GENERAL SOLUTION OF N^{th} -ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we present a formula for the general solution of n^{th} -order linear ordinary differential equation with constant coefficients as our main result. In this regard we also present two supporting results in this paper which reduce the order of the concerned differential equation by one and give the relation between the coefficients of the initial differential equation and the differential equation obtained. We also discuss about the complementary solution and homogeneous equations with regard to the main result described in this paper.

1. INTRODUCTION

The general n^{th} -order linear ordinary differential equation is given as

$$(1) \quad y_{(n)} + P_1(x)y_{(n-1)} + \cdots + P_{n-1}(x)y_{(1)} + P_n(x)y = R(x),$$

where $y_{(k)}$ represents k^{th} derivative of the dependent variable - y with respect to the independent variable - x , i.e.

$$(2) \quad y_{(k)} = \frac{d^k y}{dx^k},$$

$R(x)$, $P_i(x)$, $i = 1, 2, \dots, n$ are functions of the independent variable x .

When $R(x) = 0$, Eqn. (1) represents a homogeneous n^{th} -order linear ordinary differential equation [1] and when $R(x) \neq 0$, it represents a non-homogeneous n^{th} -order linear ordinary differential equation [2].

When $P_i(x)$, $i = 1, 2, \dots, n$, are constant, such that $P_i(x) = P_i$, $i = 1, 2, \dots, n$, then Eqn. (1) represents n^{th} -order linear ordinary differential equation with constant coefficients [3]. Such an equation is given as

$$(3) \quad y_{(n)} + P_1 y_{(n-1)} + \cdots + P_{n-1} y_{(1)} + P_n y = R(x).$$

In this paper we derive a formula for the general solution of n^{th} -order linear ordinary differential equation with constant coefficients involving the use of Integrating Factors (See Theorem-1). The formula comprises of n -successive integrals involving $R(x)$ and the Integrating Factors along with n -terms of complementary solution involving integrals of Integrating Factors where the Integrating Factors are determined using a polynomial equation corresponding to the given differential equation

(Eqn. (4)) such that it has the same coefficients as the differential equation (See Eqn. (7)).

We also present another form of our formula (See Theorem-2) where we give the general form of the complementary solution part which does not involve the Integrating Factors and rather involves only the roots of the polynomial equation (Eqn. (7)) corresponding to the given differential equation. Based on the main result, we also present a formula for the solution of a homogeneous n^{th} -order linear ordinary differential equation with constant coefficients (See Corollary-1 in Section-5).

The main result and its alternative form are presented in form of theorems while the result derived using these results are presented in form of a corollary. We also describe two supporting lemmas in Section-2 which are used in proving the main result.

We now present the main result in form of Theorem 1 here whose proof is given in Section-3 (and also in Section-4) after describing the lemmas need in Section-2.

Theorem 1. *The solution of the general n^{th} -order linear ordinary differential equation with constant coefficients,*

$$(4) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

is given by

$$(5) \quad \begin{aligned} y = & \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \cdots dx}_{n \text{ times}} \\ & + \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \cdots dx}_{(n-1) \text{ times}} \\ & + \cdots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)}, \end{aligned}$$

where

$$(6) \quad g_i(x) = e^{-\alpha_i x} \quad (i = 1, 2, \dots, n),$$

α_i are the roots of the polynomial equation

$$(7) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0$$

corresponding to the differential equation (4) and $g_i(x)$ is referred to as the i^{th} Integrating Factor ($i, n \in \mathbb{N}; i \leq n$).

2. PRELIMINARY RESULTS

In this section we describe two results in form of Lemmas which are used in the proof of Theorem-1 such that they help in reducing the order of a general linear n^{th} -order ordinary differential equation with constant coefficients by one and give relation between the coefficients of the original differential equation and the differential equation obtained.

We now present Lemma-1 here along with the proof as follows.

Lemma 1. For every n^{th} -order linear ordinary differential equation with constant coefficients,

$$(8) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

there exists a corresponding polynomial equation

$$(9) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0.$$

If its roots are α_i , $i = 1, 2, \dots, n$, then any one of its roots, α_1 (say), determines an Integrating Factor $g(x)$ such that

$$(10) \quad g(x) = e^{-\alpha_1 x}.$$

When the differential equation given by Eqn. (8) is multiplied by the Integrating Factor $g(x)$ followed by its integration then it gets transformed into a linear ordinary differential equation of order $(n-1)$ of the form given as

$$(11) \quad y_{(n-1)} + P_{n-11}y_{(n-2)} + \cdots + P_{n-1n-2}y_{(1)} + P_{n-1n-1}y = \frac{1}{g(x)} \int g(x)R(x) dx + \frac{C_1}{g(x)}.$$

Proof. Consider the differential equation given by Eqn. (4) ,

$$(12) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x).$$

Let each of its coefficients P_{ni} , $i = 1, 2, \dots, n-1$, be expressed as sum of two constants a_i and b_i such that

$$(13) \quad P_{ni} = a_i + b_i.$$

Then Eqn. (12) is transformed into

$$(14) \quad y_{(n)} + (a_1 + b_1)y_{(n-1)} + \cdots + (a_{n-1} + b_{n-1})y_{(1)} + P_{nn}y = R(x)$$

$$(15) \quad \implies y_{(n)} + (a_1y_{(n-1)} + b_1y_{(n-1)}) + \cdots + (a_{n-1}y_{(1)} + b_{n-1}y_{(1)}) + P_{nn}y = R(x).$$

The terms of the above equation can be regrouped as

$$(16) \quad (y_{(n)} + a_1y_{(n-1)}) + (b_1y_{(n-1)} + a_2y_{(n-2)}) + \cdots + (b_{n-2}y_{(2)} + a_{n-1}y_{(1)}) + (b_{n-1}y_{(1)} + P_{nn}y) = R(x).$$

Now, we multiply Eqn. (16) by a function, $g(x)$, which we call here as Integrating Factor, to get

$$(17) \quad (g(x)y_{(n)} + g(x)a_1y_{(n-1)}) + (g(x)b_1y_{(n-1)} + g(x)a_2y_{(n-2)}) + \cdots + (g(x)b_{n-1}y_{(1)} + g(x)P_{nn}y) = g(x)R(x).$$

We impose the following conditions on $g(x)$,

$$(18) \quad \begin{aligned} (g(x)y_{(n)} + g(x)a_1y_{(n-1)}) &= (g(x)y_{(n-1)})' \\ (g(x)b_1y_{(n-1)} + g(x)a_2y_{(n-2)}) &= (g(x)b_1y_{(n-2)})' \\ &\vdots \\ (g(x)b_{n-1}y_{(1)} + g(x)P_{nn}y) &= (g(x)b_{n-1}y)' \end{aligned}$$

such that Eqn. (17) becomes

$$(19) \quad (g(x)y_{(n-1)})' + (g(x)b_1y_{(n-2)})' + \cdots + (g(x)b_{n-1}y)' = g(x)R(x).$$

Consider the system of equations given in (18), it follows that

$$(20) \quad \begin{aligned} g(x)a_1y_{(n-1)} &= g'(x)y_{(n-1)} \\ g(x)a_2y_{(n-2)} &= g'(x)b_1y_{(n-2)} \\ &\vdots \\ g(x)P_{nn}y &= g'(x)b_{n-1}y. \end{aligned}$$

On comparing both the sides in the above equation, we obtain

$$(21) \quad \begin{aligned} g(x)a_1 &= g'(x) \\ g(x)a_2 &= g'(x)b_1 \\ &\vdots \\ g(x)P_{nn} &= g'(x)b_{n-1}. \end{aligned}$$

Using the relation $g'(x) = a_1g(x)$ in the above system of equations, we get

$$(22) \quad \begin{aligned} g'(x) &= a_1g(x) \\ a_1b_1 &= a_2 \\ &\vdots \\ a_1b_{n-2} &= a_{n-1} \\ a_1b_{n-1} &= P_{nn}. \end{aligned}$$

Let

$$(23) \quad \begin{aligned} A_1 &= -a_1 \\ A_2 &= -a_2 \\ &\vdots \\ A_{n-1} &= -a_{n-1}, \end{aligned}$$

then the system of equations as given in (22) transforms into

$$(24) \quad \begin{aligned} g'(x) &= -A_1g(x) \\ A_1b_1 &= A_2 \\ &\vdots \\ A_1b_{n-2} &= A_{n-1} \\ A_1b_{n-1} &= -P_{nn}. \end{aligned}$$

Thus, in order to determine the Integrating Factor - $g(x)$, the system of equations as obtained in Eqn. (24) needs to be solved.

Remark 1. Here we are trying to determine the function - $g(x)$, which we call here as the Integrating Factor, because by determining such a function we can transform the expression containing the derivative terms in the differential equation given by Eqn. (12) to a complete derivative of another such expression of order one less than the initial expression such that on integrating the corresponding differential equation we will obtain a differential equation of order one less than the original one.

From the system of equations as given in Eqn. (24), we have

$$(25) \quad g'(x) = -A_1 g(x),$$

i.e., the Integrating Factor $g(x)$ depends on the value of A_1 . If the value of A_1 is determined then Eqn. (25) can be solved to determine the Integrating Factor - $g(x)$.

Also, from the equation,

$$(26) \quad A_1 b_1 = A_2,$$

as defined in the system of equations given by Eqn. (24), it follows that

$$(27) \quad A_1(P_{n1} - a_1) = A_2$$

$$(28) \quad \implies A_1(P_{n1} + A_1) = A_2$$

$$(29) \quad \implies A_1^2 + P_{n1}A_1 - A_2 = 0.$$

Hence, we get an equation relating A_1 and A_2 in Eqn. (29). In order to determine the value of A_1 , we must know the values of P_{n1} and A_2 . But A_2 is an unknown here so, the value of A_1 cannot be determined by using Eqn. (29) alone.

However, we can substitute the value of A_2 from Eqn. (29) in the following equation,

$$(30) \quad A_1 b_2 = A_3,$$

as defined in the system of equations given by Eqn. (24) such that we obtain

$$(31) \quad A_1(P_{n2} - a_2) = A_3$$

$$(32) \quad \implies A_1(P_{n2} + A_2) = A_3$$

$$(33) \quad \implies A_1(P_{n2} + (A_1^2 + P_{n1}A_1)) = A_3$$

$$(34) \quad \implies A_1^3 + P_{n1}A_1^2 + P_{n2}A_1 = A_3.$$

Similarly, we can also substitute the value of A_3 from Eqn. (34) in the equation,

$$(35) \quad A_1 b_3 = A_4,$$

as defined in the system of equations given by Eqn. (24) to obtain

$$(36) \quad A_1(P_{n3} - a_3) = A_4$$

$$(37) \quad \implies A_1(P_{n3} + A_3) = A_4$$

$$(38) \quad \implies A_1(P_{n3} + (A_1^3 + P_{n1}A_1^2 + P_{n2}A_1)) = A_4$$

$$(39) \quad \implies A_1^4 + P_{n1}A_1^3 + P_{n2}A_1^2 + P_{n3}A_1 = A_4.$$

Hence, it follows that

$$(40) \quad A_5 = A_1^5 + P_{n1}A_1^4 + P_{n2}A_1^3 + P_{n3}A_1^2 + P_{n4}A_1$$

$$(41) \quad A_6 = A_1^6 + P_{n1}A_1^5 + P_{n2}A_1^4 + P_{n3}A_1^3 + P_{n4}A_1^2 + P_{n5}A_1$$

$\vdots$

$$(42) \quad A_{n-1} = A_1^{n-1} + P_{n1}A_1^{n-2} + \dots + P_{nn-2}A_1.$$

Also, from the system of equations as defined in Eqn. (24) we have

$$(43) \quad A_1b_{n-1} = -P_{nn}$$

$$(44) \quad \implies A_1(P_{nn-1} - a_{n-1}) = -P_{nn}$$

$$(45) \quad \implies A_1(P_{nn-1} + A_{n-1}) = -P_{nn}$$

Using Eqn. (42) we get

$$(46) \quad \implies A_1(P_{nn-1} + (A_1^{n-1} + \dots + P_{nn-2}A_1)) + P_{nn} = 0$$

$$(47) \quad \implies A_1^n + P_{n1}A_1^{n-1} + \dots + P_{nn-1}A_1 + P_{nn} = 0$$

Thus, we get a polynomial equation in single variable - A_1 having known coefficients - $P_{n1}, P_{n2}, \dots, P_{nn}$. This equation can be solved to give the values of A_1 which can be then used to determine the Integrating Factor - $g(x)$.

