SEVERAL CLOSED AND DETERMINANTAL FORMS FOR CONVOLVED FIBONACCI NUMBERS

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ABSTRACT. In the paper, with the aid of the Faà di Bruno formula, some properties of the Bell polynomials of the second kind, and a general derivative formula for a ratio of two differentiable functions, the authors establish several closed and determinantal forms for convolved Fibonacci numbers.

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1. BACKGROUND AND MOTIVATIONS

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ subject to initial values $F_0 = 0$ and $F_1 = 1$. All the Fibonacci numbers F_n for $n \geq 1$ can be generated by

$$\frac{1}{1-t-t^2} = \sum_{n=0}^{\infty} F_{n+1} t^n = 1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + \cdots$$

Numerous types of generalizations of these numbers have been offered in the literature and they have a plenty of interesting properties and applications to almost every field of science and art. For more information, please refer to [11, 18, 32] and closely related references therein.

For $r \in \mathbb{R}$, convolved Fibonacci numbers $F_n(r)$ can be defined by

$$\frac{1}{(1-t-t^2)^r} = \sum_{n=0}^{\infty} F_{n+1}(r) t^n. \quad (1)$$

See [1, 2, 6, 14, 19] and closely related references therein. It is easy to see that $F_n(1) = F_n$ for $n \geq 1$. Convolved Fibonacci numbers $F_n(r)$ for $n \in \mathbb{N}$ and their generalizations have been considered in the papers [8, 10, 12, 14, 19, 37, 38, 39]

2010 *Mathematics Subject Classification.* 05A19, 11B83, 11C20, 11Y55.

Key words and phrases. Bell polynomial of the second kind; closed form; convolved Fibonacci number; determinantal form; Faà di Bruno formula; Fibonacci number; Hessenberg determinant; tridiagonal determinant.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

in a variety of context. For instance, Ramirez defined in [37] the convolved $h(x)$ -Fibonacci numbers as an extension of classical convolved Fibonacci numbers and offered several combinatorial formulas.

In recent years, the second author of this paper and his colleagues have obtained a number of explicit and recursive formulas for some important numbers and polynomials via the Faà di Bruno formula, some properties of the Bell polynomials of the second kind, and a general derivative formula for a ratio of two differentiable functions. One may consult the papers [15, 16, 17, 21, 24, 28, 29, 30, 36], especially the review and survey article [31], and closely related references therein.

A tridiagonal determinant is a determinant whose nonzero elements locate only on the diagonal and slots horizontally or vertically adjacent the diagonal. In other words, a determinant $H = |h_{ij}|_{n \times n}$ is called a tridiagonal determinant if $h_{ij} = 0$ for all pairs (i, j) such that $|i - j| > 1$. See [7].

A determinant $H = |h_{ij}|_{n \times n}$ is called a lower (or an upper, respectively) Hessenberg determinant if $h_{ij} = 0$ for all pairs (i, j) such that $i + 1 < j$ (or $j + 1 < i$, respectively). See [13].

In analytic combinatorics and analytic number theory, explaining a sequence of numbers or a sequence of polynomials in terms of a special and simple determinant is an interesting and important direction and sometimes produces significant results. Accordingly, some sequences and polynomials, such as the Bernoulli numbers and polynomials [22, 26], the Euler numbers and polynomials [40], the Delannoy numbers [15, 20, 21, 23], the Horadam polynomials [27], and the Fibonacci numbers and polynomial [18, 20, 25, 33], have been considered in this framework. Interested readers can also refer to the studies in [9, 25, 34, 35] and closely related references therein.

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations and may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit. For systematic explanation of closed forms, please refer to the article [3] and closely related references therein.

In this paper, we will find several closed and determinantal forms for convolved Fibonacci numbers $F_{n+1}(r)$ by means of the Faà di Bruno formula, some properties and special values for the Bell polynomials of the second kind, and the generating function method, while we will give a determinantal formula for convolved Fibonacci numbers $F_{n+1}(r)$ in terms of tridiagonal determinants by aid of a general derivative formula, which is fundamental but not extensively circulated yet, for the ratio of two differentiable functions.

2. LEMMAS

In order to prove our main results, we recall several lemmas below.

