Pairwise comparison based ranking and scoring algorithms

Technical Report No. 223b

Ulf A. Hamster
orcid: 0000-0002-0440-4868

April 1, 2021

Abstract

The task is to convert a large sparse matrix of unbalanced paired comparison vote counts to scores for each item. Five procedures are presented and benchmarked. It is recommended to use the BTL algorithm for full batch computations (e.g. one-time quantitative studies), and the proposed p-value based procedure when scores must be partially updated in production deployments.

Nomenclature

symbol	type	description
C_i	-	Name of the i -th item (or resp. i -th choice)
i, j	$\mathbb{N}_{>0}$	Position index of the items C_i and C_j
N_{ij}	$\mathbb{N}_{\geq 0}$	Num. of times (count, frequency) users choose $C_i > C_j$
μ_{ij}	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Simple ratio $\mu_{ij} = \frac{N_{ij}}{N_{ij} + N_{ji}}$
m_i	$\mathbb{R}$	A metric associated with the i -th item
s_i	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Adjusted, scaled, calibrated scores
p_{ij}	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Probability of error (p-value) of a Pearson's χ^2 -test
q_{ij}	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	$1 - p_{ij}$ if $N_{ij} > N_{ji}$ else zero
A	$\mathbb{R}^{d \times d}$	Reciprocal pairwise comparison matrix $A = (a_{ij})$
m	$\mathbb{R}$	Eigenvalue
x	$\mathbb{R}^d$	Eigenvector $x = [x_1, x_2, \dots, x_d]$
G	$\mathbb{R}^{d \times d}$	The sparse generator matrix $G = (g_{kj})$
$\Pr(k j)$	$\{x \in \mathbb{R} 1 \geq x \geq 0\}^{d \times d}$	Transition probability matrix
π_j	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Item (state) probability
π_k	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Predicted probability of $C_k > C_j$
$\Pr(C_i > C_j)$	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Probability of $C_i > C_j$
γ_i	$\{x \in \mathbb{R} 1 \geq x \geq 0\}$	Score to be estimated as BTL model parameter

Table 1: *Used symbols and notations used in this report.*

1 Introduction

This reports explores procedures to rank and score items based on pairwise comparison frequencies. The paired comparison matrix is expected to be extremely sparse (i.e. no

exhaustive enumeration of all possible pairs), exhibit unbalanced frequencies or resp. unequal sample sizes (e.g. few pairs are often evaluated), and the number of items is much higher than the number of annotators (i.e. millions of items versus a couple of dozens of users). In other words, the algorithm must quickly process a large sparse matrix.

Computing *sum of simple ratios* is fast and scores might be good enough (Sec. 2). The *p-value based metric to rank* accommodates for unequal frequencies across different pairs (Sec. 3). Another idea is to *estimate and simulate a transition matrix* of the next item to be better (Sec. 5). For both the *p-value* and *transition matrix* based approach, we could not find literature. On the other side, using *eigenvectors as scores* (Sec. 4) and the *Bradley-Terry-Luce (BTL) probability model* (Sec. 6) are established algorithms to derive ranks and scores from paired comparisons.

2 Sum of Simple Ratios

The scoring procedure 1 has the following assumptions. First, the simple ratio μ_{ij} ignores the sample sizes $N_{ij} + N_{ji}$ across different pairs (i, j) . Thus, all observed pairs of the i -th item are weighted equally. Second non-observed pairs (i^*, j^*) are assumed to be $\mu_{i^*j^*} = 0$. For example diagonals are always $\mu_{ii} = 0$. Thus, the score s_i might be higher because more distinct pairs have been evaluated for the i -th item, i.e. *explorative* sampling might yield higher scores than *exploitive* sampling.

Procedure 1 Simple ratios from paired comparison frequencies.

