On the lambda algebra and Singer's cohomological transfer

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Abstract: Writing $\mathbb{A}$ for the 2-primary Steenrod algebra, which is the algebra of stable natural endomorphisms of the mod 2 cohomology functor on topological spaces. Working at the prime 2, computing the cohomology of $\mathbb{A}$ is an important problem of Algebraic topology, because it is the initial page of the Adams spectral sequence converging to stable homotopy groups of the spheres. A relatively efficient tool to describe this cohomology is the Singer algebraic transfer of rank n in [22], which passes from a certain subquotient of a divided power algebra to the cohomology of $\mathbb{A}$. Singer predicted that this transfer is a monomorphism, but this remains open for $n \geq 4$. This short note is to verify Singer's conjecture in the ranks 4 and 5 and some generic degrees.

Key words: Primary cohomology operations; Steenrod algebra; Hit problem; Cohomology of Steenrod algebra; Algebraic transfer; Lambda algebra; Adams spectral sequence.

1. Introduction It is well-known that there is a group homomorphism Sq^n for $n \geq 0$, between mod-2 cohomology groups of a topological space, called Steenrod squares of degrees n . They are stable cohomology operations, that is, they commute with suspension maps. All the Steenrod squares form an algebraic structure which is known as the 2-primary Steenrod algebra $\mathbb{A}$ subject to the Adem relations. It was applied to the vector fields on spheres and the Hopf invariant one problem, which asks for which n there exist maps of Hopf invariant ± 1 . So, the Steenrod algebra is one of the important tools in Algebraic topology. Specifically, its cohomology $\text{Ext}_{\mathbb{A}}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is an algebraic object that serves as the input to the Adams spectral sequence (the ASS) [1] and therefore, computing this cohomology is of fundamental importance to the study of the stable homotopy groups of spheres.

The cohomological transfer, defined by Singer [22], could be an useful approach to describe the mysterious structure of the cohomology algebra of $\mathbb{A}$. In order to better understand this transfer, we will use the following notations and the relevant concepts. Let denote $V^{\oplus n}$ the n -dimensional vector space over the prime field $\mathbb{Z}/2$. Then, we write $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$ for mod-2 cohomology and homology of $BV^{\oplus n}$ (the classifying space of $V^{\oplus n}$). One should note that $BV^{\oplus n}$ is homotopy equivalent to the cartesian product of n copies of the union

of the finite projective spaces. As it is known, $H^*(V^{\oplus n})$ is identified with the polynomial algebra $\mathbb{Z}/2[u_1, \dots, u_n]$ on generators of degree 1, equipped with the canonical unstable algebra structure over the Steenrod algebra (i.e., it is a commutative, associative, graded $\mathbb{Z}/2$ -algebra equipped with a structure of unstable $\mathbb{A}$ -module and satisfying two relations, one called the Cartan formula, and the other called the instability relation: $Sq^{\deg(x)}(x) = x^2$.) By dualizing, $H_*(V^{\oplus n})$ has a natural basis dual to the monomial basis of $H^*(V^{\oplus n})$. We denote by $a_1, \dots, a_n$ the basis of $H_1(V^{\oplus n})$ dual to the basis $u_1, \dots, u_n$ of $H^1(V^{\oplus n}) = \text{Hom}(V^{\oplus n}, \mathbb{Z}/2)$, so that $a_i(u_j) = 1$ if $i = j$ and is 0 if $i \neq j$. We write the dual of $u_1^{d_1} \dots u_n^{d_n}$ as $a_1^{(d_1)} \dots a_n^{(d_n)}$ where the parenthesized exponents are called *divided powers*, and be careful that in the corresponding situation over a field of characteristic 0 in place of $\mathbb{Z}/2$, $a_i^{(d_i)} = a_i^{d_i}/d_i!$, which fits with the formula $a^{(d)}a^{(e)} = \binom{d+e}{d}a^{(d+e)}$. This product gives a commutative graded algebra $\Gamma(a_1, \dots, a_n)$ over $\mathbb{Z}/2$ called a *divided power algebra*, where $\Gamma(a_1, \dots, a_n) = H_*(V^{\oplus n})$, and an element $a_1^{(d_1)} \dots a_n^{(d_n)}$ in $H_*(V^{\oplus n})$ corresponding to a monomial $u_1^{d_1} \dots u_n^{d_n}$ in $H^*(V^{\oplus n})$ is called *d-monomial*. Now, let $P_{\mathbb{A}}H_*(V^{\oplus n})$ be the subspace of $H_*(V^{\oplus n})$ consisting of all elements that are annihilated by all Steenrod squares of positive degrees. The general linear group $GL_n = GL(V^{\oplus n})$ acts regularly on the classifying space $BV^{\oplus n}$ and

