac{\delta}{\nu} T_n} A_{n+1} + Q_{n+1} R_n^{1/\nu} [-\delta + \gamma r + \gamma (\alpha - r) \pi_n] \\ &\quad - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) \pi_n^2. \end{aligned} \quad (25)$$

204 **Proof :**

205 Consider U as in equation (14) which is given by

$$U(X_{\tau_n}) = \max_{\{T_n, W_{\tau_n}, V_{\tau_n}\}} \left\{ Q_n^\nu \frac{1}{\gamma} [W_{\tau_n} - b]^\gamma + e^{-\delta T_n} E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] \right\}.$$

206 By differentiation of function U with respect to W_{τ_n} , V_{τ_n} and T_n respec-
 207 tively, and setting each of them equals to zero, then necessary conditions for
 208 (14) for all $n = 1, 2, 3, \dots$ are :

$$Q_n^\nu [W_{\tau_n} - b]^{-\nu} = (1 - \varepsilon) e^{-(\delta - r) T_n} E[U'(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}],$$

209

$$E[(\Gamma_{n+1} - e^{r T_n}) U'(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] = 0,$$

$$\delta e^{-\frac{\delta}{\nu} T_n} Q_n^{-\gamma} \frac{1}{\gamma} [W_{\tau_n} - b]^\gamma = e^{-\delta T_n} \left\{ \delta E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] - \frac{\partial E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}]}{\partial T_n} \right\}. \quad (26)$$

Application of Itô's formula³ on the function U , results in

$$U(X_{\tau_{n+1}}) - U(X_{\tau_n}) = \int_{\tau_n}^{\tau_{n+1}} [U'(X_t) dX_t + \frac{1}{2} U''(X_t) dX_t^2]. \quad (27)$$

From the definition of the wealth process $X_{\tau_{n+1}}$ in relationship (15), then for any $t \in [\tau_n, \tau_{n+1})$, we have

$$X_t = (1 - \varepsilon) [X_{\tau_n} - W_{\tau_n}] e^{r(t-\tau_n)} + V_{\tau_n} (\Gamma_t - e^{r(t-\tau_n)}).$$

Therefore, the following holds for any $t \in [\tau_n, \tau_{n+1})$,

$$dX_t = [r(X_t - G_t) + \alpha G_t] dt + \sigma G_t dB_t,$$

and $dX_t^2 = \sigma^2 G_t^2 dt$, with $G_t \equiv V_{\tau_n} \Gamma_t$, and where Γ_t is given by

$$\Gamma_t = \exp[(\alpha - \frac{1}{2} \sigma^2)(t - \tau_n) + \sigma(B_t - B_{\tau_n})].$$

By using the above results in equation (27), then

$$\begin{aligned} U(X_{\tau_{n+1}}) - U(X_{\tau_n}) &= \int_{\tau_n}^{\tau_{n+1}} \{ [r(X_t - G_t) + \alpha G_t] U'(X_t) dt \\ &\quad + \frac{1}{2} \sigma^2 G_t^2 U''(X_t) dt + \sigma G_t U'(X_t) dB_t \}. \end{aligned}$$

Let processes $\{Z_t\}$ be defined as the following :

$$Z_t = \int_{\tau_n}^t \sigma G_s U(X_s) dB_s, \quad t \in (\tau_n, \tau_{n+1}].$$

Apparently processes $\{Z_t\}$ is a martingale.⁴ Therefore,

$$\begin{aligned} \frac{\partial E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}]}{\partial T_n} &= E[[r(X_{\tau_{n+1}} - G_{\tau_{n+1}}) + \alpha G_{\tau_{n+1}}] U'(X_{\tau_{n+1}}) \\ &\quad + \frac{1}{2} \sigma^2 G_{\tau_{n+1}}^2 U''(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}]. \end{aligned} \quad (28)$$

³See for example Theorem 32 of Protter (1990)

⁴For details, see appendix 3 of Duffie and Sun (1990)