1: $\mu_{ij} \leftarrow \frac{N_{ij}}{N_{ij} + N_{ji}} \forall i, j$	▷ Compute all ratios
2: $r_i \leftarrow \sum_j \mu_{ij} \forall i$	▷ Sum rows
3: $m_i \leftarrow \frac{r_i}{\sum_j 1_{N_{ij} > 0}} \forall i$	▷ Divide by the number of observed pairs
4: $s_i \leftarrow \text{adjust}(m_1, m_2, \dots) \forall i$	▷ e.g. min-max, ordinal scaling, platt calibration
-

3 p-value based Metric to Rank

The question which opposing frequency N_{ij} or N_{ji} is larger, can be treated as hypothesis test (Eq. 1).

$$\mu_{ij} = \frac{N_{ij}}{N_{ij} + N_{ji}} \quad , \quad H_0 : \mu_{ij} = 0.5 \quad , \quad H_a : \mu_{ij} > 0.5 \quad (1)$$

Instead of the discrete binomial test, the Pearson's χ^2 -test should be used to compute the probability of error p_{ij} (p-value). If $\mu_{ij} > 0.5$ or resp. $N_{ij} > N_{ji}$, the smaller the p-value $p_{ij} \rightarrow 0$, the more significant the rejection of the null hypothesis H_0 . In other words, pairs with small p-value should yield higher scores, i.e. $q = 1 - p$.

$$q_{ij} = \begin{cases} 1 - p_{ij}, & \text{if } N_{ij} > N_{ji} \\ 0, & \text{otherwise} \end{cases} \quad \forall i, j \quad (2)$$

The Eq. 2 has the advantage that a) non-observed pairs, and b) at least half of the elements can be stored in a sparse matrix format. Similar to procedure 1, we can compute the sum of each row $r_i = \sum_j q_{ij} \forall i$, divide it by the number of observed pairs, and adjust these values to the scores.

The Hoaglin Approximation [Hoa77] can be used to compute the p-values [Beh18, Eq. 6]

what would simplify for zero degree of freedom to Eq. 4.

$$X^2 = \frac{(N_{ij} - E)^2}{E} \quad \text{with } E = (N_{ij} + N_{ji})/2 \quad (3)$$

$$p \approx (0.1)^{\frac{\sqrt{X^2+1.37266}}{2.13161}} \quad (4)$$

4 Eigenvectors as scores

[Saa03] proposed to solve a pairwise comparison matrix as Eigenvalue-problem whereas the eigenvector can be interpreted as the items' scores.

1. Create a reciprocal pairwise comparison matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} N_{ij}/N_{ji}, & \text{if } N_{ji} > 0 \\ 0, & \text{otherwise} \end{cases} \quad \forall i, j \quad (5)$$

2. Solve the sparse Eigenvalue-problem $Ax = mx$ with m the eigenvalue and $x = [x_1, x_2, \dots, x_d]$ the eigenvector.

The sparse Eigenvalue-problem can be solved with the scipy implementation [Vir+20] of ARPACK [LSY98].

5 Estimate and simulate a transition matrix

The idea is to estimate transition probability of the next item to be better. In a first step the generator matrix $G = (g_{ij})$ is derived from the observed paired comparison frequencies N_{ij} (Line 1-4 in procedure 2). The transition probability matrix $\text{Pr}(i|j)$ of items being evaluated $C_i > C_j$ is the matrix exponential of the generator matrix $\text{expm}(g_{ij})$ what can be computed with scipy's implementation [Vir+20; AH09] (Line 5). In a final step, the state probabilities π_i that $C_i > C_j$ are simulated (Line 6-7) and used as scores.

Procedure 2 Compute the generator matrix $G = (g_{ij})$, the transition matrix $\text{Pr}(i|j)$, and simulate the probability that π_i .