therefore on $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$. This action commutes with that of the algebra $\mathbb{A}$ and so acts $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n})$ and $P_{\mathbb{A}}H_*(V^{\oplus n})$. For each $n \geq 0$, Singer constructed in [22] a linear transformation from $P_{\mathbb{A}}H_*(V^{\oplus n})$ to the n -th cohomology group $\text{Ext}_{\mathbb{A}}^{n,n+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ of $\mathbb{A}$, which commutes with two Sq^0 's on $P_{\mathbb{A}}H_*(V^{\oplus n})$ and $\text{Ext}_{\mathbb{A}}^{n,n+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ (see Boardman [2] and Minami [13] for more about this). He shows that this map factors through the quotient of its domain's GL_n -coinvariants to give rise the so-called *cohomological transfer of rank n*

$$\begin{aligned} \phi_n^*(\mathbb{Z}/2) : \mathbb{Z}/2 \otimes_{GL_n} P_{\mathbb{A}}H_*(V^{\oplus n}) \\ \longrightarrow \text{Ext}_{\mathbb{A}}^{n,n+*}(\mathbb{Z}/2, \mathbb{Z}/2). \end{aligned}$$

The domain of this transfer is dual to the space of GL_n -invariants $(\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n}))^{GL_n}$. It is to be noted that $\phi_n^*(\mathbb{Z}/2)$ is induced over the E_2 -term of the ASS by the geometrical transfer map $\Sigma^\infty(B(V^{\oplus n})_+) \rightarrow \Sigma^\infty(\mathbb{S}^0)$ in stable homotopy theory (see also Mitchell [14]). The work of Minami [13] indicated that these transfers play a key role in finding permanent cycles in the ASS. In the second cohomology groups of $\mathbb{A}$, following Mahowald [11] and Lin-Mahowald [9], the classes h_1h_j for $j \geq 3$ and h_j^2 for $0 \leq j \leq 5$, are known to be the permanent cycles in the ASS. In 2016, Hill, Hopkins, and Ravenel [6] showed that when $j \geq 7$, the class h_j^2 is not a permanent cycle in the ASS. It is surprising that so far there is no answer for $j = 6$. The question of whether these h_j^2 are the permanent cycles in the ASS or not is called *Kervaire invariant problem* in literature [4]. This is one of the oldest unresolved issues in Differential and Algebraic topology.

Direct calculating the value of $\phi_n^*(\mathbb{Z}/2)$ on any non-zero element is a hard work. It has been demonstrated that $\phi_n^*(\mathbb{Z}/2)$ is an isomorphism for $n \leq 2$ by Singer himself [22], and $n = 3$ by Boardman [2]. Most notably, Singer sets up a hypothesis in the same paper [22] that $\phi_n^*(\mathbb{Z}/2)$ is a *monomorphism*, but this is still not confirmed, for all cohomological degrees $n \geq 4$. The cases $n = 4, 5$ are our concern in this paper. Besides Singer's transfer homomorphism, the lambda algebra Λ of Bousfield et al. [3] is also a relatively efficient tool to compute the cohomology of the Steenrod algebra. Recall that Λ is the quotient of the graded tensor algebra over $\mathbb{Z}/2$ on symbols λ_i for $i \geq -1$, modulo the two-sided ideal generated by $\lambda_s\lambda_k - \sum_j \binom{j-k-1}{2j-s} \lambda_{s+k-j}\lambda_j$, for any $s, k \geq -1$ by the right ideal generated by λ_{-1} . An interesting representation in the algebra Λ of the algebraic transfer,

established by Chơn and Hà [5], is a $\mathbb{Z}/2$ -linear map ψ_n from $P_{\mathbb{A}}H_*(V^{\oplus n})$ to a subspace of Λ spanned by all monomials of length n in all the monomials in λ_i . The authors showed that the image of an element $\zeta \in P_{\mathbb{A}}H_*(V^{\oplus n})$ under ψ_n is a cycle in Λ and $[\psi_n(\zeta)] = \phi_n^*(\mathbb{Z}/2)([\zeta])$. Note also that this result is a dual version of the one in Hưng [7].