Lemma 1 ([5, pp. 134 and 139]). *The Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \geq k \geq 0$, is defined by*

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{\ell-k+1} \ell_i!} \prod_{i=1}^{\ell-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \quad (2)$$

Lemma 2 ([5, p. 135]). *For $n \geq k \geq 0$, we have*

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \quad (3)$$

where a and b are any complex number.

Lemma 3 ([31, Section 1.4]). *For $n \geq k \geq 0$, we have*

$$B_{n,k}(x, 1, 0, \dots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \binom{k}{n-k} x^{2k-n}. \quad (4)$$

Lemma 4 ([22, Lemma 2.4]). *Let $f(t) = 1 + \sum_{k=1}^{\infty} a_k t^k$ and $g(t) = 1 + \sum_{k=1}^{\infty} b_k t^k$ be formal power series such that $f(t)g(t) = 1$. Then*

$$b_n = (-1)^n \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}.$$

Lemma 5 ([4, p. 40, Entry 5]). *For two differentiable functions $p(x)$ and $q(x) \neq 0$, we have*

$$\frac{d^k}{dx^k} \left[\frac{p(x)}{q(x)} \right] = \frac{(-1)^k}{q^{k+1}} \begin{vmatrix} p & q & 0 & \cdots & 0 & 0 \\ p' & q' & q & \cdots & 0 & 0 \\ p'' & q'' & \binom{2}{1} q' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1} q^{(k-3)} & \cdots & q & 0 \\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1} q^{(k-2)} & \cdots & \binom{k-1}{k-2} q' & q \\ p^{(k)} & q^{(k)} & \binom{k}{1} q^{(k-1)} & \cdots & \binom{k}{k-2} q'' & \binom{k}{k-1} q' \end{vmatrix}.$$

In other words,

$$\frac{d^k}{dx^k} \left[\frac{p(x)}{q(x)} \right] = \frac{(-1)^k}{q^{k+1}(x)} |U_{(k+1) \times 1}(x) \quad V_{(k+1) \times k}(x)|,$$

where the matrix $U_{(k+1) \times 1}(x) = (u_{\ell,1}(x))$ has the elements $u_{\ell,1}(x) = p^{(\ell-1)}(x)$ for $1 \leq \ell \leq k+1$ and the matrix $V_{(k+1) \times k}(x)$ has the entries of the form

$$v_{ij}(x) = \begin{cases} \binom{i-1}{j-1} q^{(i-j)}(x), & i-j \geq 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \leq i \leq k+1$ and $1 \leq j \leq k$.

3. MAIN RESULTS

In this section, we state and prove our main results.

Theorem 1. *For $k \geq 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r) = \sum_{\ell=0}^k \binom{\ell}{k-\ell} \frac{(r)_\ell}{\ell!},$$

where

$$(x)_n = \prod_{k=0}^{n-1} (x+k) = \begin{cases} x(x+1) \cdots (x+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

denotes the rising factorial, or say, Pochhammer symbol, of $x \in \mathbb{C}$. Consequently, the Fibonacci numbers F_{k+1} for $k \geq 0$ can be computed by

$$F_{k+1} = \sum_{\ell=0}^k \binom{\ell}{k-\ell}.$$

Proof. By virtue of (2), (3), and (4), we have

$$\begin{aligned} \frac{d^k}{dt^k} \left[\frac{1}{(1-t-t^2)^r} \right] &= \sum_{\ell=0}^k \langle -r \rangle_\ell (1-t-t^2)^{-r-\ell} B_{k,\ell}(-1-2t, -2, 0, \dots, 0) \\ &= \sum_{\ell=0}^k \langle -r \rangle_\ell (1-t-t^2)^{-r-\ell} (-2)^\ell B_{k,\ell} \left(t + \frac{1}{2}, 1, 0, \dots, 0 \right) \\ &\rightarrow \sum_{\ell=0}^k \langle -r \rangle_\ell (-2)^\ell B_{k,\ell} \left(\frac{1}{2}, 1, 0, \dots, 0 \right), \quad t \rightarrow 0 \\ &= \sum_{\ell=0}^k \langle -r \rangle_\ell (-2)^\ell \frac{(k-\ell)!}{2^{k-\ell}} \binom{k}{\ell} \binom{\ell}{k-\ell} \left(\frac{1}{2} \right)^{2\ell-k} \\ &= k! \sum_{\ell=0}^k \langle -r \rangle_\ell (-1)^\ell \frac{1}{\ell!} \binom{\ell}{k-\ell} \\ &= k! \sum_{\ell=0}^k (r)_\ell \frac{1}{\ell!} \binom{\ell}{k-\ell}, \end{aligned}$$

where

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of $x \in \mathbb{C}$. So, it follows from (1) that

$$F_{k+1}(r) = \frac{1}{k!} \lim_{t \rightarrow 0} \frac{d^k}{dt^k} \left[\frac{1}{(1-t-t^2)^r} \right] = \sum_{\ell=0}^k \frac{(r)_\ell}{\ell!} \binom{\ell}{k-\ell}. \quad (5)$$