1: $g_{ij} \leftarrow N_{ji}$	▷ use the transposed paired comparison frequency matrix
2: $r_i \leftarrow \sum_j g_{ij}$	▷ sum each row
3: $g_{ii} \leftarrow -r_i$	▷ overwrite diagonals
4: $g_{ij} \leftarrow g_{ij} / \max(1, r_i) \quad \forall j$	▷ scale each row
5: $\text{Pr}(i j) = \text{expm}(g_{ij})$	▷ Compute the matrix exponential of the generator matrix
6: $\pi_j \leftarrow 1/d \quad \forall j$	▷ The initial items have an equally distributed item probability
7: $\pi_i \leftarrow \pi_j \cdot \text{Pr}(i j)$	▷ Predict the probability π_i that $C_i > C_j$
-

6 Bradley-Terry-Luce (BTL) probability model

Given d the number of items, and each item has a parameters $\gamma_i \in \{x \in \mathbb{R} | 1 \geq x \geq 0\}$, the Bradley-Terry-Luce probability model [BT52] is

$$\text{Pr}(C_i > C_j) = \frac{\gamma_i}{\gamma_i + \gamma_j} \quad (6)$$

According to [Hun04, pp. 386-387] we can maximize the log-likelihood

$$\max_{\gamma_1, \dots, \gamma_d} \sum_{i=1}^d \sum_{j=1}^d N_{ij} [\ln(\gamma_i) - \ln(\gamma_i + \gamma_j)] \quad (7)$$

by updating in $h = 1, 2, \dots, H$ iteration steps

$$\gamma_i^{(h+1)} = \text{norm} \left[\sum_{j=1}^d N_{ij} \cdot \left(\sum_{j \neq i} \frac{N_{ij} + N_{ji}}{\gamma_i^{(h)} + \gamma_j^{(h)}} \right)^{-1} \right] \quad (8)$$

with

$$\text{norm}(x_1, \dots, x_d) = \frac{x_i}{\sum_{j=1}^d x_j} \quad (9)$$

to ensure $\sum_{i=1}^d \gamma_i^{(h+1)} = 1$.

7 Experiment

We extracted a paired comparison matrix from the sentiment dataset (see "*BWS annotations*") in [KM17] by using the PyPi package `bwsample` (v0.6.1) [Ham21]. The five different scores were compared with the reported scores in [KM17] (see "*Scores obtained by BWS*") that were computed according to [Orm09]. The Spearman rank correlations in Tab. 2 indicate that the Eigenvector procedure might have troubles what can be confirmed by manual inspection. The eigenvector values are very tiny what leads to numerical instabilities. To our surprise the sparse matrix based implementation of the BTL model with the MM algorithm is very fast.

	(b)	(c)	(d)	(e)	(f)	time
(a) simple ratio	.9997	-0.0133	1.	1.	1.	1.000
(b) approx p-value	-	-0.0110	1.	1.	1.	0.730
(c) eigenvector	-	-	1.	1.	1.	0.733
(d) transitions	-	-	-	1.	1.	12.077
(e) BTL model	-	-	-	-	1.	0.743
(f) scores from [KM17]	-	-	-	-	-	-

Table 2: Spearman rank correlations between the five procedures and the reference scores. The computation run time is represented as a multiple of the run time for computing the simple ratios based scores.

8 Conclusion

Among the transition matrix procedure (Sec. 5), eigenvector procedure (Sec. 4) and BTL model (Sec. 6), the latter should be used because it delivers numerical stable results, is fast to compute, and theoretical sound. However, all three procedures would require a complete re-computation if the frequency of just one pair was updated, i.e. an update is still $\mathcal{O}(n^2)$ complex. In case of simple ratios (Sec. 2) and p-value based metrics (Sec. 3) the intermediate results can be stored, e.g. the (q_{ij}) matrix, and only the pairs of i -th row and j -th columns must be re-computed and summed up, i.e. an update is only $\mathcal{O}(2n)$ complex. Therefore, it depends on the use case which procedure should be used. For a quantitative study, use the BTL model to compute the scores in on go. And deploy the p-value based metric as online algorithm in an production deployment.