Moreover, in Eqn. (47), the coefficients of $A_1^{n-1}, A_1^{n-2}, \dots, A_1^0$ also correspond to the derivative terms - $y_{(n-1)}, y_{(n-2)}, \dots, y$, respectively, of the differential equation given by Eqn. (12). Therefore, we can conclude that for a general linear ordinary differential equation with constant coefficients given by Eqn. (12), there exists a corresponding polynomial equation,

$$(48) \quad X^n + P_{n1}X^{n-1} + \dots + P_{nn-1}X + P_{nn} = 0,$$

where Eqn. (48) is an equivalent representation of Eqn. (47), such that the roots of Eqn. (48) are also the roots of Eqn. (47) and the required value of A_1 can also be determined using Eqn. (48).

If the roots of the polynomial equation (48) are $\alpha_1, \alpha_2, \dots, \alpha_n$, then it follows that

$$(49) \quad X^n + P_{n1}X^{n-1} + \dots + P_{nn} = (X - \alpha_1)(X - \alpha_2) \dots (X - \alpha_n).$$

Let α_1 represent one of the roots of Eqn. (48) which is used to determine the Integrating Factor - $g(x)$ such that $A_1 = \alpha_1$.

Then using the value of A_1 in Eqn. (25), we get

$$(50) \quad g'(x) = -\alpha_1 g(x),$$

such that on solving the above equation, we obtain

$$(51) \quad g(x) = e^{-\alpha_1 x}.$$

Remark 2. *In the above equations for determining the Integrating Factor - $g(x)$, we have considered the root - α_1 of the polynomial equation (48) such that it represents any one root out of the n roots of the polynomial equation. This implies that for determining Integrating Factor for a given linear ordinary differential equation with constant coefficients of order n , we can use any one root out of the n roots of the corresponding polynomial equation and this choice is possible since we do not have any restriction on choosing the root for determining the Integrating Factor as it is shown by the equations above.*

Now, we consider Eqn. (19). On integrating it, we obtain

$$(52) \quad g(x)y_{(n-1)} + g(x)b_1y_{(n-2)} + \cdots + g(x)b_{n-1}y = \int g(x)R(x) dx + C$$

$$(53) \quad \begin{aligned} &\Rightarrow y_{(n-1)} + b_1y_{(n-2)} + \cdots + b_{n-2}y_{(1)} + b_{n-1}y = \\ &\frac{1}{g(x)} \int g(x)R(x) dx + \frac{C}{g(x)}. \end{aligned}$$

Let

$$(54) \quad \begin{aligned} b_1 &= P_{n-11} \\ b_2 &= P_{n-12} \\ &\vdots \\ b_{n-1} &= P_{n-1n-1}, \end{aligned}$$

where we have equivalently replaced b_i with P_{n-1i} , ($i = 1, 2, \dots, n$) such that P_{n-1i} represent coefficients of the differential equation of order $n - 1$ obtained in Eqn. (53).

Then Eqn. (53) becomes

$$(55) \quad \begin{aligned} &y_{(n-1)} + P_{n-11}y_{(n-2)} + \cdots + P_{n-1n-2}y_{(1)} + P_{n-1n-1}y = \\ &\frac{1}{g(x)} \int g(x)R(x) dx + \frac{C_1}{g(x)}. \end{aligned}$$

Therefore, we conclude by stating that a linear ordinary differential equation with constant coefficients as given by Eqn. (12) has a corresponding polynomial equation (48) whose roots can be represented as $-\alpha_1, \alpha_2, \dots, \alpha_n$ such that one of the roots - α_1 (say) determines the Integrating Factor - $g(x)$ which helps in transforming the given differential equation (12) into another such differential equation of order one less than it of the form as given by Eqn. (55). $\square$

The above Lemma gives the polynomial equation corresponding to a general linear ordinary differential equation with constant coefficients such that any one of its roots can be used to determine an Integrating Factor which can convert it into another similar differential equation of order one less. Hence, it is useful for reducing the order of a given linear ordinary differential equation with constant coefficients by one.

Lemma-1 is complemented by Lemma-2 such that it gives the relation between coefficients of the given differential equation and the differential equation obtained by the application of Lemma-1 in the given differential equation and also gives the information about the roots of the polynomial equation corresponding to the obtained differential equation. We present Lemma-2 along with the proof here as follows.

Lemma 2. *The differential equations (12) and (55), obtained by application of Lemma-1 in differential equation (12), as given in Lemma-1 have their coefficients related by the equation*

$$(56) \quad (X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1})(X - \alpha_1) = (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn})$$

where α_1 represents one of the roots of the polynomial equation (48) corresponding to the differential equation given by Eqn. (12) which is used in determining the Integrating Factor - $g(x)$ and is given as $g(x) = e^{-\alpha_1 x}$ such that the roots of the polynomial equation,

$$(57) \quad X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1} = 0,$$

corresponding to the differential equation (55) are - $\alpha_2, \alpha_3, \dots, \alpha_n$ when the roots of the polynomial equation (48) are $\alpha_1, \alpha_2, \dots, \alpha_n$.

Proof. Consider the differential equations (12),

$$y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

and (55),

$$y_{(n-1)} + P_{n-11}y_{(n-2)} + \cdots + P_{n-1n-2}y_{(1)} + P_{n-1n-1}y = \frac{1}{g(x)} \int g(x)R(x) dx + \frac{C_1}{g(x)},$$

obtained by the application of Lemma-1 in the differential equation (12), as described in Lemma-1.

Eqn. (12) has corresponding polynomial equation (Eqn. (48)) given as

$$X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0.$$

The above equation has roots - $\alpha_1, \alpha_2, \dots, \alpha_n$ as given in Lemma-1. Then it follows that

$$(58) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn} = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n).$$

By application of Lemma-1 in Eqn. (55), we can express its corresponding polynomial equation as

$$(59) \quad X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1} = 0.$$

For equations (48) and (59), we express the corresponding polynomials as

$$(60) \quad F_1(X) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn},$$

$$(61) \quad F_2(X) = X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-2}X + P_{n-1n-1}.$$

Consider the polynomial $F_2(X)$ as defined in Eqn. (61),

$$F_2(X) = X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-2}X + P_{n-1n-1}.$$

Using Eqn. (54) and Eqn. (13) in the above equation, we get

$$(62) \quad \begin{aligned} F_2(X) &= X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X + b_{n-1} \\ &= X^{n-1} + (P_{n1} - a_1)X^{n-2} + \cdots + (P_{nn-1} - a_{n-1}) \end{aligned}$$

Multiplying the above equation by X , we obtain

$$(63) \quad \begin{aligned} XF_2(X) &= (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X) - \\ &\quad (a_1X^{n-1} + a_2X^{n-2} + \cdots + a_{n-1}X). \end{aligned}$$

Using Eqn. (22) in the above equation, it transforms into

$$(64) \quad \begin{aligned} XF_2(X) &= (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X) - \\ &\quad a_1(X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X) \end{aligned}$$

$$(65) \quad \begin{aligned} \implies XF_2(X) &= (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X) - \\ &\quad a_1(X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X) + P_{nn} - P_{nn} \end{aligned}$$

$$(66) \quad \begin{aligned} \implies XF_2(X) &= (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn}) - \\ &\quad a_1(X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X + b_{n-1}) \end{aligned}$$

Using Eqn. (23), we obtain

$$(67) \quad \begin{aligned} \implies XF_2(X) &= (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn}) + \\ &\quad A_1(X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X + b_{n-1}) \end{aligned}$$

But

$$F_1(X) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn},$$

$$\begin{aligned} F_2(X) &= X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-2}X + P_{n-1n-1} \\ &= X^{n-1} + b_1X^{n-2} + \cdots + b_{n-2}X + b_{n-1} \text{ (By using Eqn. (54)).} \end{aligned}$$

Then Eqn. (67) becomes

$$(68) \quad XF_2(X) = F_1(X) + A_1F_2(X)$$

$$(69) \quad \implies (X - A_1)F_2(X) = F_1(X).$$

We have considered the value of A_1 to be α_1 , i.e. $A_1 = \alpha_1$, for determining the Integrating Factor - $g(x)$, as described in Eqn. (25) and Eqn. (50), for the differential equation given by Eqn. (12).

Then the above equation becomes

$$(70) \quad (X - \alpha_1)F_2(X) = F_1(X).$$

On using the expressions for the polynomials $F_1(X)$ and $F_2(X)$, the above equation transforms into

$$(71) \quad (X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1})(X - \alpha_1) = (X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn}).$$

Therefore, we get a relation (Eqn. (71)) between polynomials - $F_1(X)$ and $F_2(X)$ respectively corresponding to the differential equations - (12) and (55) such that it also describes the relation between the coefficients of the concerned differential equations.

On using Eqn. (58) in Eqn. (71), it follows that

$$(72) \quad (X - \alpha_1)(X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1}) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n).$$

On comparing both sides of the above equation, we obtain

$$(73) \quad (X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1}) = (X - \alpha_2)(X - \alpha_3) \cdots (X - \alpha_n).$$

Then Eqn. (59) becomes

$$(74) \quad (X - \alpha_2)(X - \alpha_3) \cdots (X - \alpha_n) = 0$$

such that the roots of the polynomial equation (59) corresponding to the differential equation (55) are - $\alpha_2, \alpha_3, \dots, \alpha_n$.

Therefore, we conclude by stating that the coefficients of the differential equations (12) and (55) are related by the means of Eqn. (71) such that the roots of the polynomial equation (59) corresponding to the differential equation (55) are $\alpha_2, \alpha_3, \dots, \alpha_n$ when $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of the polynomial equation given by Eqn. (48) corresponding to the differential equation (12) and α_1 is the root of Eqn. (48) used in determining the Integrating Factor - $g(x)$. $\square$

The above two results given in form of Lemma-1 and Lemma-2 are used in the proof of the main result given in form of Theorem-1 which is described in Section-3 as well as in Section-4 where Theorem-1 is proved by using Mathematical Induction.

3. PROOF OF THEOREM-1

In this section we restate Theorem-1, as given in Section-1, along with the proof here as follows.

Theorem 1. *The solution of the general n^{th} -order linear ordinary differential equation with constant coefficients,*

$$(75) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

is given by

$$(76) \quad \begin{aligned} y = & \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \cdots dx}_{n \text{ times}} \\ & + \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \cdots dx}_{(n-1) \text{ times}} \\ & + \cdots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)}, \end{aligned}$$

where

$$(77) \quad g_i(x) = e^{-\alpha_i x} \quad (i = 1, 2, \dots, n),$$

α_i are the roots of the polynomial equation

$$(78) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0$$

corresponding to the differential equation (75) and $g_i(x)$ is referred to as the i^{th} Integrating Factor ($i, n \in \mathbb{N}; i \leq n$).

Proof. Consider the general n^{th} -order linear ordinary differential equation with constant coefficients given as

$$(79) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x).$$

Using Lemma-1 in the above equation we get its corresponding polynomial equation as

$$(80) \quad G_1(X) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0.$$

Let its roots be - $\alpha_i, i = 1, 2, \dots, n$, then we have the corresponding Integrating Factor - $g_1(x)$ given as

$$(81) \quad g_1(x) = e^{-\alpha_1 x},$$

where α_1 represents one of the roots of the polynomial equation (80) chosen for determining the Integrating Factor - $g_1(x)$.