The proof of Theorem 1 is complete. $\square$

Theorem 2 ([19, Theorem 1]). *For $k \geq 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r) = \frac{1}{k!} \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^\ell (r)_\ell (r)_{k-\ell}. \quad (6)$$

Consequently, the Fibonacci numbers F_{k+1} for $k \geq 0$ can be computed by

$$F_{k+1} = \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^k} \sum_{\ell=0}^k (-1)^\ell \left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^\ell.$$

Proof. It is easy to see that

$$\frac{1}{1-t-t^2} = \frac{1}{\left(t + \frac{\sqrt{5}+1}{2}\right)\left(\frac{\sqrt{5}-1}{2} - t\right)}.$$

It is not difficult to see that

$$\begin{aligned} \frac{d^k}{dt^k} \left[\frac{1}{(t+a)^r (b-t)^r} \right] &= \sum_{\ell=0}^k \binom{k}{\ell} \left[\frac{1}{(t+a)^r} \right]^{(\ell)} \left[\frac{1}{(b-t)^r} \right]^{(k-\ell)} \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \frac{\langle -r \rangle_\ell}{(t+a)^{r+\ell}} \frac{\langle -r \rangle_{k-\ell} (-1)^{k-\ell}}{(b-t)^{r+k-\ell}} \\ &= \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \frac{(r)_\ell}{(t+a)^{r+\ell}} \frac{(r)_{k-\ell}}{(b-t)^{r+k-\ell}} \\ &\rightarrow \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \frac{(r)_\ell}{a^{r+\ell}} \frac{(r)_{k-\ell}}{b^{r+k-\ell}}, \quad t \rightarrow 0. \end{aligned}$$

Accordingly, considering the first equality in (5), taking $a = \frac{\sqrt{5}+1}{2}$ and $b = \frac{\sqrt{5}-1}{2}$, and simplifying yield the formula (6). The proof of Theorem 2 is complete. $\square$

Theorem 3. *For $k \geq 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r) = (-1)^k \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & 1 & 0 \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_2 & a_1 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}$$

and satisfy the identity

$$\sum_{\ell=0}^k (-1)^\ell \frac{\langle r \rangle_\ell}{\ell!} \binom{\ell}{k-\ell} = (-1)^k \begin{vmatrix} F_1(r) & 1 & 0 & \cdots & 0 & 0 & 0 \\ F_2(r) & F_1(r) & 1 & \cdots & 0 & 0 & 0 \\ F_3(r) & F_2(r) & F_1(r) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ F_{k-2}(r) & F_{k-3}(r) & F_{k-4}(r) & \cdots & F_1(r) & 1 & 0 \\ F_{k-1}(r) & F_{k-2}(r) & F_{k-3}(r) & \cdots & F_2(r) & F_1(r) & 1 \\ F_k(r) & F_{k-1}(r) & F_{k-2}(r) & \cdots & F_3(r) & F_2(r) & F_1(r) \end{vmatrix}.$$

where

$$a_k = \sum_{\ell=0}^k (-1)^\ell \frac{\langle r \rangle_\ell}{\ell!} \binom{\ell}{k-\ell}.$$

Proof. For $|t^2 + t| < 1$, using the binomial theorem leads to

$$\begin{aligned} (1 - t - t^2)^r &= \sum_{k=0}^{\infty} \langle r \rangle_k \frac{(-1)^k}{k!} (t^2 + t)^k \\ &= \sum_{k=0}^{\infty} \langle r \rangle_k \frac{(-1)^k}{k!} \sum_{\ell=0}^k \binom{k}{\ell} t^{k+\ell} \\ &= \sum_{k=0}^{\infty} \sum_{m=k}^{2k} \langle r \rangle_k \frac{(-1)^k}{k!} \binom{k}{m-k} t^m \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \langle r \rangle_k \frac{(-1)^k}{k!} \binom{k}{m-k} t^m \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \langle r \rangle_\ell \frac{(-1)^\ell}{\ell!} \binom{\ell}{k-\ell} t^k. \end{aligned}$$