The Singer transfer we are discussing is closely related to the *hit problem* in literature [16] of determination of a minimal generating set for the unstable $\mathbb{A}$ -module $H^*(V^{\oplus n})$. The reader can find an excellent list of publications about this problem in the works by Kameko [8], Mothebe-Uys [15], the present writer [17, 18, 21], Singer [23], Sum [24, 26], Walker-Wood [28], Wood [29] and others. Hit problems are motivated by several problems in Topology and Algebra. It was completely studied for the cases $n \leq 4$ by Peterson [16], Kameko's thesis [8] and Sum [24]. Nevertheless, the general answer seems to be out of reach with the present techniques. Therefore, it is renowned as a difficult problem, even with the help of a computer. In fact, when $\mathbb{Z}/2$ is a trivial $\mathbb{A}$ -module, solving the hit problem is equivalent to determining the "cohits" $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n})$ as a graded vector space, or more generally as a graded module over the group algebra $\mathbb{Z}/2[GL_n]$. Frank Peterson [16] conjectured that $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n}) = 0$ unless $\alpha(n+*) \leq n$, where $\alpha(k)$ is the number of 1's in the dyadic expansion of a positive integer k . His motivation for this was to prove that if $\mathcal{M}$ is a smooth manifold of dimension $*$ such that all products of length n of Stiefel-Whitney classes of its normal bundle vanish, then either $\alpha(*) \leq n$ or $\mathcal{M}$ is cobordant to zero. The conjecture was established by Wood [29]. Therefrom, to study $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n})$ in each n and degree $*$ ≥ 0 , it suffices by Peterson's conjecture and iteration of the Kameko map [8] to consider degrees $*$ in the following "generic" form:

$$(1) \quad k(2^t - 1) + \ell \cdot 2^t, \quad \text{for } k, t, \ell \geq 0, \mu(\ell) < k \leq n,$$

where $\mu(\ell)$ is a smallest number $r \in \mathbb{N}$ such that $\alpha(\ell + r) \leq r$. By Minami [12], hit problems are also considered as an useful tool for studying permanent cycles in the ASS.

In the present work, we explicitly determine the structure of the coinvariant $\mathbb{Z}/2 \otimes_{GL_n} P_{\mathbb{A}}H_*(V^{\oplus n})$ and the behavior of the cohomological transfer of ranks 4 and 5 in some generic degrees of the form (1) by using techniques of the hit problem and the representation in the algebra Λ of these transfers.

2. Main results To begin with, we remark that by Sum [24], it is enough to depict the behavior of the fourth transfer in the following degrees d :

- (i) $d = 2^{t+1} - r$, for $1 \leq r \leq 3$,
- (ii) $d = 2^{t+s+1} + 2^{t+1} - 3$,
- (iii) $d = 2^{t+s} + 2^t - 2$,
- (iv) $d = 2^{t+s+u} + 2^{t+s} + 2^t - 3$,

whenever r, t, s , and u are positive integers. It is not difficult to check that the above degrees can be rewritten as (1). The cases of (i) are known by Sum [25]. The results for (ii) and (iii) have been partially probed in [19, 20]. This note is to investigate the case (iv).

Now, let us state the main results of this text. We first study the fourth cohomological transfer in degrees $d_{t,s,u} := 2^{t+s+u} + 2^{t+s} + 2^t - 3$. To make this, we give an explicit description of a minimal generating set for the domain of $\phi_4^*(\mathbb{Z}/2)$ in degree $d_{t,s,u}$, and obtain the following, which is proved in many steps by using computational techniques of the hit problem of four variables and some previous results by the present author [19, 20] and Sum [25].