On multiplying Eqn. (79) with $g_1(x)$ and integrating, we obtain

$$(82) \quad \begin{aligned} & \int (g_1(x)y_{(n)} + g_1(x)P_{n1}y_{(n-1)} + \cdots + g_1(x)P_{nn}y) dx \\ & = \int g_1(x)R(x) dx + C_1. \end{aligned}$$

By Lemma-1, the above equation is transformed into

$$(83) \quad \begin{aligned} & y_{(n-1)} + P_{n-11}y_{(n-2)} + \cdots + P_{n-1n-2}y_{(1)} + P_{n-1n-1}y = \\ & \frac{1}{g_1(x)} \int g_1(x)R(x) dx + \frac{C_1}{g_1(x)}. \end{aligned}$$

Using Lemma-1 in the above equation, we get its corresponding polynomial equation as

$$(84) \quad G_2(X) = X^{n-1} + P_{n-11}X^{n-2} + \cdots + P_{n-1n-1} = 0.$$

Using Lemma-2 in Eqn. (79) and Eqn. (83), we get the relation between their coefficients by the equation

$$(85) \quad (X_1^{n-1} + P_{n-11}X_1^{n-2} + \cdots + P_{n-1n-1})(X_1 - \alpha_1) = (X_1^n + P_{n1}X_1^{n-1} + \cdots + P_{nn-1}X_1 + P_{nn}),$$

such that,

$$(86) \quad (X_1^{n-1} + P_{n-11}X_1^{n-2} + \cdots + P_{n-1n-1}) = (X_1 - \alpha_2)(X_1 - \alpha_3) \cdots (X_1 - \alpha_n).$$

Also

$$(87) \quad X_1^{n-1} + P_{n-11}X_1^{n-2} + \cdots + P_{n-1n-1} \equiv G_2(X),$$

$$(88) \quad X_1^n + P_{n1}X_1^{n-1} + \cdots + P_{nn-1}X_1 + P_{nn} \equiv G_1(X).$$

Hence, the roots of Eqn. (84) are $\alpha_i, i = 2, 3, \dots, n$ where $\alpha_i, i = 1, 2, \dots, n$ are the roots of Eqn. (80) and α_1 represents the root of the Eqn. (80) used for determining the Integrating Factor - $g_1(x)$.

Now, we can equivalently consider Eqn. (83) as a linear ordinary differential equation with constant coefficients of order $n - 1$ having corresponding polynomial equation (84) with roots - $\alpha_2, \alpha_3, \dots, \alpha_n$.

Using Lemma-1 in Eqn. (83), we define an Integrating Factor - $g_2(x)$ such that

$$(89) \quad g_2(x) = e^{-\alpha_2 x},$$

where α_2 represents one of the roots of the polynomial equation (84).

On multiplying Eqn. (83) with $g_2(x)$ and integrating, we obtain

$$(90) \quad \int (g_2(x)y_{(n-1)} + g_2(x)P_{n-11}y_{(n-2)} + \cdots + g_2(x)P_{n-1n-1}y) dx = \int \frac{g_2(x)}{g_1(x)} \int g_1(x)R(x) dx dx + C_1 \int \frac{g_2(x)}{g_1(x)} + C_2.$$

By Lemma-1, the above equation is transformed into

$$(91) \quad y_{(n-2)} + P_{n-21}y_{(n-3)} + \cdots + P_{n-2n-3}y_{(1)} + P_{n-2n-2}y = \frac{1}{g_2(x)} \int \frac{g_2(x)}{g_1(x)} \int g_1(x)R(x) dx dx + \frac{C_1}{g_2(x)} \int \frac{g_2(x)}{g_1(x)} + \frac{C_2}{g_2(x)}.$$

Using Lemma-1 in Eqn. (91), we get its corresponding polynomial equation as

$$(92) \quad G_3(X) = X^{n-2} + P_{n-21}X^{n-3} + \cdots + P_{n-2n-3}X + P_{n-2n-2} = 0.$$

Using Lemma-2 in Eqn. (83) and Eqn. (91), we get the relation between their coefficients by the equation

$$(93) \quad (X_2^{n-2} + P_{n-21}X_2^{n-3} + \cdots + P_{n-2n-2})(X_2 - \alpha_2) = (X_2^{n-1} + P_{n-11}X_2^{n-2} + \cdots + P_{n-1n-2}X_2 + P_{n-1n-1}),$$

such that

$$(94) \quad (X_2^{n-2} + P_{n-21}X_2^{n-3} + \cdots + P_{n-2n-2}) = (X_2 - \alpha_3)(X_2 - \alpha_4) \cdots (X_2 - \alpha_n).$$

Also

$$(95) \quad X_2^{n-2} + P_{n-21}X_2^{n-3} + \cdots + P_{n-2n-2} \equiv G_3(X),$$

$$(96) \quad X_2^{n-1} + P_{n-11}X_2^{n-2} + \cdots + P_{n-1n-2}X_2 + P_{n-1n-1} \equiv G_2(X).$$

Hence, the roots of Eqn. (92) are $\alpha_i, i = 3, 4, \dots, n$ where $\alpha_i, i = 2, 3, \dots, n$ are the roots of Eqn. (84) and α_2 represents the root of Eqn. (84) used in determining the Integrating Factor - $g_2(x)$.

Consider Eqn. (93) equivalently expressed as

$$(97) \quad G_3(X)(X - \alpha_2) = G_2(X),$$

and also Eqn. (86) equivalently expressed as

$$(98) \quad G_2(X)(X - \alpha_1) = G_1(X).$$

Using Eqn. (97) in Eqn. (98), we get

$$(99) \quad G_3(X)(X - \alpha_2)(X - \alpha_1) = G_1(X)$$

$$(100) \quad \Rightarrow (X^{n-2} + P_{n-21}X^{n-3} + \cdots + P_{n-2n-2})(X - \alpha_2)(X - \alpha_1) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0.$$

The above observations support the fact that Eqn. (92) has roots - $\alpha_3, \alpha_4, \dots, \alpha_n$ which are the roots of not only Eqn. (84) but also of Eqn. (80) such that the roots α_1 and α_2 are used to determine the Integrating Factors - $g_1(x)$ and $g_2(x)$ respectively.

Thus, if we consider Eqn. (79) to be the main differential equation, then we can say that Eqn. (92) has roots - $\alpha_3, \alpha_4, \dots, \alpha_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of Eqn. (80) corresponding to the main differential equation (79) and α_1, α_2 are used to determine the Integrating Factors - $g_1(x)$ and $g_2(x)$, which are used to transform the main differential equation given by Eqn. (79) to the obtained differential equation of order 2 less than the main differential equation which is given by Eqn. (91).

It is worthwhile to discuss about the coefficients $P_{ij}, j = 1, 2, \dots, i$, corresponding to a linear ordinary differential equation with constant coefficients of order i , for differential equations given by Eqn. (79), Eqn. (83) and Eqn. (91).

Consider the main differential equation given by Eqn. (79),

$$y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x).$$

It has corresponding polynomial equation (80),

$$G_1(X) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0,$$

with the same coefficients as the differential equation (Eqn. (79)) it corresponds to.

We have considered the roots of the above polynomial equation to be $\alpha_i, i = 1, 2, \dots, n$, such that we have the following corresponding relation -

$$(101) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn} = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n).$$

Since it (Eqn. (80)) is a polynomial equation (having n roots), we can express the relation between roots and coefficients by the following system of equations -

$$\begin{aligned} P_{n1} &= -(\text{Sum of roots taken 1 at a time}) \\ &= -(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\ &= -\sum \alpha_i (1 \leq i \leq n), \\ P_{n2} &= (-1)^2(\text{Sum of roots taken 2 at a time}) \\ (102) \quad &= (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \cdots) \\ &= \sum \alpha_i\alpha_j (1 \leq i, j \leq n; i \neq j), \\ &\vdots \\ P_{nn} &= (-1)^n(\text{Product of roots}) \\ &= (-1)^n(\alpha_1\alpha_2 \cdots \alpha_n). \end{aligned}$$

Similarly for the differential equations given by Eqn. (83) and Eqn. (91), the relation between the roots of the corresponding polynomial equations - Eqn. (84) and Eqn. (92) and their coefficients (of the corresponding differential equations too) can be described by the following systems of equations -

$$\begin{aligned} P_{n-11} &= -(\text{Sum of roots taken 1 at a time}) \\ &= -(\alpha_2 + \alpha_3 + \cdots + \alpha_n) \\ &= -\sum \alpha_i (2 \leq i \leq n), \\ P_{n-12} &= (-1)^2(\text{Sum of roots taken 2 at a time}) \\ (103) \quad &= (\alpha_2\alpha_3 + \alpha_2\alpha_4 + \cdots + \alpha_2\alpha_n + \alpha_3\alpha_4 + \cdots) \\ &= \sum \alpha_i\alpha_j (2 \leq i, j \leq n; i \neq j), \\ &\vdots \\ P_{n-1n-1} &= (-1)^{n-1}(\text{Product of roots}) \\ &= (-1)^{n-1}(\alpha_2\alpha_3 \cdots \alpha_n), \end{aligned}$$

$$\begin{aligned}
P_{n-21} &= -(\text{Sum of roots taken 1 at a time}) \\
&= -(\alpha_3 + \alpha_4 + \cdots + \alpha_n) \\
&= -\sum \alpha_i \ (3 \leq i \leq n), \\
P_{n-22} &= (-1)^2(\text{Sum of roots taken 2 at a time}) \\
&= (\alpha_3\alpha_4 + \alpha_3\alpha_5 + \cdots + \alpha_3\alpha_n + \alpha_4\alpha_5 + \cdots) \\
(104) \quad &= \sum \alpha_i\alpha_j \ (3 \leq i, j \leq n; i \neq j), \\
&\vdots \\
P_{n-2n-2} &= (-1)^{n-2}(\text{Product of roots}) \\
&= (-1)^{n-2}(\alpha_3\alpha_4 \cdots \alpha_n),
\end{aligned}$$

such that coefficients of the polynomial equation (84) have no involvement of α_1 (used for determining Integrating Factor - $g_1(x)$) and that of the polynomial equation (92) have no involvement of α_1, α_2 (used for determining Integrating Factors - $g_1(x)$ and $g_2(x)$ respectively), which is also justified by Eqn. (97) and Eqn. (99).

Therefore, from the above observations we can say that for our main differential equation of order n given by Eqn. (79), there is a corresponding main polynomial equation given by Eqn. (80) having roots - $\alpha_1, \alpha_2, \dots, \alpha_n$ such that on transformation to differential equation of order $n - 1$ (one less than the main differential equation), one of the roots - α_1 is used to determine the Integrating Factor - $g_1(x)$ and the other roots determine the coefficients of the differential equation obtained (Eqn. (83)). Similarly when the obtained differential equation of order $n - 1$ given by Eqn. (83) is transformed to another such differential equation of order $n - 2$ (one less than the previous equation), one of the remaining roots - α_2 from $\alpha_2, \alpha_3, \dots, \alpha_n$ is used to determine another Integrating Factor - $g_2(x)$ while the other roots determine the coefficients of the differential equation obtained (Eqn. (91)).

To sum up, by using Lemma-1 and Lemma-2, the main differential equation of order n given by Eqn. (79) is consecutively transformed into differential equation of order $n - 2$ given by Eqn. (91), through differential equation of order $n - 1$ given by Eqn. (83), with the help of the two Integrating Factors - $g_1(x)$ and $g_2(x)$ determined using two roots - α_1 and α_2 of the main polynomial equation given by Eqn. (80) (corresponding to the main differential equation given by Eqn. (79)) such that the remaining roots of the main polynomial equation determine the coefficients of the final differential equation obtained (Eqn. (91)).

Hence it follows that for k consecutive transformations ($k \leq n, k, n \in \mathbb{N}$) of the main differential equation, by using Lemma-1 and Lemma-2, k number of Integrating Factors - $g_1(x), g_2(x), \dots, g_k(x)$ are required where each Integrating Factor - $g_i(x)$ is determined using one of the roots of the main polynomial equation (Eqn. (80)), i.e. k number of

roots are used to determine the required Integrating Factors such that each Integrating Factor can be represented as

$$(105) \quad g_i(x) = e^{-\alpha_i x}, \quad i = 1, 2, \dots, k$$

where $\alpha_i, i = 1, 2, \dots, k$ represent k number of roots arbitrarily chosen from the n roots of the main polynomial equation. This is so because after i^{th} consecutive transformation of the main differential equation, the corresponding polynomial equation of the obtained differential equation has $n - i$ number of roots and any one of those roots may be chosen without any restriction to determine the $(i + 1)^{th}$ Integrating Factor such that in total i number of roots of the main polynomial equation are arbitrarily chosen from the n roots of the main polynomial equation to determine the i number of corresponding Integrating Factors where for each step up to the i^{th} step, the corresponding polynomial is a factor of the main polynomial as the roots of the corresponding polynomial equation are also the roots of the main polynomial equation.