From the equation (1), it follows that

$$\begin{aligned} 1 &= (1 - t - t^2)^r \sum_{k=0}^{\infty} F_{k+1}(r) t^k \\ &= \left(\sum_{k=0}^{\infty} \left[\sum_{\ell=0}^k \langle r \rangle_\ell \frac{(-1)^\ell}{\ell!} \binom{\ell}{k-\ell} \right] t^k \right) \sum_{k=0}^{\infty} F_{k+1}(r) t^k. \end{aligned}$$

Further regarding coefficients in the above bracket as a_k and $F_{k+1}(r)$ as b_k in Lemma 4, interchanging the roles of a_k and b_k , and simplifying arrive at desired expressions. $\square$

Theorem 4. For $k \geq 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by

$$F_{k+1}(r) = (-1)^k \begin{vmatrix} 1 & \gamma_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_1 & \gamma_0 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \gamma_0 & \cdots & 0 & 0 \\ 0 & \gamma_3 & \binom{3}{1}\gamma_2 & \binom{3}{2}\gamma_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \binom{k-2}{2}\gamma_{k-4} & \cdots & \gamma_0 & 0 \\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \binom{k-1}{2}\gamma_{k-3} & \cdots & \binom{k-1}{k-2}\gamma_1 & \gamma_0 \\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \binom{k}{2}\gamma_{k-2} & \cdots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1 \end{vmatrix},$$

where

$$\gamma_k = \gamma_k(r) = \sum_{\ell=0}^k \frac{\langle r \rangle_\ell}{\ell!} \binom{\ell}{k-\ell}.$$

Proof. Proceeding as in the proof of Theorem 1 reveals

$$\lim_{t \rightarrow 0} \frac{d^k}{dt^k} [(1-t-t^2)^r] = k! \sum_{\ell=0}^k \frac{\langle r \rangle_\ell}{\ell!} \binom{\ell}{k-\ell} = k! \gamma_k(r).$$

By aid of Lemma 5 for $p(t) = 1$ and $q(t) = (1-t-t^2)^r$, we have

$$\begin{aligned} & \frac{d^k}{dt^k} \left[\frac{1}{(1-t-t^2)^r} \right] = \frac{(-1)^k}{(1-t-t^2)^{r(k+1)}} \\ & \times \begin{vmatrix} 1 & q(t) & 0 & \cdots & 0 & 0 \\ 0 & q'(t) & q(t) & \cdots & 0 & 0 \\ 0 & q''(t) & \binom{2}{1}q'(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & q^{(k-2)}(t) & \binom{k-2}{1}q^{(k-3)}(t) & \cdots & q(t) & 0 \\ 0 & q^{(k-1)}(t) & \binom{k-1}{1}q^{(k-2)}(t) & \cdots & \binom{k-1}{k-2}q'(t) & q(t) \\ 0 & q^{(k)}(t) & \binom{k}{1}q^{(k-1)}(t) & \cdots & \binom{k}{k-2}q''(t) & \binom{k}{k-1}q'(t) \end{vmatrix} \\ & \rightarrow (-1)^k \begin{vmatrix} 1 & \gamma_0 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_1 & \gamma_0 & \cdots & 0 & 0 \\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \cdots & \gamma_0 & 0 \\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \cdots & \binom{k-1}{k-2}\gamma_1 & \gamma_0 \\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \cdots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1 \end{vmatrix} \end{aligned}$$

as $t \rightarrow 0$. Using the first equality in (5), we find that

$$F_{k+1}(r) = \frac{1}{k!} \lim_{t \rightarrow 0} \frac{d^k}{dt^k} \left[\frac{1}{(1-t-t^2)^r} \right]$$

$$= (-1)^k \begin{vmatrix} 1 & \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_1 & \gamma_0 & \dots & 0 & 0 \\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \dots & \gamma_0 & 0 \\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \dots & \binom{k-1}{k-2}\gamma_1 & \gamma_0 \\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \dots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1 \end{vmatrix}.$$

The proof of Theorem 4 is complete. $\square$

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