219 Substitution of relation (28) into (26) results in

$$\begin{aligned} \delta e^{-\frac{\delta}{\nu} T_n} Q_n^{-\gamma} \frac{1}{\gamma} [W_{\tau_n} - b]^\gamma = \\ \delta e^{-\delta T_n} E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] - e^{-\frac{\delta}{\nu} T_n} E[r(X_{\tau_{n+1}} - G_{\tau_{n+1}}) \\ + \alpha G_{\tau_{n+1}}] U'(X_{\tau_{n+1}}) + \frac{1}{2} \sigma^2 G_{\tau_{n+1}}^2 U''(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}]. \end{aligned} \quad (29)$$

220 **Remark 3.2** *Duffie and Sun (1990) have used $V_{\tau_{n+1}}$ in place of $G_{\tau_{n+1}}$.*

221 Notice that by insertion of (17) and (18) into (15), then the total wealth
222 process $X_{\tau_{n+1}}$ may be re-written as

$$X_{\tau_{n+1}} = (1 - \varepsilon)(1 - A_n)(X_{\tau_n} - Y_n)[e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})] + Y_{n+1}. \quad (30)$$

223 Since $G_{\tau_{n+1}} = V_{\tau_n} \Gamma_{n+1}$, then by application of (18) we also have

$$G_{\tau_{n+1}} = (1 - \varepsilon)(1 - A_n)(X_{\tau_n} - Y_n)\pi_n \Gamma_{n+1}. \quad (31)$$

224 Applying (30) in (16) then we have the following equations :

$$E[U(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] = (1 - \varepsilon)^\gamma (1 - A_n)^\gamma \frac{1}{\gamma} (X_{\tau_n} - Y_n)^\gamma \quad (32)$$

$$\begin{aligned} & \times E([e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^\gamma) Q_{n+1}^\nu A_{n+1}^{-\nu}, \\ E[U'(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] &= (1 - \varepsilon)^{-\nu} (1 - A_n)^{-\nu} (X_{\tau_n} - Y_n)^{-\nu} \quad (33) \end{aligned}$$

$$\begin{aligned} & \times E([e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu}) Q_{n+1}^\nu A_{n+1}^{-\nu}, \\ E[U''(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] &= (1 - \varepsilon)^{-\nu-1} (1 - A_n)^{-\nu-1} (X_{\tau_n} - Y_n)^{-\nu-1} \quad (34) \\ & \times (-\nu) E([e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu-1}) Q_{n+1}^\nu A_{n+1}^{-\nu}. \end{aligned}$$

225 From the definition of Ω_n in (21), then its derivative with respect to π_n
226 gives

$$E([e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu} [\Gamma_{n+1} - e^{rT_n}]) = 0.$$

227 This implies that the following relations hold :

$$\Omega_n = E(e^{rT_n} [e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu}), \quad (35)$$

$$= E(\Gamma_{n+1} [e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu}), \quad (36)$$

$$= E(\Gamma_{n+1}^2 [e^{rT_n} + \pi_n(\Gamma_{n+1} - e^{rT_n})]^{-\nu-1}). \quad (37)$$

228 By substituting previous equations and (17), (31), (35), (36), (37) into (29),
229 and by dividing all terms by $(1/\gamma)(X_{\tau_n} - Y_n)^{\gamma-1}$, and by rearranging the
230 terms, then we have the following equation :

$$\begin{aligned} (X_{\tau_n} - Y_n) \{ \delta e^{-\frac{\delta}{\nu} T_n} Q_n^{-\gamma} A_n^\gamma + Q_{n+1}^\nu A_{n+1}^{-\nu} (1 - A_n)^\gamma R_n [-\delta + \gamma r \\ + \gamma(\alpha - r)\pi_n - \frac{1}{2} \sigma^2 \gamma(1 - \gamma)\pi_n^2] \} \\ + \gamma r (1 - \varepsilon)^{-1} e^{-rT_n} Q_{n+1}^\nu A_{n+1}^{-\nu} (1 - A_n)^{-\nu} R_n Y_{n+1} = 0. \end{aligned} \quad (38)$$