Theorem 2.1. *The domain of $\phi_4^*(\mathbb{Z}/2)$ in degree $d_{t,s,u}$ is determined by*

$$\mathbb{Z}/2 \otimes_{GL_4} P_{\mathbb{A}} H_{d_{t,s,u}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } s = 1, u = 1 \text{ and } t \geq 1, \\ 0 & \text{if } s = 2, u = 1 \text{ and } t \geq 2, \\ 0 & \text{if } s \geq 3, u = 1 \text{ and } t \geq 1, \\ 0 & \text{if } s = 1, u = 2 \text{ and } t = 1, \\ \langle [\zeta_{t,1,2}] \rangle, & \text{if } s = 1, u = 2 \text{ and } t \geq 2, \\ 0 & \text{if } s = 1, u \geq 3 \text{ and } t \geq 1, \\ \langle [\zeta_{1,2,u}] \rangle, & \text{if } s = 2, u \geq 1 \text{ and } t = 1, \\ 0 & \text{if } s \geq 3, u \geq 2 \text{ and } t = 1, \\ \langle [\zeta_{t,s,u}] \rangle, & \text{if } s \geq 2, u \geq 2 \text{ and } t \geq 2, \end{cases}$$

where

$$\begin{aligned} \zeta_{t,1,2} &= a_1^{(0)} a_2^{(2^{t+2}-1)} a_3^{(2^{t+2}-1)} a_4^{(3 \cdot 2^t - 1)} \\ &\quad + a_1^{(0)} a_2^{(2^{t+2}-1)} a_3^{(5 \cdot 2^t - 1)} a_4^{(2^{t+1}-1)} \\ &\quad + a_1^{(0)} a_2^{(6 \cdot 2^t - 1)} a_3^{(3 \cdot 2^t - 1)} a_4^{(2^{t+1}-1)} \\ &\quad + a_1^{(0)} a_2^{(7 \cdot 2^t - 1)} a_3^{(2^{t+1}-1)} a_4^{(2^{t+1}-1)}, \end{aligned}$$

$$\begin{aligned} \zeta_{1,2,u} &= a_1^{(2^{u+3}-1)} a_2^{(3)} a_3^{(3)} a_4^{(2)} \\ &\quad + a_1^{(2^{u+3}-1)} a_2^{(3)} a_3^{(4)} a_4^{(1)} \\ &\quad + a_1^{(2^{u+3}-1)} a_2^{(5)} a_3^{(2)} a_4^{(1)} \\ &\quad + a_1^{(2^{u+3}-1)} a_2^{(6)} a_3^{(1)} a_4^{(1)}, \end{aligned}$$

$$\zeta_{t,s,u} = a_1^{(0)} a_2^{(2^t-1)} a_3^{(2^{s+t}-1)} a_4^{(2^{s+t+u}-1)}.$$

It would be interesting also to see that by this

theorem, we have isomorphisms:

$$\begin{aligned} \mathbb{Z}/2 \otimes_{GL_4} P_{\mathbb{A}} H_{d_{t,1,2}}(V^{\oplus 4}) &\cong \mathbb{Z}/2, \quad (t \geq 2), \\ \mathbb{Z}/2 \otimes_{GL_4} P_{\mathbb{A}} H_{d_{1,2,u}}(V^{\oplus 4}) &\cong \mathbb{Z}/2, \quad (u \geq 1), \\ \mathbb{Z}/2 \otimes_{GL_4} P_{\mathbb{A}} H_{d_{t,s,u}}(V^{\oplus 4}) &\cong \mathbb{Z}/2, \quad (t \geq 2, s \geq 2, u \geq 2). \end{aligned}$$

Since the elements $\zeta_{t,1,2}$, $\zeta_{1,2,u}$ and $\zeta_{t,s,u}$ belong to $P_{\mathbb{A}} H_{d_{t,1,2}}(V^{\oplus 4})$, $P_{\mathbb{A}} H_{d_{1,2,u}}(V^{\oplus 4})$ and $P_{\mathbb{A}} H_{d_{t,s,u}}(V^{\oplus 4})$, respectively, $\psi_4(\zeta_{t,1,2})$, $\psi_4(\zeta_{1,2,u})$ and $\psi_4(\zeta_{t,s,u})$ are cycles in the lambda algebra Λ . Furthermore, using the representation in Λ of the fourth transfer map, it may be concluded that