Consequently, after the k^{th} consecutive transformation, the main differential equation given by Eqn. (79) becomes

$$(106) \quad \begin{aligned} & y_{(n-k)} + P_{n-k1}y_{(n-k-1)} + \dots + P_{n-kn-k+1}y_{(1)} + P_{n-kn-k}y \\ &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\ &+ \frac{C_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\ &+ \dots + \frac{C_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{C_k}{g_k(x)}, \end{aligned}$$

where

$$(107) \quad g_i(x) = e^{-\alpha_i x} \quad (i = 1, 2, \dots, k)$$

and $\alpha_i, i = 1, 2, \dots, k$ are the roots of the main polynomial equation given by Eqn. (80).

The coefficients of the above differential equation are determined using the remaining roots of the main polynomial equation - $\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n$ that can also be described by the using the following relation involving the corresponding polynomial of the differential equation (106) obtained by using Lemma-1 in the differential equation (106)

$$(108) \quad X^{n-k} + P_{n-k1}X^{n-k-1} + \dots + P_{n-kn-k} = 0$$

$$(109) \quad \begin{aligned} &\implies X^{n-k} + P_{n-k1}X^{n-k-1} + \dots + P_{n-kn-k} = \\ &(X - \alpha_{k+1})(X - \alpha_{k+2}) \dots (X - \alpha_n). \end{aligned}$$

Here, we call $g_i(x)$ as the i^{th} Integrating Factor for a linear ordinary differential equation with constant coefficients of order n ($i \leq n; n, i \in \mathbb{N}$).

Since each consecutive transformation of the main differential equation, given by Eqn. (79), by Lemma-1 and Lemma-2 transforms it into another such differential equation of order one less than the previous one, so, in order to solve the main differential equation of order n , we need n consecutive transformations to obtain differential equation of order 0 which is just the equation of the form $y = Q(x)$ that gives the complete solution of the main differential equation (Eqn. (79)) in terms of $R(x)$, the Integrating Factors and arbitrary constants.

Based on Eqn. (106), after n consecutive transformations of the main differential equation given by Eqn. (79), by using Lemma-1 and Lemma-2, we obtain

$$\begin{aligned}
 (110) \quad y = & \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
 & + \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}} \\
 & + \dots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)},
 \end{aligned}$$

such that it gives the solution of the main differential equation (as given in Eqn. (79)), where $C_1, C_2, \dots, C_n$ are arbitrary constants, $g_i(x)$ ($i = 1, 2, \dots, n$) is the i^{th} Integrating Factor given by

$$(111) \quad g_i(x) = e^{-\alpha_i x}$$

and α_i , ($i = 1, 2, \dots, n$) are the roots of the main polynomial equation (Eqn. (80)) corresponding to the main differential equation given by Eqn. (79) ($i \leq n; i, n \in \mathbb{N}$). $\square$

4. PROOF OF THEOREM-1 BY MATHEMATICAL INDUCTION

This section presents another proof of Theorem-1 by making use of Principle of Mathematical Induction which gives a stronger argument for the validity of Theorem-1 besides justifying the motivation involved in deriving the required result.

We again restate Theorem-1 along with the proof here as follows.

Theorem 1. *The solution of the general n^{th} -order linear ordinary differential equation with constant coefficients,*

$$(112) \quad y_{(n)} + P_{n1}y_{(n-1)} + \dots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

is given by

$$\begin{aligned}
(113) \quad y &= \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
&+ \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}} \\
&+ \cdots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)},
\end{aligned}$$

where

$$(114) \quad g_i(x) = e^{-\alpha_i x} \quad (i = 1, 2, \dots, n),$$

α_i are the roots of the polynomial equation

$$(115) \quad X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0$$

corresponding to the differential equation (112) and $g_i(x)$ is referred to as the i^{th} Integrating Factor ($i, n \in \mathbb{N}; i \leq n$).

Proof. We give the proof of Theorem-1 by Mathematical Induction. Consider the equations given by Eqn. (112) and Eqn. (113) such that when $n = 1$, we have from Eqn. (112),

$$(116) \quad y_{(1)} + P_{11}y = R(x)$$

and from Eqn. (113),

$$(117) \quad y = \frac{1}{g_1(x)} \int g_1(x) R(x) dx + \frac{C_1}{g_1(x)},$$

where $g_1(x) = e^{-\alpha_1 x}$ and α_1 is the root of $X + P_{11} = 0$ such that

$$(118) \quad g_1(x) = e^{-(-P_{11}x)} = e^{P_{11}x}.$$

Using the above equation in Eqn. (117) we obtain

$$(119) \quad y = \frac{1}{e^{P_{11}x}} \int e^{P_{11}x} R(x) dx + \frac{C_1}{e^{P_{11}x}}.$$

Moreover, Eqn. (4) is a first order linear ordinary differential equation with constant coefficients such that its solution is given by (See [5])

$$(120) \quad y = \frac{1}{e^{\int P_{11} dx}} \int e^{\int P_{11} dx} R(x) dx + \frac{C_1}{e^{\int P_{11} dx}}$$

$$(121) \quad \implies y = \frac{1}{e^{P_{11}x}} \int e^{P_{11}x} R(x) dx + \frac{C_1}{e^{P_{11}x}}.$$

Clearly Eqn. (119) and Eqn. (121) are equivalent. Hence, Theorem-1 is true for $n = 1$.

Let $T(n)$ represent the equivalent statement corresponding to the statement of Theorem-1 such that it is given as

T(n): The solution of the general n^{th} -order linear ordinary differential equation with constant coefficients,

$$(122) \quad y_{(n)} + P_{n1}y_{(n-1)} + \cdots + P_{nn-1}y_{(1)} + P_{nn}y = R(x),$$

is given by

$$(123) \quad \begin{aligned} y = & \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\ & + \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}} \\ & + \cdots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)}, \end{aligned}$$

where

$$(124) \quad g_i(x) = e^{-\alpha_i x} \quad (i = 1, 2, \dots, n),$$

α_i are the roots of the polynomial equation

$$(125) \quad H(X) = X^n + P_{n1}X^{n-1} + \cdots + P_{nn-1}X + P_{nn} = 0$$

corresponding to the differential equation (122) ($i, n \in \mathbb{N}; i \leq n$).

We assume that $T(k)$ is true for $k \in \mathbb{N}$ such that Eqn. (122), Eqn. (123) and Eqn. (125) from $T(n)$ corresponding to $T(k)$ can be equivalently expressed respectively as

$$(126) \quad y_{(k)} + p_{k1}y_{(k-1)} + \cdots + p_{kk-1}y_{(1)} + p_{kk}y = R_0(x),$$

$$(127) \quad \begin{aligned} y = & \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R_0(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\ & + \frac{C_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \cdots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\ & + \cdots + \frac{C_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{C_k}{g_k(x)}, \end{aligned}$$

$$(128) \quad H_1(X) = X^k + p_{k1}X^{k-1} + \cdots + p_{kk-1}X + p_{kk} = 0,$$

where $g_i(x) = e^{-\alpha_i x}$, $i = 1, 2, \dots, k$ and α_i are the roots of the polynomial equation given by Eqn. (128) and Eqn. (127) is the main result corresponding to $T(k)$ such that it gives the solution of the differential equation given by Eqn. (126) corresponding to $T(k)$.

Then we need to show that $T(k+1)$ is also true when $T(k)$ is true for $T(n)$ to be true $\forall n \in \mathbb{N}$.

Consider $T(k+1)$ such that $n = k+1$ and Eqn. (122) from $T(n)$ corresponding to $T(k+1)$ can be expressed as

$$(129) \quad y_{(k+1)} + P_{k+11}y_{(k)} + \cdots + P_{k+1k}y_{(1)} + P_{k+1k+1}y = R(x).$$

Using Lemma-1 in the above equation, we get its corresponding polynomial equation as

$$(130) \quad H_2(X) = X^{k+1} + P_{k+11}X^k + \cdots + P_{k+1k}X + P_{k+1k+1} = 0.$$

Let the roots of the above polynomial equation be - $\beta_i, i = 1, 2, \dots, k+1$, then we define the corresponding Integrating Factor as $g(x) = e^{-\beta_1 x}$, where β_1 represents one of the roots of the polynomial equation given by Eqn. (130).

On multiplying Eqn. (129) with $g(x)$ and integrating we obtain

$$(131) \quad \int (g(x)y_{(k+1)} + g(x)P_{k+11}y_{(k)} + \cdots + g(x)P_{k+1k+1}y) dx = \int g(x)R(x) dx + c.$$

By Lemma-1 the above equation is transformed into

$$(132) \quad y_{(k)} + P_{k1}y_{(k-1)} + \cdots + P_{kk-1}y_{(1)} + P_{kk}y = \frac{1}{g(x)} \int g(x)R(x) dx + \frac{c}{g(x)}.$$

Let

$$(133) \quad \frac{1}{g(x)} \int g(x)R(x) dx + \frac{c}{g(x)} = Q(x),$$

then Eqn. (132) is transformed into

$$(134) \quad y_{(k)} + P_{k1}y_{(k-1)} + \cdots + P_{kk-1}y_{(1)} + P_{kk}y = Q(x).$$

Using Lemma-1 in the above equation, we get its corresponding polynomial equation as

$$(135) \quad H_3(X) = X^k + P_{k1}X^{k-1} + \cdots + P_{kk-1}X + P_{kk} = 0.$$

Using Lemma-2 in Eqn. (129) and Eqn. (134), we get the relation between their coefficients by the equation given as

$$(136) \quad (X_1^k + P_{k1}X_1^{k-1} + \cdots + P_{kk-1}X_1 + P_{kk})(X_1 - \beta_1) = (X_1^{k+1} + P_{k+11}X_1^k + \cdots + P_{k+1k}X_1 + P_{k+1k+1}),$$

such that

$$(137) \quad (X_1^k + P_{k1}X_1^{k-1} + \cdots + P_{kk-1}X_1 + P_{kk}) = (X_1 - \beta_2)(X_1 - \beta_3) \cdots (X_1 - \beta_{k+1}).$$

Also

$$(138) \quad X_1^k + P_{k1}X_1^{k-1} + \cdots + P_{kk-1}X_1 + P_{kk} \equiv H_3(X),$$

$$(139) \quad X_1^{k+1} + P_{k+11}X_1^k + \cdots + P_{k+1k}X_1 + P_{k+1k+1} \equiv H_2(X).$$

Hence, the roots of polynomial equation given by Eqn (135) are - $\beta_2, \beta_3, \dots, \beta_{k+1}$ where $\beta_1, \beta_2, \dots, \beta_{k+1}$ are the roots of the polynomial equation given by Eqn. (130) and β_1 represents the root of the polynomial equation (130) used in determining the Integrating Factor - $g(x)$.

Since we get a linear ordinary differential equation with constant coefficients of order k in Eqn. (134), we can use the result corresponding to $T(k)$ in Eqn. (134) which gives the solution of k^{th} -order linear ordinary differential equation with constant coefficients of the form as given in Eqn. (126) through equations - Eqn. (127) and Eqn. (128).

Therefore, on using result corresponding to $T(k)$ in Eqn. (134), we obtain

$$\begin{aligned}
 (140) \quad y &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) Q(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\
 &+ \frac{c_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\
 &+ \dots + \frac{c_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{c_k}{g_k(x)},
 \end{aligned}$$

where $g_i(x) = e^{-\beta_i x}$ and $\beta_i, i = 2, 3, \dots, k+1$ are the roots of the polynomial equation (135) corresponding to the differential equation (134) such that Eqn. (135) is of the same form as Eqn. (128).