References

- [BT52] Ralph Allan Bradley and Milton E. Terry. “Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons”. In: *Biometrika* 39.3/4 (1952). Publisher: [Oxford University Press, Biometrika Trust], pp. 324–345. ISSN: 0006-3444. DOI: 10.2307/2334029. (Visited on 03/25/2021).
- [Hoa77] David C. Hoaglin. “Direct Approximations for Chi-Squared Percentage Points”. In: *Journal of the American Statistical Association* 72.359 (1977), pp. 508–515. DOI: 10.2307/2286208. (Visited on 04/01/2021).
- [LSY98] R. B. Lehoucq, D. C. Sorensen, and C. Yang. *ARPACK Users’ Guide*. Software, Environments and Tools. Society for Industrial and Applied Mathematics, Jan. 1998. ISBN: 978-0-89871-407-4. DOI: 10.1137/1.9780898719628. URL: <https://epubs.siam.org/doi/abs/10.1137/1.9780898719628>.
- [Saa03] Thomas L. Saaty. “Decision-making with the AHP: Why is the principal eigenvector necessary”. In: *European Journal of Operational Research* 145.1 (Feb. 2003), pp. 85–91. ISSN: 03772217. DOI: 10.1016/S0377-2217(02)00227-8.
- [Hun04] David R. Hunter. “MM algorithms for generalized Bradley-Terry models”. In: *The Annals of Statistics* 32.1 (Feb. 2004). Publisher: Institute of Mathematical Statistics, pp. 384–406. ISSN: 0090-5364, 2168-8966. DOI: 10.1214/aos/1079120141.
- [AH09] Awad H. Al-Mohy and Nicholas J. Higham. “A New Scaling and Squaring Algorithm for the Matrix Exponential”. In: *SIAM Journal on Matrix Analysis and Applications* 31.3 (Aug. 2009). Publisher: Society for Industrial and Applied Mathematics, pp. 970–989. ISSN: 0895-4798. DOI: 10.1137/09074721X. URL: <https://epubs.siam.org/doi/10.1137/09074721X>.
- [Orm09] Bryan Orme. “MaxDiff Analysis: Simple Counting, Individual-Level Logit, and HB”. en. 2009. URL: <https://sawtoothsoftware.com/uploads/sawtoothsoftware/originals/f89a6537-1cae-4fb5-afad-9d325c2a3143.pdf>.
- [KM17] Svetlana Kiritchenko and Saif Mohammad. “Best-Worst Scaling More Reliable than Rating Scales: A Case Study on Sentiment Intensity Annotation”. In: *Proceedings of the 55th Annual Meeting of the Association for Computational Linguistics (Volume 2: Short Papers)*. Vancouver, Canada: Association for Computational Linguistics, July 2017, pp. 465–470. DOI: 10.18653/v1/P17-2074.
- [Beh18] Eric Beh. “Exploring How to Simply Approximate the P-value of a Chi-squared Statistic”. In: *Austrian Journal of Statistics* 47.3 (May 2018), pp. 63–75. ISSN: 1026-597X. DOI: 10.17713/ajs.v47i3.757. (Visited on 03/30/2021).
- [Vir+20] Pauli Virtanen et al. “SciPy 1.0: fundamental algorithms for scientific computing in Python”. en. In: *Nature Methods* 17.3 (Mar. 2020). Number: 3 Publisher: Nature Publishing Group, pp. 261–272. ISSN: 1548-7105. DOI: 10.1038/s41592-019-0686-2. URL: <https://www.nature.com/articles/s41592-019-0686-2>.
- [Ham21] Ulf Hamster. *bwsample: Sampling and Evaluation of Best-Worst Scaling sets*. Version 0.6.1. Mar. 2021. DOI: 10.5281/zenodo.4603520. URL: <https://doi.org/10.5281/zenodo.4603520>.