231 By applications of A_n of (19) and Y_n of (20) in (38), then we have

$$\begin{aligned}
& (X_{\tau_n} - Y_n) \left\{ \delta e^{-\frac{\delta}{\nu} T_n} \frac{A_{n+1}^\gamma}{[A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}]^\gamma} + Q_{n+1}^\nu A_{n+1}^{-\nu} R_n \right. \\
& \times \frac{Q_{n+1}^\gamma R_n^{\gamma/\nu}}{[A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}]^\gamma} \left[-\delta + \gamma r + \gamma(\alpha - r)\pi_n - \frac{1}{2}\sigma^2\gamma(1 - \gamma)\pi_n^2 \right] \Big\} \\
& + \frac{\gamma r Y_{n+1} R_n}{(1 - \varepsilon) e^{r T_n}} Q_{n+1}^\nu A_{n+1}^{-\nu} \frac{Q_{n+1}^{-\nu} R_n^{-\nu/\nu}}{[A_{n+1} Q_n + Q_{n+1} R_n]^{-\nu}} = 0. \tag{39}
\end{aligned}$$

232 Finally, by dividing all terms of (39) by $\frac{A_{n+1}^{-\nu}}{[A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}]^\gamma}$ results in the
 233 following relation :

$$g(T_n) + (X_{\tau_n} - Y_n) h(T_n) = 0, \tag{40}$$

234 where g and h are defined as the following :

$$g(T_n) = \gamma r (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1} [A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}], \tag{41}$$

235

$$\begin{aligned}
h(T_n) &= \delta e^{-\frac{\delta}{\nu} T_n} A_{n+1} + Q_{n+1} R_n^{1/\nu} [-\delta + \gamma r + \gamma(\alpha - r)\pi_n] \\
&- \frac{1}{2} \sigma^2 \gamma (1 - \gamma) \pi_n^2. \tag{42}
\end{aligned}$$

236 Therefore the proof of the theorem has been completed ♠

237 **Theorem 3.3** Suppose that $\alpha, \delta, \varepsilon, \gamma, r, \sigma$ satisfying the following con-
 238 ditions :

239 1. $\alpha - r \geq \frac{1}{2} \sigma^2 (1 - \gamma)$

240 2. $\max(\gamma\alpha, \gamma r) < \delta < \min(\frac{\gamma r}{(1-\varepsilon)^{\gamma/\nu}}, r - \gamma r)$

241 If A_n as given by (19) has a property such that $A_n \geq Q_n (1 - R_n^{1/\nu})$, then
 242 the equation

$$g(T_n) + (X_{\tau_n} - Y_n) h(T_n) = 0$$

243 as defined in Theorem 3.1 has a solution.

244 **Proof :** Let $H(T_n) = g(T_n) + (X_{\tau_n} - Y_n) h(T_n)$, where g and h are
 245 defined by (24) and (25) respectively. Note that H is a continuous function

of T_n . It will be shown that $H > 0$ as $T_n \rightarrow 0^+$ and $H < 0$ as $T_n \rightarrow +\infty$. From the definition of functions g and h , then we have

$$\begin{aligned} \lim_{T_n \rightarrow 0^+} H(T_n) &= \lim_{T_n \rightarrow 0^+} \gamma r (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1} [A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}] \\ &+ \lim_{T_n \rightarrow 0^+} Q_{n+1} (X_{\tau_n} - Y_n) \left\{ \delta e^{-\delta/\nu T_n} \frac{A_{n+1}}{Q_{n+1}} \right. \\ &+ \left. R_n^{1/\nu} [-\delta + \gamma r + \gamma (\alpha - r) \pi_n - 1/2 \sigma^2 \gamma (1 - \gamma) \pi_n^2] \right\}. \end{aligned}$$

But A_n has the property that either $A_n \geq Q_n (1 - R_n^{1/\nu})$, or $A_n < Q_n (1 - R_n^{1/\nu})$. Since we have assumed that $A_n \geq Q_n (1 - R_n^{1/\nu})$, therefore, $\frac{A_{n+1}}{Q_{n+1}} \geq (1 - R_n^{1/\nu}) R_n^{1/\nu}$. Furthermore, $Y_n = b + (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1}$ by relation (20). Therefore,

$$\begin{aligned} \lim_{T_n \rightarrow 0^+} H(T_n) &\geq \lim_{T_n \rightarrow 0^+} \gamma r (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1} [A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}] \\ &+ \lim_{T_n \rightarrow 0^+} Q_{n+1} (X_{\tau_n} - b - (1 - \varepsilon)^{-1} e^{-r T_n} Y_{n+1}) \\ &\times \left\{ \delta e^{-\delta/\nu T_n} (1 - R_n^{1/\nu}) R_n^{1/\nu} + R_n^{1/\nu} [-\delta + \gamma r \right. \\ &+ \left. \gamma (\alpha - r) \pi_n - 1/2 \sigma^2 \gamma (1 - \gamma) \pi_n^2] \right\}. \end{aligned}$$