On putting the expression of $Q(x)$, as given in Eqn. (133), in the above equation (Eqn. (140)), we obtain

$$\begin{aligned}
 (141) \quad y &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_1(x)}{g(x)} \int g(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
 &+ \frac{c}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_1(x)}{g(x)} \underbrace{dx dx \dots dx}_{k \text{ times}} \\
 &+ \dots + \frac{c_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{c_k}{g_k(x)}.
 \end{aligned}$$

In the above equation, we have $c, c_1, c_2, \dots, c_k$ as arbitrary constants, $g(x) = e^{-\beta_1 x}$ and $g_i(x) = e^{-\beta_i x}, i = 2, 3, \dots, k+1$ where β_i are the roots of Eqn. (135) and β_1 is the root of Eqn. (130). But $\beta_i, i = 2, 3, \dots, k+1$ are also the roots of Eqn. (130), then we have $g(x) = e^{-\beta_1 x}$ and $g_i(x) = e^{-\beta_i x}, i = 2, 3, \dots, k+1$ such that $\beta_i, i = 1, 2, \dots, k+1$ are the roots of the polynomial equation given by Eqn. (130) corresponding to the differential equation given by Eqn. (129).

Let

$$(142) \quad G_1(x) = g(x), G_2(x) = g_1(x), \dots, G_{k+1}(x) = g_k(x);$$

$$(143) \quad C_1 = c, C_2 = c_1, \dots, C_{k+1} = c_k.$$

Then by using Eqn. (142) and Eqn. (143) in Eqn. (141), we get the result

$$\begin{aligned}
(144) \quad y &= \frac{1}{G_{k+1}(x)} \int \frac{G_{k+1}(x)}{G_k(x)} \int \dots \int \frac{G_2(x)}{G_1(x)} \int G_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&+ \frac{C_1}{G_{k+1}(x)} \int \frac{G_{k+1}(x)}{G_k(x)} \int \dots \int \frac{G_2(x)}{G_1(x)} \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ \dots + \frac{C_k}{G_{k+1}(x)} \int \frac{G_{k+1}(x)}{G_k(x)} dx + \frac{C_{k+1}}{G_{k+1}(x)},
\end{aligned}$$

where $G_i(x) = e^{-\beta_i x}$ and $\beta_i, i = 1, 2, \dots, k+1$ are the roots of the polynomial equation given by Eqn. (130) corresponding to the differential equation given by Eqn. (129) such that Eqn. (144) gives the solution of the concerned differential equation.

Also, the desirable form of the result for the solution of the concerned differential equation, according to $T(k+1)$, can be stated as:

$$\begin{aligned}
(145) \quad y &= \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&+ \frac{C_1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ \dots + \frac{C_k}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} dx + \frac{C_{k+1}}{g_{k+1}(x)},
\end{aligned}$$

where $g_i(x) = e^{-\alpha_i x}$ and $\alpha_i, i = 1, 2, \dots, k+1$ are the roots of the polynomial equation,

$$(146) \quad X^{k+1} + P_{k+11}X^k + \dots + P_{k+1k}X + P_{k+1k+1} = 0,$$

corresponding to the differential equation given by Eqn. (129) such that Eqn. (145) gives the solution of the concerned differential equation.

We observe that the result obtained in Eqn. (144) is equivalent to the result stated in Eqn. (145), which is the desired result according to $T(k+1)$.

Thus, $T(k+1)$ is true such that when $T(k)$ is true, $T(k+1)$ is also true.

Hence, we have shown the following:

- (1) $T(1)$ is true i.e. Theorem-1 follows for $n = 1$.
- (2) and $T(k+1)$ is also true when $T(k)$ is true for $k \in \mathbb{N}$.

Therefore, by the Principle of Mathematical Induction, $T(n)$ is true $\forall n \in \mathbb{N}$ such that Theorem-1 follows $\forall n \in \mathbb{N}$.

Remark 3. In the above proof since there are no restrictions on choosing the roots for determining each of the Integrating Factors, the solution expression for the concerned differential equation is equivalent

irrespective of the order of the roots chosen for the parameters α_i and $\beta_i, i = 1, 2, \dots, k+1$ that represent them.

□

5. OTHER RESULTS BASED ON THE MAIN RESULT

In this section we give two other important results related to the main result given by Theorem-1 in form of a theorem and its corollary. The results pertain to alternative form of solution of the general n^{th} -order linear ordinary differential equation of the form as given in Eqn. (3) and to the solution of homogeneous differential equation corresponding to the concerned differential equation which is commonly known (See [4]).

The first result of this section is given in Theorem-2 here along with the proof as follows.

Theorem 2. *The solution of the general n^{th} -order linear ordinary differential equation with constant coefficients, of the form as given in Eqn. (4), given through Eqn. (5) in Theorem-1 can be equivalently expressed as*

$$(147) \quad y = \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\ + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x} \\ + \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x},$$

where $g_i(x) = e^{-\alpha_i x}$, $\alpha_i, i = 1, 2, \dots, n$ are the roots of the polynomial equation given by Eqn. (7) corresponding to the differential equation given by Eqn. (4) and the roots $\alpha_1, \alpha_2, \dots, \alpha_n$ can be equivalently represented as $\lambda_1, \lambda_2, \dots, \lambda_m$ where each λ_i has corresponding multiplicity - $k_i, i = 1, 2, \dots, m$ such that

$$(148) \quad \sum_{i=1}^m k_i = n, \\ \alpha_1 = \alpha_2 = \dots = \alpha_{k_1} = \lambda_1, \\ \alpha_{k_1+1} = \alpha_{k_1+2} = \dots = \alpha_{k_1+k_2} = \lambda_2, \\ \vdots \\ \alpha_{n-k_m+1} = \alpha_{n-k_m+2} = \dots = \alpha_n = \lambda_m.$$

Proof. Consider the n^{th} -order linear ordinary differential equation with constant coefficients given by Eqn. (4) described here as follows -

$$(150) \quad y_{(n)} + P_{n1} y_{(n-1)} + \dots + P_{nn-1} y_{(1)} + P_{nn} y = R(x),$$

along with its corresponding polynomial equation (Eqn. (7)) given here as

$$(151) \quad X^n + P_{n1} X^{n-1} + \dots + P_{nn-1} X + P_{nn} = 0,$$

such that the solution of the given differential equation is given by Theorem-1 through Eqn. (5) as

$$\begin{aligned}
 (152) \quad y &= \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
 &+ \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}} \\
 &+ \dots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)},
 \end{aligned}$$

where $g_i(x) = e^{-\alpha_i x}$ and $\alpha_i, i = 1, 2, \dots, n$ are the roots of the polynomial equation given by Eqn. (7) corresponding to the differential equation given by Eqn. (4).

We shall prove this theorem by Mathematical Induction and in this regard we consider a statement $P(n)$ corresponding to the statement of Theorem-2 given as follows -

P(n): For the general n^{th} -order linear ordinary differential equation with constant coefficients of the form as given in Eqn. (4), the solution is given by Theorem-1 such that the solution expression is of the form as given in Eqn. (5) and is expressed as

$$\begin{aligned}
 (153) \quad y = Y_{n1}(x) &= \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
 &+ \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}} \\
 &+ \dots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)},
 \end{aligned}$$

which is equivalent to the solution expression given by the equation

$$\begin{aligned}
 (154) \quad y = Y_{n2}(x) &= \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
 &+ C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x} \\
 &+ \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x},
 \end{aligned}$$

such that

$$(155) \quad Y_{n1}(x) \equiv Y_{n2}(x),$$

where $g_i(x) = e^{-\alpha_i x}$, $\alpha_i, i = 1, 2, \dots, n$ are the roots of the polynomial equation given by Eqn. (7) corresponding to the differential equation of the form as given by Eqn. (5) and the roots $\alpha_1, \alpha_2, \dots, \alpha_n$ can be equivalently represented as $\lambda_1, \lambda_2, \dots, \lambda_m$ where each λ_i has corresponding

multiplicity - $k_i, i = 1, 2, \dots, m$ such that

$$\begin{aligned}
 (156) \quad & \sum_{i=1}^m k_i = n, \\
 & \alpha_1 = \alpha_2 = \dots = \alpha_{k_1} = \lambda_1, \\
 (157) \quad & \alpha_{k_1+1} = \alpha_{k_1+2} = \dots = \alpha_{k_1+k_2} = \lambda_2, \\
 & \vdots \\
 & \alpha_{n-k_m+1} = \alpha_{n-k_m+2} = \dots = \alpha_n = \lambda_m.
 \end{aligned}$$

Consider the statement $P(n)$ for $n = 1$ such that corresponding to $P(1)$, we have the differential equation

$$(158) \quad y_{(1)} + P_{11}y = R(x),$$

having solution given by Theorem-1 as

$$(159) \quad y = \frac{1}{g_1(x)} \int g_1(x) R(x) dx + \frac{C_1}{g_1(x)},$$

where $g_1(x) = e^{-\alpha_1 x}$ and α_1 is the root of the equation

$$(160) \quad X + P_{11} = 0.$$

Consider Eqn. (159) which gives the solution of the differential equation given by Eqn. (158) such that it can be evaluated further as

$$(161) \quad y = \frac{1}{g_1(x)} \int g_1(x) R(x) dx + C_1 e^{\alpha_1 x}.$$

Also, according to $P(1)$, the alternative form of solution of the differential equation given by Eqn. (158) is given as

$$(162) \quad y = \frac{1}{g_1(x)} \int g_1(x) R(x) dx + C_1 e^{\lambda_1 x},$$

where $g_1(x) = e^{\alpha_1 x}$ and α_1 is the root of Eqn. (160) corresponding to the differential equation given by Eqn. (158) such that the root α_1 can be equivalently represented as λ_1 having multiplicity k_1 ($k_1 = 1$) such that

$$\sum_{i=1}^1 k_i = 1 \text{ and } \alpha_1 = \lambda_1.$$

Thus, the solution of the differential equation given by Eqn. (158) obtained in Eqn. (161) is equivalent to the required form of solution as given by Eqn. (162) according to $P(1)$. So, $P(1)$ is true.

We assume that $P(k)$ is true for $k \in \mathbb{N}$ such that the result corresponding to it can be expressed as

$$(163) \quad Y_{k1}(x) \equiv Y_{k2}(x),$$

where

$$\begin{aligned}
(164) \quad Y_{k1}(x) &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R_0(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ \frac{C_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\
&+ \dots + \frac{C_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{C_k}{g_k(x)},
\end{aligned}$$

$$\begin{aligned}
(165) \quad Y_{k2}(x) &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R_0(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{K_1} x^{K_1-1} e^{\lambda_1 x} \\
&+ \dots + C_{k-K_m+1} e^{\lambda_m x} + \dots + C_k x^{K_m-1} e^{\lambda_m x},
\end{aligned}$$

$g_i(x) = e^{-\alpha_i x}$ and $\alpha_i, i = 1, 2, \dots, k$ are the roots of the polynomial equation

$$(166) \quad X^k + P_{k1}X^{k-1} + \dots + P_{kk-1}X + P_{kk} = 0$$

corresponding to a general k^{th} -order linear ordinary differential equation with constant coefficients of the form given by equation

$$(167) \quad y_{(k)} + P_{k1}y_{(k-1)} + \dots + P_{kk-1}y_{(1)} + P_{kk}y = R_0(x)$$

and the roots $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equivalently represented as $\lambda_1, \lambda_2, \dots, \lambda_m$ having corresponding multiplicities $K_1, K_2, \dots, K_m$ such that

$$\begin{aligned}
(168) \quad \sum_{i=1}^m K_i &= k, \\
\alpha_1 &= \alpha_2 = \dots = \alpha_{K_1} = \lambda_1, \\
\alpha_{K_1+1} &= \alpha_{K_1+2} = \dots = \alpha_{K_1+K_2} = \lambda_2, \\
(169) \quad &\vdots \\
\alpha_{k-K_m+1} &= \alpha_{k-K_m+2} = \dots = \alpha_k = \lambda_m.
\end{aligned}$$

Then we need to show that $P(k+1)$ is also true when $P(k)$ is true for $P(n)$ to be true $\forall n \in \mathbb{N}$.

Before discussing about the statement $P(k+1)$, we discuss about the integration of product of polynomial and exponential functions.

Consider a polynomial of degree n , ($n \in \{0, 1, 2, 3, \dots\}$) given by $f(x, n)$ as

$$(170) \quad f(x, n) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n,$$

where $a_0, a_1, \dots, a_n$ are constants.