$$\begin{aligned} \phi_4^*(\mathbb{Z}/2)([\zeta_{t,1,2}]) &= [\psi_4(\zeta_{t,1,2})] \\ &= [\lambda_0 \lambda_{2^{t+2}-1} \lambda_{2^{t+1}+2^t-1}] \\ &= h_0 c_t \in \text{Ext}_{\mathbb{A}}^{4,4+d_{t,1,2}}(\mathbb{Z}/2, \mathbb{Z}/2), \\ \phi_4^*(\mathbb{Z}/2)([\zeta_{1,2,u}]) &= [\psi_4(\zeta_{1,2,u})] \\ &= [\lambda_{2^{u+3}-1} \lambda_3^2 \lambda_2] \\ &= h_{u+3} c_0 \in \text{Ext}_{\mathbb{A}}^{4,4+d_{1,2,u}}(\mathbb{Z}/2, \mathbb{Z}/2), \\ \phi_4^*(\mathbb{Z}/2)([\zeta_{t,s,u}]) &= [\psi_4(\zeta_{t,s,u})] \\ &= [\lambda_0 \lambda_{2^t-1} \lambda_{2^{s+t}-1} \lambda_{2^{s+t+u}-1}] \\ &= h_0 h_t h_{s+t} h_{s+t+u} \in \text{Ext}_{\mathbb{A}}^{4,4+d_{t,s,u}}(\mathbb{Z}/2, \mathbb{Z}/2). \end{aligned}$$

These, together with Theorem 2.1 and a fact of the fourth cohomology groups $\text{Ext}_{\mathbb{A}}^{4,4+d_{t,s,u}}(\mathbb{Z}/2, \mathbb{Z}/2)$ (see Lin [10]), yields the following.

Corollary 2.2. *The algebraic transfer is an isomorphism in bidegree $(4, 4 + d_{t,s,u})$ for all t, s, u .*

Next, we are going to survey the behavior of the transfer homomorphism of rank 5 in degrees of the form (1) with $k = n = 5$ and $\ell = 50$. To accomplish this, the domains of $\phi_5^*(\mathbb{Z}/2)$ at these degrees are computed as follows.

Theorem 2.3. *Let $d_t := 5(2^t - 1) + 50 \cdot 2^t$ with t an arbitrary non-negative integer. Then, the coinvariant spaces $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathbb{A}} H_{d_t}(V^{\oplus 5})$ are trivial.*

The proof of the theorem is quite long and complicated. We give a brief description for this: Firstly, we compute explicitly the monomial bases for the spaces of indecomposable elements $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus 5})$ in degrees d_t . An effective approach, based on the iterated Kameko squaring operation [8] coupled with the results by Mothebe-Uys [15], Tn [27] and some linear transformations in Sum [24], is applied in our computations. Additionally, we also used the MAGMA computer algebra to verify the results. In particular, these calculations confirmed Sum's conjecture [26] for the relationship between $\mathbb{A}$ -generators of the algebras $H^*(V^{\oplus 4})$ and $H^*(V^{\oplus 5})$ in the above degrees. Next, using those monomial bases, we may claim that the invariants $(\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus 5}))^{GL_5}$ are

trivial in degrees d_t , from which the theorem follows from the fact that $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathbb{A}} H_*(V^{\oplus 5})$ is isomorphic to $(\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus 5}))^{GL_5}$.

Now according to [10], it is easily seen that $\text{Ext}_{\mathbb{A}}^{5,5+d_t}(\mathbb{Z}/2, \mathbb{Z}/2) = \langle h_0 h_{t+1} h_{t+2} h_{t+4} h_{t+5} \rangle = 0$, for arbitrary $t \geq 0$, and so, by Theorem 2.3, we immediately obtain

Corollary 2.4. *The Singer transfer is a trivial isomorphism in bidegree $(5, 5 + d_t)$ for every non-negative integer t .*

Thus, Corollaries 2.2 and 2.4 favor the Singer conjecture in bidegrees $(4, 4 + d_{t,s,u})$ and $(5, 5 + d_t)$ for any t, s, u .

Detailed proofs of all the results of this note will be published elsewhere.

3. Conclusion Although our work does not apparently lead to either a proof or a refutation of the Singer conjecture in general, we feel that it represents an interesting note about an application of hit problem and the lambda algebra for studying this conjecture. Perhaps a continuation of our methods may prove fruitful. It is our belief that further research can spring from these ideas.

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