By the assumption of the theorem, $(\alpha - r) \geq \frac{1}{2} \sigma^2 (1 - \gamma)$. In addition, by its definition, $0 \leq \pi_n \leq 1$. These imply that

$$-\delta + \gamma r + \gamma (\alpha - r) \pi_n - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) \pi_n^2 \geq -\delta + \gamma r.$$

And since $\lim_{T_n \rightarrow 0^+} R_n^{1/\nu} = (1 - \varepsilon)^{\gamma/\nu}$, then we have that

$$\begin{aligned} \lim_{T_n \rightarrow 0^+} H(T_n) &\geq \gamma r (1 - \varepsilon)^{-1} Y_{n+1} Q_{n+1} (1 - \varepsilon)^{\gamma/\nu} \\ &+ Q_{n+1} (X_{\tau_n} - b - (1 - \varepsilon)^{-1} Y_{n+1}) \\ &\times \left\{ \delta [1 - (1 - \varepsilon)^{\gamma/\nu}] (1 - \varepsilon)^{\gamma/\nu} + (1 - \varepsilon)^{\gamma/\nu} [-\delta + \gamma r] \right\}. \end{aligned}$$

By the assumption of the Theorem, $\delta < \frac{\gamma r}{(1 - \varepsilon)^{\gamma/\nu}}$. This implies that

$$\delta [1 - (1 - \varepsilon)^{\gamma/\nu}] (1 - \varepsilon)^{\gamma/\nu} + (1 - \varepsilon)^{\gamma/\nu} [-\delta + \gamma r] > 0.$$

Therefore,

$$\lim_{T_n \rightarrow 0^+} H(T_n) > \gamma r (1 - \varepsilon)^{-1 + \gamma/\nu} Q_{n+1} Y_{n+1}.$$

Since $\gamma, r, \varepsilon, Y_{n+1}, Q_{n+1}$ are nonnegatives, hence

$$\lim_{T_n \rightarrow 0^+} H(T_n) > 0.$$

On the other hand, since A_{n+1} , Q_{n+1} , Q_n are nonnegatives and less than one, and $R_n \leq (1 - \varepsilon)^\gamma$, then

$$\begin{aligned}
H(T_n) &= g(T_n) + (X_{\tau_n} - Y_n) h(T_n) \\
&= \frac{\gamma r}{(1 - \varepsilon) e^{r T_n}} Y_{n+1} [A_{n+1} Q_n + Q_{n+1} R_n^{1/\nu}] \\
&\quad + (X_{\tau_n} - Y_n) \left\{ \delta e^{-\frac{\delta}{\nu} T_n} A_{n+1} + Q_{n+1} R_n^{1/\nu} \right. \\
&\quad \times \left. [-\delta + \gamma r + \gamma(\alpha - r) \pi_n - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) \pi_n^2] \right\} \\
&\leq \frac{\gamma r Y_{n+1} [1 + (1 - \varepsilon)^{\gamma/\nu}]}{(1 - \varepsilon) e^{r T_n}} \\
&\quad + (X_{\tau_n} - Y_n) \left\{ \delta e^{-\frac{\delta}{\nu} T_n} + Q_{n+1} R_n^{1/\nu} [-\delta + \max(\gamma \alpha, \gamma r)] \right\}.
\end{aligned}$$

By re-arranging all terms, then

$$\begin{aligned}
H(T_n) &\leq e^{-\frac{\delta}{\nu} T_n} (X_{\tau_n} - Y_n) \left\{ \frac{\gamma r Y_{n+1} e^{\frac{\delta}{\nu} T_n} [1 + (1 - \varepsilon)^{\gamma/\nu}]}{(X_{\tau_n} - Y_n) (1 - \varepsilon) e^{r T_n}} \right. \\
&\quad \left. + \delta + e^{\delta/\nu T_n} Q_{n+1} R_n^{1/\nu} [-\delta + \max(\gamma \alpha, \gamma r)] \right\}.
\end{aligned}$$