Consider the integral

$$(171) \quad I = \int e^{kx} f(x, n) dx.$$

On applying Integration by parts rule in the above equation, we obtain

$$(172) \quad I = f(x, n) \frac{e^{kx}}{k} - \frac{d}{dx} (f(x, n)) \frac{e^{kx}}{k^2} + \dots$$

$$\dots + (-1)^n \frac{d^n}{dx^n} (f(x, n)) \frac{e^{kx}}{(k)^{n+1}} + C$$

$$(173) \quad \Rightarrow I = e^{kx} \left\{ \sum_{i=1}^{n+1} (-1)^{i-1} \frac{d^{i-1}}{dx^{i-1}} (f(x, n)) \frac{1}{k^i} \right\} + C.$$

As the derivative of a polynomial function of degree n is another polynomial function with degree $n-1$, the above integral represents equivalently the product of the same exponential function with another polynomial of the same degree such that

$$(174) \quad I = \int e^{kx} f(x, n) dx = e^{kx} F(x, n) + C,$$

where

$$(175) \quad F(x, n) = b_0 x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n$$

$$= \left\{ \sum_{i=1}^{n+1} (-1)^{i-1} \frac{d^{i-1}}{dx^{i-1}} (f(x, n)) \frac{1}{k^i} \right\}$$

and $b_0, b_1, \dots, b_n$ are constant values.

Therefore, from the above observations, we conclude that the integration of product of exponential function and polynomial function of degree n gives in result product of the same exponential function and another polynomial function of same degree such that the coefficients of the two polynomials are related by the equation given by Eqn. (175).

Consider the statement $P(n)$ for $n = k+1$, $k \in \mathbb{N}$, such that corresponding to $P(k+1)$ we have the differential equation,

$$(176) \quad y_{(k+1)} + P_{k+11} y_{(k)} + \dots + P_{k+1k} y_{(1)} + P_{k+1k+1} y = R(x),$$

having solution given by Theorem-1 as

$$(177) \quad y = Y_{k+11}(x)$$

$$= \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}}$$

$$+ \frac{c_1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{k \text{ times}}$$

$$+ \dots + \frac{c_k}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} dx + \frac{c_{k+1}}{g_{k+1}(x)},$$

where $g_i(x) = e^{-\beta_i x}$ and $\beta_i, i = 1, 2, \dots, k+1$ are the roots of the polynomial equation

$$(178) \quad X^{k+1} + P_{k+11} X^k + \dots + P_{k+1k} X + P_{k+1k+1} = 0$$

corresponding to the differential equation given by Eqn. (176).

Consider the solution expression for the differential equation (176) given by $Y_{k+11}(x)$ in Eqn. (177)

$$(179) \quad Y_{k+11}(x) = \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ + \frac{c_1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{k \text{ times}} \\ + \dots + \frac{c_k}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} dx + \frac{c_{k+1}}{g_{k+1}(x)},$$

$$(180) \quad \Rightarrow Y_{k+11}(x) = \frac{1}{g_{k+1}(x)} \int \left[g_{k+1}(x) \times \left\{ \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \right. \right. \\ + \frac{c_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\ \left. \left. + \dots + \frac{c_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{c_k}{g_k(x)} \right\} \right] dx + \frac{c_{k+1}}{g_{k+1}(x)}.$$

In Eqn. (180), consider the expression

$$(181) \quad \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\ + \frac{c_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\ + \dots + \frac{c_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{c_k}{g_k(x)} = Y_0(x) \text{ (Say).}$$

With regard to the above equation, we can say that it is solution of a k^{th} -order linear ordinary differential equation with constant coefficients,

$$(182) \quad y_{(k)} + p_1 y_{(k-1)} + \dots + p_{k-1} y_{(1)} + p_k y = R(x),$$

having corresponding polynomial equation

$$(183) \quad X^k + p_1 X^{k-1} + \dots + p_{k-1} X + p_k = 0$$

with roots $\beta_1, \beta_2, \dots, \beta_k$ such that the Integrating Factors $g_i(x), i = 1, 2, \dots, k$ are defined as

$$(184) \quad g_i(x) = e^{-\beta_i x} \quad (i = 1, 2, \dots, k),$$

which is in accordance with Theorem-1.

Since Eqn. (181) deals with k^{th} -order linear ordinary differential equation with constant coefficients, we can use the result corresponding to $P(k)$, given by Eqn. (163), in Eqn. (181) to get

$$\begin{aligned}
(185) \quad Y_0(x) &= \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R_0(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ \frac{c_1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(k-1) \text{ times}} \\
&+ \dots + \frac{c_{k-1}}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} dx + \frac{c_k}{g_k(x)} \\
&\equiv \frac{1}{g_k(x)} \int \frac{g_k(x)}{g_{k-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R_0(x) \underbrace{dx dx \dots dx}_{k \text{ times}} \\
&+ c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + \dots + c_{M_1} x^{M_1-1} e^{\lambda_1 x} \\
&+ \dots + c_{k-M_p+1} e^{\lambda_p x} + \dots + c_k x^{M_p-1} e^{\lambda_p x},
\end{aligned}$$

where the roots $\beta_1, \beta_2, \dots, \beta_k$ are equivalently represented as $\lambda_1, \lambda_2, \dots, \lambda_p$ having multiplicities $M_1, M_2, \dots, M_p$ respectively such that

$$\begin{aligned}
(186) \quad &\sum_{i=1}^p M_i = k, \\
&\beta_1 = \beta_2 = \dots = \beta_{M_1} = \lambda_1, \\
&\beta_{M_1+1} = \beta_{M_1+2} = \dots = \beta_{M_1+M_2} = \lambda_2, \\
(187) \quad &\vdots \\
&\beta_{k-M_p+1} = \beta_{k-M_p+2} = \dots = \beta_k = \lambda_p.
\end{aligned}$$

On using Eqn. (185) in Eqn. (180), we obtain

$$\begin{aligned}
(188) \quad Y_{k+1}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&+ \frac{1}{g_{k+1}(x)} \int \left\{ g_{k+1}(x) (c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \right. \\
&+ \dots + c_{M_1} x^{M_1-1} e^{\lambda_1 x} + \dots + c_{k-M_p+1} e^{\lambda_p x} \\
&+ \dots + c_k x^{M_p-1} e^{\lambda_p x}) dx \left. \right\} + \frac{c_{k+1}}{g_{k+1}(x)},
\end{aligned}$$

such that we can express the above equation equivalently as

$$\begin{aligned}
Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
& + e^{\beta_{k+1}x} \int \left\{ e^{-\beta_{k+1}x} \left(e^{\lambda_1 x} \sum_{i=1}^{M_1} (c_i x^{i-1}) \right. \right. \\
(189) \quad & + e^{\lambda_2 x} \sum_{i=1}^{M_2} (c_{M_1+i} x^{i-1}) + \dots \\
& \left. \left. \dots + e^{\lambda_p x} \sum_{i=1}^{M_p} (c_{k-M_p+i} x^{i-1}) \right) \right\} dx + \frac{c_{k+1}}{g_{k+1}(x)}.
\end{aligned}$$

In the above equation we can express

$$(190) \quad \sum_{i=1}^{M_j} (c_{M_i+\dots+M_{j-1}+i} x^{i-1}) = f_j(x, M_j - 1) \quad (j = 1, 2, \dots, p),$$

where $f_j(x, M_j - 1)$ represents a polynomial function in x of degree $(M_j - 1)$ with arbitrary constants as coefficients, such that the expression given in Eqn. (189) becomes

$$\begin{aligned}
Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
(191) \quad & + e^{\beta_{k+1}x} \int \left\{ e^{-\beta_{k+1}x} \left(e^{\lambda_1 x} f_1(x, M_1 - 1) \right. \right. \\
& + e^{\lambda_2 x} f_2(x, M_2 - 1) + \dots + e^{\lambda_p x} f_p(x, M_p - 1) \left. \right\} dx \\
& + \frac{c_{k+1}}{g_{k+1}(x)}.
\end{aligned}$$

For the above equation, we can consider two cases -

- (1) β_{k+1} is a non-repeated root of the polynomial equation (178).
- or
- (2) β_{k+1} is a repeated root of the polynomial equation (178).

Case 1. β_{k+1} is a non-repeated root of the polynomial equation (178).

In this case we have

$$(192) \quad \beta_{k+1} \neq \lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_p$$

such that $\lambda_i - \beta_{k+1} \neq 0, i = 1, 2, \dots, p$. For this case consider the expression of $Y_{k+11}(x)$ given in Eqn. (191) expressed as

$$\begin{aligned}
(193) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&+ e^{\beta_{k+1}x} \int \{ e^{(\lambda_1 - \beta_{k+1})x} f_1(x, M_1 - 1) \\
&+ e^{(\lambda_2 - \beta_{k+1})x} f_2(x, M_2 - 1) \\
&+ \dots + e^{(\lambda_p - \beta_{k+1})x} f_p(x, M_p - 1) \} dx + \frac{c_{k+1}}{g_{k+1}(x)}.
\end{aligned}$$

In the above equation consider the integral

$$\begin{aligned}
(194) \quad &\int \{ e^{(\lambda_1 - \beta_{k+1})x} f_1(x, M_1 - 1) + e^{(\lambda_2 - \beta_{k+1})x} f_2(x, M_2 - 1) \\
&+ \dots + e^{(\lambda_p - \beta_{k+1})x} f_p(x, M_p - 1) \} dx = I_1 \text{ (say)}
\end{aligned}$$

In the integral I_1 given in the above equation, each term is a product of exponential and polynomial functions which is of the form

$$(195) \quad e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1) \equiv e^{mx} f(x, n - 1) \quad (j = 1, 2, \dots, p),$$

such that $m = \lambda_j - \beta_{k+1}$ and $M_j = n$.

Hence, the each term of integral I_1 can be evaluated based on the result given through Eqn. (174) as

$$(196) \quad \int e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1) dx = e^{(\lambda_j - \beta_{k+1})x} F_j(x, M_j - 1)$$

where $j = 1, 2, \dots, p$, $F_j(x, M_j - 1)$ represents a different polynomial than $f_j(x, M_j - 1)$ but with the same degree such that

$$(197) \quad F_j(x, M_j - 1) = \sum_{i=1}^{M_j} (-1)^{i-1} \frac{d^{i-1}}{dx^{i-1}} (f_j(x, M_j - 1)) \frac{1}{(\lambda_j - \beta_{k+1})^i}.$$

Remark 4. In the above integral we have not considered the constant of integration since in the solution expression given by $Y_{k+11}(x)$ each integral is considered without the involvement of constant of integration as they have been already accounted for through the means of arbitrary constants - $c_1, c_2, \dots, c_{k+1}$ in Eqn. (179). So, it will be superfluous to consider another constant of integration in this case for the integral I_1 such that the constant of integration has been accounted for by the arbitrary constant c_{k+1} and this can be easily verified by considering Eqn. (193) and substituting the value of I_1 in it followed by its evaluation.

Since each $f_j(x, M_j - 1)$ has coefficients as arbitrary constants, the integral given in Eqn. (196) for each term, $e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1)$, will evaluate to product of the same exponential function, $e^{(\lambda_j - \beta_{k+1})x}$, and a different polynomial function, $F_j(x, M_j - 1)$, with the same degree such

that the coefficients of each $F_j(x, M_j - 1)$ will be linear combinations of the coefficients of $f_j(x, M_j - 1)$ (which are arbitrary constants) in accordance to Eqn. (197). As the coefficients of $F_j(x, M_j - 1)$ are linear combinations of arbitrary constants (which are the coefficients of $f_j(x, M_j - 1)$), the coefficients of $F_j(x, M_j - 1)$ will also be equivalently arbitrary constants such that we can express $F_j(x, M_j - 1)$ as

$$(198) \quad F_j(x, M_j - 1) = \sum_{i=1}^{M_j} (C_{M_i+\dots+M_{j-1}+i} x^{i-1}) \quad (j = 1, 2, \dots, p),$$

where each $C_{M_i+\dots+M_{j-1}+i}$ represents an arbitrary constant for all $i = 1, 2, \dots, M_j$ and $j = 1, 2, \dots, p$.

Form Eqn. (198) we also observe that the form of $F_j(x, M_j - 1)$ is equivalent to the form of $f_j(x, M_j - 1)$, where the corresponding coefficients of each term in both the polynomial functions are arbitrary constants. Hence we can state that

$$(199) \quad F_j(x, M_j - 1) \equiv f_j(x, M_j - 1),$$

such that

$$(200) \quad \int e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1) dx \equiv e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1),$$

subject to condition that the coefficients of $f_j(x, M_j - 1)$ are arbitrary constants.

So, we can express I_1 as

$$(201) \quad I_1 = e^{(\lambda_1 - \beta_{k+1})x} F_1(x, M_1 - 1) + e^{(\lambda_2 - \beta_{k+1})x} F_2(x, M_2 - 1) \\ + \dots + e^{(\lambda_p - \beta_{k+1})x} F_p(x, M_p - 1),$$

where $F_j(x, M_j - 1)$, ($j = 1, 2, \dots, p$) is as given in Eqn. (198).

Putting back the expression of I_1 in the solution expression given by $Y_{k+11}(x)$ in Eqn. (193) we obtain

$$(202) \quad Y_{k+11}(x) \equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ + \{ e^{\lambda_1 x} F_1(x, M_1 - 1) + e^{\lambda_2 x} F_2(x, M_2 - 1) \\ + \dots + e^{\lambda_p x} F_p(x, M_p - 1) \} + c_{k+1} e^{\beta_{k+1} x}.$$

On using the expression of $F_j(x, M_j - 1)$ from Eqn. (198) in the above equation, we obtain

$$\begin{aligned}
(203) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + e^{\lambda_1 x} \sum_{i=1}^{M_1} (C_i x^{i-1}) + e^{\lambda_2 x} \sum_{i=1}^{M_2} (C_{M_1+i} x^{i-1}) \\
&\quad + \dots + e^{\lambda_p x} \sum_{i=1}^{M_p} (C_{k-M_p+i} x^{i-1}) + c_{k+1} e^{\beta_{k+1} x}. \\
(204) \quad \Rightarrow Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{M_1} e^{\lambda_1 x} x^{M_1-1} \\
&\quad + \dots + C_k x^{M_p-1} e^{\lambda_p x} + c_{k+1} e^{\beta_{k+1} x}.
\end{aligned}$$

In this case let $\beta_{k+1} = \lambda_{p+1}$ such that λ_{p+1} has multiplicity $M_{p+1} = 1$. Then we have $Y_{k+11}(x)$ as

$$\begin{aligned}
(205) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{M_1} e^{\lambda_1 x} x^{M_1-1} \\
&\quad + \dots + C_k x^{M_p-1} e^{\lambda_p x} + c_{k+1} e^{\lambda_{p+1} x},
\end{aligned}$$

which gives the result for Case-1 where $Y_{k+11}(x)$, as given above, gives the equivalent expression of the solution given by Eqn. (177) for the differential equation given by Eqn. (176) such that $\beta_1, \beta_2, \dots, \beta_{k+1}$ are the roots of the polynomial equation given by Eqn. (178) corresponding to the concerned differential equation and these roots are equivalently expressed as $\lambda_1, \lambda_2, \dots, \lambda_{p+1}$ having respective multiplicities $M_1, M_2, \dots, M_{p+1}$ such that

$$\begin{aligned}
(206) \quad &\sum_{i=1}^{p+1} M_i = k + 1, \\
&\beta_1 = \beta_2 = \dots = \beta_{M_1} = \lambda_1, \\
&\beta_{M_1+1} = \beta_{M_1+2} = \dots = \beta_{M_1+M_2} = \lambda_2, \\
(207) \quad &\vdots \\
&\beta_{k-M_p+1} = \beta_{k-M_p+2} = \dots = \beta_k = \lambda_p, \\
&\beta_{k+1} = \lambda_{p+1}
\end{aligned}$$

and $g_i(x) = e^{-\beta_i x}$ ($i = 1, 2, \dots, k + 1$).

Case 2. β_{k+1} is a repeated root of the polynomial equation (178).

In this case we consider

$$(208) \quad \beta_{k+1} = \lambda_p \neq \lambda_{p-1} \neq \dots \neq \lambda_1, \text{ (which is an arbitrary choice)}$$

such that β_{k+1} is a repeated root of the polynomial equation given by Eqn. (178) and

$$(209) \quad \lambda_i - \beta_{k+1} \neq 0 \ (i = 1, 2, \dots, p-1),$$

$$(210) \quad \lambda_p - \beta_{k+1} = 0.$$

For this case consider the expression for $Y_{k+11}(x)$ as given in Eqn. (191)

$$(211) \quad \begin{aligned} Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ & + e^{\beta_{k+1}x} \int \{ e^{(\lambda_1 - \beta_{k+1})x} f_1(x, M_1 - 1) \\ & + e^{(\lambda_2 - \beta_{k+1})x} f_2(x, M_2 - 1) \\ & + \dots + e^{(\lambda_p - \beta_{k+1})x} f_p(x, M_p - 1) \} dx + \frac{c_{k+1}}{g_{k+1}(x)}. \end{aligned}$$

Using Eqn. (210) in the above equation, we obtain

$$(212) \quad \begin{aligned} Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ & + e^{\beta_{k+1}x} \int \{ e^{(\lambda_1 - \beta_{k+1})x} f_1(x, M_1 - 1) \\ & + e^{(\lambda_2 - \beta_{k+1})x} f_2(x, M_2 - 1) \\ & + \dots + e^{(\lambda_{p-1} - \beta_{k+1})x} f_{p-1}(x, M_{p-1} - 1) \} dx \\ & + e^{\beta_{k+1}x} \int f_p(x, M_p - 1) dx + \frac{c_{k+1}}{g_{k+1}(x)}. \end{aligned}$$

In the above equation consider the following integrals,

$$(213) \quad \begin{aligned} & \int \{ e^{(\lambda_1 - \beta_{k+1})x} f_1(x, M_1 - 1) + e^{(\lambda_2 - \beta_{k+1})x} f_2(x, M_2 - 1) \\ & + \dots + e^{(\lambda_{p-1} - \beta_{k+1})x} f_{p-1}(x, M_{p-1} - 1) \} dx = I_2 \text{ (say)}, \end{aligned}$$

$$(214) \quad \int f_p(x, M_p - 1) dx = I_3 \text{ (say)}.$$

Consider the integral I_2 in which each term is a product of exponential and polynomial functions such that the polynomial functions $f_j(x, M_j - 1)$ ($j = 1, 2, \dots, p-1$) have arbitrary constants as coefficients. Hence, in this case I_2 is evaluated similarly as I_1 is evaluated in Case-1.

For evaluating I_2 , we use the result corresponding to Eqn. (200) to obtain

$$(215) \quad \int e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1) dx \equiv e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1),$$

then, based on the result corresponding to Eqn. (174), we can also express the above integral as

$$(216) \quad \int e^{(\lambda_j - \beta_{k+1})x} f_j(x, M_j - 1) dx = e^{(\lambda_j - \beta_{k+1})x} F_j(x, M_j - 1),$$

where

$$(217) \quad F_j(x, M_j - 1) = \sum_{i=1}^{M_j} (C_{M_i + \dots + M_{j-1} + i} x^{i-1})$$

$$= \sum_{i=1}^{M_j} (-1)^{i-1} \frac{d^{i-1}}{dx^{i-1}} (f_j(x, M_j - 1)) \frac{1}{(\lambda_j - \beta_{k+1})^i},$$

$$(218) \quad F_j(x, M_j - 1) \equiv f_j(x, M_j - 1),$$

$C_{M_1 + \dots + M_{j-1} + i}$ represents arbitrary constant for all $i = 1, 2, \dots, M_j$ and $j = 1, 2, \dots, p-1$ such that each arbitrary constant, $C_{M_1 + \dots + M_{j-1} + i}$, of the corresponding term in $F_j(x, M_j - 1)$ is linear combination of coefficients of $f_j(x, M_j - 1)$, which are also arbitrary constants, in accordance to Eqn. (217).

Then, by using Eqn. (216), we can express I_2 as

$$(219) \quad I_2 = e^{(\lambda_1 - \beta_{k+1})x} F_1(x, M_1 - 1) + e^{(\lambda_2 - \beta_{k+1})x} F_2(x, M_2 - 1) \\ + \dots + e^{(\lambda_{p-1} - \beta_{k+1})x} F_{p-1}(x, M_{p-1} - 1),$$

where $F_j(x, M_j - 1)$ is as defined in Eqn. (217) for all $j = 1, 2, \dots, p-1$. The above equation can be further expressed as

$$(220) \quad I_2 = e^{(\lambda_1 - \beta_{k+1})x} \sum_{i=1}^{M_1} (C_i x^{i-1}) + e^{(\lambda_2 - \beta_{k+1})x} \sum_{i=1}^{M_2} (C_{M_1 + i} x^{i-1}) \\ + \dots + e^{(\lambda_{p-1} - \beta_{k+1})x} \sum_{i=1}^{M_{p-1}} (C_{M_1 + \dots + M_{p-2} + i} x^{i-1}).$$

Consider the integral I_3 given as

$$(221) \quad I_3 = \int f_p(x, M_p - 1) dx = \int \sum_{i=1}^{M_p} (c_{k-M_p+i} x^{i-1}) dx$$

$$(222) \quad \Rightarrow I_3 = \int (c_{k-M_p+1} + c_{k-M_p+2}x + \dots + c_k x^{M_p-1}) dx \\ = \sum_{i=1}^{M_p} \left(\frac{c_{k-M_p+i}}{i} x^i \right).$$

Since c_{k-M_p+i} is an arbitrary constant, $\frac{c_{k-M_p+i}}{i}$ can be equivalently replaced by another arbitrary constant C_{k-M_p+i} for all $i = 1, 2, \dots, M_p$

such that I_3 can be expressed as

$$(223) \quad I_3 = \sum_{i=1}^{M_p} (C_{k-M_p+i} x^i).$$

On putting the expressions of I_2 and I_3 in Eqn. (212), we obtain the solution expression $Y_{k+11}(x)$ as

$$(224) \quad \begin{aligned} Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ & + e^{\lambda_1 x} \sum_{i=1}^{M_1} (C_i x^{i-1}) + e^{\lambda_2 x} \sum_{i=1}^{M_2} (C_{M_1+i} x^{i-1}) \\ & + \dots + e^{\lambda_{p-1} x} \sum_{i=1}^{M_{p-1}} (C_{M_1+\dots+M_{p-2}+i} x^{i-1}) \\ & + e^{\lambda_p x} \sum_{i=1}^{M_p} (C_{k-M_p+i} x^i) + c_{k+1} e^{\beta_{k+1} x}. \end{aligned}$$

In the above equation we can replace c_{k+1} by C_{k+1} such that the equation becomes

$$(225) \quad \begin{aligned} Y_{k+11}(x) \equiv & \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\ & + e^{\lambda_1 x} \sum_{i=1}^{M_1} (C_i x^{i-1}) + e^{\lambda_2 x} \sum_{i=1}^{M_2} (C_{M_1+i} x^{i-1}) \\ & + \dots + e^{\lambda_{p-1} x} \sum_{i=1}^{M_{p-1}} (C_{M_1+\dots+M_{p-2}+i} x^{i-1}) \\ & + e^{\lambda_p x} \sum_{i=1}^{M_p} (C_{k-M_p+i} x^i) + C_{k+1} e^{\beta_{k+1} x}. \end{aligned}$$

But we also have the relation from Eqn. (210), $\beta_{k+1} = \lambda_p$ and also from the above equation, we have

$$(226) \quad \begin{aligned} \sum_{i=1}^{M_p} (C_{k-M_p+i} x^i) + C_{k+1} e^{\beta_{k+1} x} &= C_{k-M_p+1} x e^{\lambda_p x} \\ &+ \dots + C_k x^{M_p} e^{\lambda_p x} + C_{k+1} e^{\lambda_p x}. \end{aligned}$$

Using the above equation in Eqn. (225), we get

$$\begin{aligned}
(227) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{M_1} e^{\lambda_1 x} x^{M_1-1} \\
&\quad + \dots + C_{k-M_p+1} x e^{\lambda_p x} + \dots + C_k x^{M_p} e^{\lambda_p x} + C_{k+1} e^{\lambda_{p+1} x}.
\end{aligned}$$