Since, by assumptions of Theorem 3.3 that $\delta > \max(\gamma \alpha, \gamma r)$, and $r > \delta/\nu$, then $H(T_n) < 0$ for large T_n . It has been shown that $H(T_n) > 0$ as $T_n \rightarrow 0^+$, and $H(T_n) < 0$ for large T_n . Since $H(T_n)$ is a continuous function in T_n , then there exists $\hat{T}_n$, such that $H(\hat{T}_n) = 0$. Therefore, the proof of Theorem 3.3 has been completed ♠

Theorem 3.4 Suppose that α , δ , ε , γ , r , σ satisfying the following conditions :

1. $\alpha - r \geq \frac{1}{2} \sigma^2 (1 - \gamma)$
2. $\max(\gamma \alpha, \gamma r) < \delta < \min(\frac{\gamma r}{(1 - \varepsilon)^{\gamma/\nu}}, r - \gamma r)$

If A_n as given by (19) has the property such that $A_n \geq Q_n (1 - R_n^{1/\nu})$, then an optimal policy (T, W, V, C) exists.

Proof : We have that $\frac{A_{n+1}}{Q_{n+1}} \geq (1 - R_n^{1/\nu}) R_n^{1/\nu}$ for $A_n \geq Q_n (1 - R_n^{1/\nu})$.

By Theorem 3.3, then there exists a scalar $\hat{T}_n > 0$ such that $H(\hat{T}_n) = 0$. The arguments in the proof of Theorem 3.3 show that one of the solutions, say $\hat{T}_n$, corresponds to the maximum of (14).

276 Now consider the following withdrawal and investment policy and func-
 277 tion f , for $n = 1, 2, 3, \dots$

$$f(X_{\tau_n}) = Q_n^\nu A_n^{-\nu} \frac{1}{\gamma} (X_{\tau_n} - Y_n)^\gamma, \quad (43)$$

$$W_{\tau_n} = A_n (X_{\tau_n} - Y_n) + b, \quad (44)$$

$$V_{\tau_n} = (1 - \varepsilon) (1 - A_n) (X_{\tau_n} - Y_n) \pi_n, \quad (45)$$

$$T_n = \hat{T}_n. \quad (46)$$

278 From the preceeding calculations, we know that $f(X_{\tau_n})$ satisfies (14)
 279 $\forall n$. It will be shown that $\lim_{n \rightarrow \infty} E[e^{-\delta \tau_n} f(X_{\tau_n})] = 0$. By using (43),
 280 then

$$E[e^{-\delta \tau_{n+1}} f(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] = Q_{n+1}^\nu A_{n+1}^{-\nu} e^{-\delta \tau_{n+1}} \frac{1}{\gamma} E[(X_{\tau_{n+1}} - Y_{n+1})^\gamma | \mathcal{H}_{\tau_n}], \quad (47)$$

281 with the total wealth $X_{\tau_{n+1}}$ is given by (30).

282 By substitution of (44) and (45) into the total wealth $X_{\tau_{n+1}}$ of (30),
 283 then the expectation factor on the right hand side of (47) may be written as

$$E[(X_{\tau_{n+1}} - Y_{n+1})^\gamma | \mathcal{H}_{\tau_n}] = (1 - \varepsilon)^\gamma \Omega_n (1 - A_n)^\gamma [(X_{\tau_n} - Y_n)^\gamma | \mathcal{H}_{\tau_{n-1}}].$$

284 Since $R_n = (1 - \varepsilon)^\gamma e^{-\delta T_n} \Omega_n$, and $\tau_{n+1} = T_n + \tau_n$, then the right hand side
 285 of (47) may be written as

$$\begin{aligned} & \frac{1}{\gamma} Q_{n+1}^\nu A_{n+1}^{-\nu} e^{-\delta \tau_{n+1}} E[(X_{\tau_{n+1}} - Y_{n+1})^\gamma | \mathcal{H}_{\tau_n}] \\ &= \frac{1}{\gamma} Q_{n+1}^\nu A_{n+1}^{-\nu} e^{-\delta T_n} (1 - \varepsilon)^\gamma \Omega_n (1 - A_n)^\gamma e^{-\delta \tau_n} [(X_{\tau_n} - Y_n)^\gamma | \mathcal{H}_{\tau_{n-1}}] \\ &= \frac{1}{\gamma} Q_{n+1}^\nu A_{n+1}^{-\nu} R_n (1 - A_n)^\gamma e^{-\delta \tau_n} [(X_{\tau_n} - Y_n)^\gamma | \mathcal{H}_{\tau_{n-1}}]. \end{aligned}$$

286 Hence, we have

$$E[e^{-\delta \tau_{n+1}} f(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] = \frac{1}{\gamma} Q_{n+1}^\nu A_{n+1}^{-\nu} R_n (1 - A_n)^\gamma e^{-\delta \tau_n} [(X_{\tau_n} - Y_n)^\gamma | \mathcal{H}_{\tau_{n-1}}].$$

287 Let $D = \frac{1}{\gamma} Q_{n+1}^\nu A_{n+1}^{-\nu}$.

288 Then by induction we have the following :

$$\begin{aligned} & E[e^{-\delta \tau_{n+1}} f(X_{\tau_{n+1}})] \\ &= D R_n (1 - A_n)^\gamma e^{-\delta \tau_n} [(X_{\tau_n} - Y_n)^\gamma | \mathcal{H}_{\tau_{n-1}}] \\ &= D R_n (1 - A_n)^\gamma R_{n-1} (1 - A_{n-1})^\gamma e^{-\delta \tau_{n-1}} [(X_{\tau_{n-1}} - Y_{n-1})^\gamma | \mathcal{H}_{\tau_{n-2}}] \\ &= D R_n (1 - A_n)^\gamma \dots R_1 (1 - A_1)^\gamma e^{-\delta \tau_1} [(X_{\tau_1} - Y_1)^\gamma | \mathcal{H}_{\tau_1}] \\ &\leq D X_0^\gamma \prod_{k=1}^n [R_k (1 - A_k)^\gamma] \leq D X_0^\gamma \prod_{k=1}^n [(1 - \varepsilon)^{\gamma/\nu}], \end{aligned}$$

as $0 < 1 - A_n < 1$, and $R_n \leq (1 - \varepsilon)^{\gamma/\nu} \forall n$. Since D is bounded, then
 $\lim_{n \rightarrow \infty} E[e^{-\delta \tau_n} f(X_{\tau_n})] = 0$.

Therefore, the proof of Theorem 3.4 has been completed ♠

4 Equal Intervals

Now consider the case of transaction costs $b = 0$. Let transaction intervals $T_1 = T_2 = \dots = K$. This implies that $\tau_n = (n - 1)K$. With T_n are equals for all n , then $A_{n+1} = A_n$, $Q_{n+1} = Q_n$, $Y_{n+1} = Y_n$. By replacing A_{n+1} with A_n and Q_{n+1} with Q_n in (19), and replacing Y_{n+1} with Y_n in (20), respectively, result in

$$A_n = 1 - [(1 - \varepsilon)^\gamma e^{-\delta T_n} \Omega_n]^{1/\nu},$$

and

$$Y_n = \frac{(1 - \varepsilon) e^{r T_n} b}{(1 - \varepsilon) e^{r T_n} - 1}.$$

Note that transaction costs function $b = 0$ implies $Y_n = 0$. This implies that $g(T_n) + (X_{\tau_n} - Y_n) h(T_n) = 0$ is equivalent to $h(T_n) = 0$, where h is given by

$$\begin{aligned} h(T_n) &= \delta e^{-\frac{\delta}{\nu} T_n} A_n + Q_n R_n^{1/\nu} [-\delta + \gamma r + \gamma (\alpha - r) \pi_n \\ &\quad - \frac{1}{2} \sigma^2 \gamma \nu \pi_n^2]. \end{aligned}$$

The following result is proved in Duffie and Sun(1990), however the proof is incomplete.

Theorem 4.1 Suppose that $\delta > \max(\gamma\alpha, \gamma r)$, and that $b = 0$. Then $h(T_n) = 0$ has a solution.