We can express the above equation equivalently as

$$\begin{aligned}
(228) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + K_1 e^{\lambda_1 x} + K_2 x e^{\lambda_1 x} + \dots + K_{M_1} e^{\lambda_1 x} x^{M_1-1} \\
&\quad + \dots + K_{k-M_p+1} e^{\lambda_{p+1} x} + \dots + K_{k+1} x^{M_p} e^{\lambda_p x},
\end{aligned}$$

where $K_1, K_2, \dots, K_{k+1}$ are arbitrary constants such that

$$\begin{aligned}
(229) \quad K_i &= C_i, \quad i = 1, 2, \dots, k - M_p, \\
K_{j+1} &= C_j, \quad j = k - M_p + 1, k - M_p + 2, \dots, k, \\
K_{k-M_p+1} &= C_{k+1}.
\end{aligned}$$

Thus, Eqn. (228) gives the result for Case-2 where $Y_{k+11}(x)$ gives the equivalent expression of the solution given by Eqn. (177) for the differential equation given by Eqn. (176) such that $\beta_1, \beta_2, \dots, \beta_{k+1}$ are the roots of the polynomial equation given by Eqn. (178) corresponding to the concerned differential equation and these roots are equivalently expressed as $\lambda_1, \lambda_2, \dots, \lambda_p$ having respective multiplicities $M_1, M_2, \dots, M_p + 1$ respectively such that

$$(230) \quad M_1 + M_2 + \dots + M_{p-1} + (M_p + 1) = k + 1,$$

$$\beta_1 = \beta_2 = \dots = \beta_{M_1} = \lambda_1,$$

$$\beta_{M_1+1} = \beta_{M_1+2} = \dots = \beta_{M_1+M_2} = \lambda_2,$$

$$(231) \quad \vdots$$

$$\beta_{k-M_p+1} = \beta_{k-M_p+2} = \dots = \beta_k = \beta_{k+1} = \lambda_p$$

and $g_i(x) = e^{-\beta_i x}$ ($i = 1, 2, \dots, k + 1$).

We now consider the desirable form of the result for the equivalent solution of the differential equation given by Eqn. (176), corresponding to $P(k + 1)$, which is given as

$$\begin{aligned}
(232) \quad Y_{k+11}(x) &\equiv \frac{1}{g_{k+1}(x)} \int \frac{g_{k+1}(x)}{g_k(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{(k+1) \text{ times}} \\
&\quad + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{m_1} e^{\lambda_1 x} x^{m_1-1} \\
&\quad + \dots + C_{(k+1)-m_l+1} e^{\lambda_l x} + \dots + C_{k+1} x^{m_l-1} e^{\lambda_l x},
\end{aligned}$$

where $Y_{k+11}(x)$ is as defined in Eqn. (177), $g_i(x) = e^{-\alpha_i x}$, $i = 1, 2, \dots, k+1$, α_i are the roots of the polynomial equation given by Eqn. (178) corresponding to the differential equation given by Eqn. (176) and these roots can be equivalently represented as $\lambda_1, \lambda_2, \dots, \lambda_l$ having corresponding multiplicities $m_1, m_2, \dots, m_l$ such that

$$\begin{aligned}
 (233) \quad & \sum_{i=1}^l m_i = k+1, \\
 & \alpha_1 = \alpha_2 = \dots = \alpha_{m_1} = \lambda_1, \\
 (234) \quad & \alpha_{m_1+1} = \alpha_{m_1+2} = \dots = \alpha_{m_1+m_2} = \lambda_2, \\
 & \vdots \\
 & \alpha_{(k+1)-m_l+1} = \alpha_{(k+1)-m_l+2} = \dots = \alpha_{k+1} = \lambda_l.
 \end{aligned}$$

We observe that the results obtained in Eqn. (205) and Eqn. (228) for Case-1 and Case-2 respectively, corresponding to $P(k+1)$, are equivalent to the result stated in Eqn. (232) i.e. the desired result according to $P(k+1)$.

Therefore, $P(k+1)$ is true such that when $P(k)$ is true, $P(k+1)$ is also true.

Hence, we have shown the following:

- (1) $P(1)$ is true i.e. Theorem-2 follows for $n = 1$.
- (2) and $P(k+1)$ is true when $P(k)$ is true for $k \in \mathbb{N}$.

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true $\forall n \in \mathbb{N}$ such that Theorem-2 follows $\forall n \in \mathbb{N}$. $\square$

The second result of this section deals with the solution of homogeneous n^{th} -order linear ordinary differential equation with constant coefficients such that it gives the commonly known form of solution of such equation [4]. The result is given in form of a corollary of Theorem-2 such that it is derived by using the result given by Theorem-2.

The second result is given by Corollary-1 along with the proof here as follows.

Corollary 1. *The solution of the homogeneous n^{th} -order linear ordinary differential equation with constant coefficients,*

$$(235) \quad y_{(n)} + P_1 y_{(n-1)} + \dots + P_{n-1} y_{(1)} + P_n y = 0,$$

is given by

$$\begin{aligned}
 (236) \quad y = & C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x} \\
 & + \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x},
 \end{aligned}$$

where $\alpha_i, i = 1, 2, \dots, n$ are the roots of the polynomial equation,

$$(237) \quad X^n + P_1 X^{n-1} + \dots + P_{n-1} X + P_n = 0,$$

corresponding to the differential equation given by Eqn. (235) and the roots of this equation can be expressed equivalently as $\lambda_1, \lambda_2, \dots, \lambda_m$ having corresponding multiplicities as $k_1, k_2, \dots, k_m$ such that

$$(238) \quad \sum_{i=1}^m k_i = n,$$

$$\alpha_1 = \alpha_2 = \dots = \alpha_{k_1} = \lambda_1,$$

$$(239) \quad \alpha_{k_1+1} = \alpha_{k_1+2} = \dots = \alpha_{k_1+k_2} = \lambda_2,$$

$$\vdots$$

$$\alpha_{n-k_m+1} = \alpha_{n-k_m+2} = \dots = \alpha_n = \lambda_m.$$

Proof. Consider a general n^{th} -order linear ordinary differential equation with constant coefficients given as

$$(240) \quad y_{(n)} + P_1 y_{(n-1)} + \dots + P_{n-1} y_{(1)} + P_n y = R(x).$$

By Theorem-1, the solution of the above equation is given as

$$(241) \quad y = Y_{n1}(x) = \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}}$$

$$+ \frac{C_1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \underbrace{dx dx \dots dx}_{(n-1) \text{ times}}$$

$$+ \dots + \frac{C_{n-1}}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} dx + \frac{C_n}{g_n(x)},$$

where $g_i(x) = e^{-\alpha_i x}$, $i = 1, 2, \dots, n$ and α_i are the roots of the polynomial equation,

$$(242) \quad X^n + P_1 X^{n-1} + \dots + P_{n-1} X + P_n = 0,$$

corresponding to the differential equation given by Eqn. (240).

Also by Theorem-2, the alternative expression of the solution of the concerned differential equation (Eqn. (240)) is given as

$$(243) \quad y = Y_{n2}(x) = \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) R(x) \underbrace{dx dx \dots dx}_{n \text{ times}}$$

$$+ C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x}$$

$$+ \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x},$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the equivalent representation of the roots given as $\alpha_1, \alpha_2, \dots, \alpha_n$ of the polynomial equation given by Eqn. (242) having

respective multiplicities $k_1, k_2, \dots, k_m$ such that

$$\begin{aligned}
 (244) \quad & \sum_{i=1}^m k_i = n, \\
 & \alpha_1 = \alpha_2 = \dots = \alpha_{k_1} = \lambda_1, \\
 (245) \quad & \alpha_{k_1+1} = \alpha_{k_1+2} = \dots = \alpha_{k_1+k_2} = \lambda_2, \\
 & \vdots \\
 & \alpha_{n-k_m+1} = \alpha_{n-k_m+2} = \dots = \alpha_n = \lambda_m
 \end{aligned}$$

and $Y_{n1}(x) \equiv Y_{n2}(x)$.

For the solution of homogeneous n^{th} -order linear ordinary differential equation with constant coefficients we put $R(x) = 0$ in equations - Eqn. (240) and Eqn. (243) such that we have the corresponding differential equation given as

$$(246) \quad y_{(n)} + P_1 y_{(n-1)} + \dots + P_{n-1} y_{(1)} + P_n y = 0$$

and the corresponding solution given as

$$\begin{aligned}
 (247) \quad y = Y_{n2}(x) = & \frac{1}{g_n(x)} \int \frac{g_n(x)}{g_{n-1}(x)} \int \dots \int \frac{g_2(x)}{g_1(x)} \int g_1(x) (0) \underbrace{dx dx \dots dx}_{n \text{ times}} \\
 & + C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x} \\
 & + \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x}.
 \end{aligned}$$

In the above equation, the first term comprising of n -successive integrals having $R(x) = 0$ evaluates to 0 since the integrals are considered without the involvement of arbitrary constants as they have been accounted for through the means of arbitrary constants $C_1, C_2, \dots, C_n$ such that it will be superfluous to consider constants of integration for the concerned integral term.

Therefore, we get the solution for the differential equation given by Eqn. (246) through Eqn. (247), given as

$$\begin{aligned}
 (248) \quad y = & C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \dots + C_{k_1} x^{k_1-1} e^{\lambda_1 x} \\
 & + \dots + C_{n-k_m+1} e^{\lambda_m x} + \dots + C_n x^{k_m-1} e^{\lambda_m x},
 \end{aligned}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of the polynomial equation given by Eqn. (242) which corresponds to the differential equation given by Eqn. (240) (and equivalently to Eqn. (246) as well) and these roots are equivalently expressed as $\lambda_1, \lambda_2, \dots, \lambda_m$ having respective multiplicities

$k_1, k_2, \dots, k_m$ such that

$$(249) \quad \sum_{i=1}^m k_i = n,$$

$$\alpha_1 = \alpha_2 = \dots = \alpha_{k_1} = \lambda_1,$$

$$(250) \quad \alpha_{k_1+1} = \alpha_{k_1+2} = \dots = \alpha_{k_1+k_2} = \lambda_2,$$

$$\vdots$$

$$\alpha_{n-k_m+1} = \alpha_{n-k_m+2} = \dots = \alpha_n = \lambda_m.$$

Thus, we obtain the required result through Eqn. (248) as described above. $\square$

Besides the above two results another important result which follows is that the complete solution of the general n^{th} -order linear ordinary differential equation with constant coefficients is given as sum of two parts - the solution of the corresponding homogeneous equation, which is also known as complementary solution [1], and the particular solution [2].

The complementary solution of the general n^{th} -order linear ordinary differential equation with constant coefficients is the solution of the corresponding homogeneous equation such that the parts of the solution expressions as given in Eqn. (5) and Eqn. (147) having arbitrary constants represent the complementary solution parts. This follows from Corollary-1 wherein it gives the solution of the homogeneous differential equation corresponding to the concerned differential equation such that the solution obtained is the part of the solution given by Theorem-2 which contains arbitrary constants and is also correspondingly the part of the solution given by Theorem-1 containing arbitrary constants as the solutions given by Theorem-1 and Theorem-2 are equivalent. Moreover, the complementary solution does not involve the function $R(x)$ but only involves the Integrating Factors and the roots of the polynomial equation corresponding to the concerned differential equation such that this part of solution is equivalent for the family of concerned differential equations having the same coefficients.

The particular solution of the general n^{th} -order linear ordinary differential equation with constant coefficients is the solution which satisfies the non-homogeneous differential equation such that it depends on the function $R(x)$ and is given by the first term in each of the equations - Eqn. (5) and Eqn. (147) corresponding to Theorem-1 and Theorem-2 respectively such that it does not contain arbitrary constants and involves n -successive integrals, Integrating factors and the function $R(x)$.

Therefore, the general solution of the n^{th} -order linear ordinary differential equation with constant coefficients comprises of the complementary solution part and the particular solution part such that it

satisfies the given differential equation completely for all valid values of arbitrary constants.

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