Proof : From its definition, h is a continuous function in T_n . It will be shown that $h > 0$, as T_n approaches zero, and $h < 0$, as T_n goes toward infinity. Since $Q_n = 1 - e^{-\frac{\delta}{\nu} T_n}$, then

$$\lim_{T_n \rightarrow 0^+} h(T_n) = \lim_{T_n \rightarrow 0^+} \delta e^{-\frac{\delta}{\nu} T_n} A_n.$$

Note that

$$\lim_{T_n \rightarrow 0^+} \exp[(-\delta + \gamma r) T_n] = 1, \quad \text{and} \quad \lim_{T_n \rightarrow 0^+} \exp[(-\delta + \max(\gamma\alpha, \gamma r)) T_n] = 1.$$

Since

$$e^{(-\delta + \gamma r) T_n} \leq e^{-\delta T_n} \Omega_n \leq e^{[-\delta + \max(\gamma\alpha, \gamma r)] T_n},$$

311 then $\lim_{T_n \rightarrow 0^+} e^{-\delta T_n} \Omega_n = 1$. Therefore,

$$\lim_{T_n \rightarrow 0^+} h(T_n) = \delta [1 - (1 - \varepsilon)^{\gamma/\nu}] > 0.$$

312 On the other hand,

$$\begin{aligned} h(T_n) &\leq \delta e^{-\frac{\delta}{\nu} T_n} A_n + (1 - A_n) Q_n \\ &\times [\gamma r + \gamma(\alpha - r) \pi_n - \delta] \\ &\leq e^{-\frac{\delta}{\nu} T_n} [\delta + (\max(\gamma\alpha, \gamma r) - \delta) (1 - \varepsilon)^{\gamma/\nu} e^{\frac{\gamma r}{\nu} T_n}]. \end{aligned}$$

313 Since by the assumption of Theorem 4.1 that $\delta > \max(\gamma\alpha, \gamma r)$, then for
314 large T_n , $h(T_n) < 0$. Since h is continuous in T_n , therefore there
315 exists $\bar{T}_n$ such that $h(\bar{T}_n) = 0$. Hence, the proof of Theorem 4.1 has been
316 completed ♠

317 **Theorem 4.2** Suppose that $\delta > \max(\gamma\alpha, \gamma r)$ and that $b = 0$. Then
318 an optimal policy (T, W, V, C) exists and the optimal transaction intervals
319 $\{T_n\}$ are equal.

320 **Proof :** The proof is as in Duffie and Sun(1990). With $b = 0$, $H(T_n) = 0$
321 is equivalent to $h(T_n) = 0$, for all n . By Theorem 4.1, $h(T_n) = 0$, has a
322 solution. Therefore, there exists $\bar{T}_n$ such that $h(\bar{T}_n) = 0$.

323 Now consider the following withdrawal and investment policy and func-
324 tion f , for $n=1,2,3,\dots$

$$\begin{aligned} f(X_{\tau_n}) &= Q_n^\nu A_n^{-\nu} \frac{1}{\gamma} X_{\tau_n}^\gamma, \\ W_{\tau_n} &= A_n X_{\tau_n}, \\ V_{\tau_n} &= (1 - \varepsilon) (1 - A_n) X_{\tau_n} \pi_n, \\ T_n &= \bar{K}. \end{aligned}$$

325 From the preceeding calculations, we know that that $f(X_{\tau_n})$ satisfies (??)
326 and that $(T_n, W_{\tau_n}, V_{\tau_n})$ achieves the maximum in (??) $\forall n$.

327 Let $D \equiv (1/\gamma) Q_{n+1}^\nu A_{n+1}^{-\nu}$. Therefore,

$$\begin{aligned} E[e^{-\delta\tau_{n+1}} f(X_{\tau_{n+1}}) | \mathcal{H}_{\tau_n}] &= e^{-\delta(T_n + \tau_n)} D E[X_{\tau_{n+1}}^\gamma | \mathcal{H}_{\tau_n}] \\ &= D e^{-\delta T_n} (1 - \varepsilon)^\gamma \Omega_n (1 - A_n)^\gamma e^{-\delta\tau_n} [X_{\tau_n}^\gamma | \mathcal{H}_{\tau_{n-1}}] \\ &= D R_n (1 - A_n)^\gamma e^{-\delta\tau_n} [X_{\tau_n}^\gamma | \mathcal{H}_{\tau_{n-1}}] \\ &= D R_n (1 - A_n)^\gamma (1 - \varepsilon)^\gamma \Omega_{n-1} (1 - A_{n-1})^\gamma e^{-\delta\tau_{n-1}} [X_{\tau_{n-1}}^\gamma | \mathcal{H}_{\tau_{n-2}}] \\ &= D R_n (1 - A_n)^\gamma R_{n-1} (1 - A_{n-1})^\gamma \dots R_1 (1 - A_1)^\gamma X_0^\gamma \\ &= D X_0^\gamma \prod_{k=1}^n R_k (1 - A_k)^\gamma = D X_0^\gamma [e^{-\delta\bar{K}} (1 - \varepsilon)^\gamma \Omega_n]^{1/\nu}]^n, \end{aligned}$$

as $1 - A_n = R_n^{1/\nu} = [(1 - \varepsilon)^\gamma \Omega_n e^{-\delta \bar{K}}]$, $\forall n$. Since $[(1 - \varepsilon)^\gamma \Omega_n e^{-\delta \bar{K}}] < (1 - \varepsilon)^\gamma$, for any feasible policy, then $\lim_{n \rightarrow \infty} E[e^{-\delta \tau_n} f(X_{\tau_n})] = 0$. Hence, the proof of Theorem 4.1 has been completed ♠

5 Conclusion

We have established an optimal transaction interval for a consumption and investment selection problem for an individual who seeks to maximize the expected utility of consumption. The individual has available a riskless asset with fixed interest rate and a risky one with logarithmic Brownian motion price fluctuations. The individual observes current wealth when making transaction, and decisions to transact can be made at any time, but not without costs. The individual is charged a fixed fraction $\epsilon > 0$ of the current wealth as a portfolio management fee plus fixed transaction costs. The problem was formulated in discrete-continuous-time stochastic optimal control problem.

It has been shown that transaction intervals satisfied a nonlinear equation, and it has been shown the existence of the nonlinear equation. The general result is that the optimal transaction intervals T_n depend on the amount of total wealth at the beginning of each intervals.

If, for any reason that transactions do not incur costs other than those of management fee, it was shown that the optimal interval between transactions are independent of wealth at the beginning of the interval. This result verifies those of Duffie and Sun (1990). This is so because Duffie and Sun (1990) actually treated linear transaction costs by transforming it to fixed transaction costs problem. Then by taking $b = 0$, Duffie and Sun (1990) provide result only for the proportional transaction costs problem.

References

- [1] Constantinides, G.M., 1979, Multiperiod Consumption and Investment Behaviour with Transaction Costs, *Management Science*, 25(11), pp. 1127-1137
- [2] Constantinides, G.M., 1986, Capital Market Equilibrium with Transaction Costs, *Journal of Political Economy*, 94(4), pp. 842-862
- [3] Cox, J.C and C.F Huang, 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, pp. 33-83
- [4] Davis, M.H.A and A.R Norman, 1990, Portfolio Selection with Transaction Costs, *Mathematics of Operations Research*, 15(4), pp. 676-713

- 363 [5] Duffie, D and T.S Sun, 1990, Transaction Costs and Portfolio Choice in a
 364 Discrete-Continuous-Time Setting, *Journal of Economic Dynamics and Con-*
 365 *trol*, 14, pp. 35-51
- 366 [6] Dumas, B. and E. Luciano, 1991, An Exact Solution to a Dynamic Portfolio
 367 Choice Problem Under Transaction Costs, *Journal of Finance*, XLVI(2), pp.
 368 577-595
- 369 [7] Karatzas, I. and S.E. Shreve, 1988, *Brownian Motion and Stochastic Calculus*,
 370 Springer-Verlag, New York
- 371 [8] Lelands, H.E., 1985, Option Pricing and Replication with Transaction Costs,
 372 *The Journal of Finance*, XL(5), pp. 1283-1301
- 373 [9] Magill M.J.P and G.M. Constantinides, 1976, Portfolio Selection with Trans-
 374 action Costs, *Journal of Economic Theory*, 13, pp. 245-263
- 375 [10] Merton, R.C., 1990, *Continuous-Time Finance*, Blackwell, Cambridge
- 376 [11] Merton, R.C., 1971, Optimum Consumption and Portfolio Rules in Continu-
 377 ous Time Model, *Journal of Economic Theory*, 3, pp. 373-413
- 378 [12] Protter, P., 1990, *Stochastic Intergration and Differential Equations*, Springer-
 379 Verlag, Berlin