GLOBALLY CONVERGENT SERIES FOR THE RIEMANN ZETA FUNCTION AND THE DIRICHLET BETA FUNCTION

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ABSTRACT. We derive the following globally convergent series for the Riemann zeta function and the Dirichlet beta function

$$\zeta(s) = \frac{1}{2^s - 2} \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \binom{2k+1}{k+1} \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s} \quad (\text{where } s \neq 1 + \frac{2\pi i n}{\ln 2}),$$

$$\beta(s) = \frac{1}{4^s} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\left(\frac{3}{4} \right)^{(k+1)} - \left(\frac{1}{4} \right)^{(k+1)} \right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s}$$

using a globally convergent series for the polylogarithm function, and integrals representing the Riemann zeta function and the Dirichlet beta function. To the best of our knowledge, these series representations are new. Additionally, we give another proof of Hasse's series representation for the Riemann zeta function.

1. MAIN RESULTS

Let $\zeta(s)$ represent the Riemann zeta function. H. Hasse ([1], [2]) proved that

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^s} \quad (\text{where } s \neq 1 + \frac{2\pi i n}{\ln 2}),$$

and

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^{s-1}} \quad (\text{where } s \neq 1). \quad (1)$$

We prove the following.

Theorem 1. *Let $\zeta(s)$ and $\beta(s)$ represent the Riemann zeta function, and the Dirichlet beta function, respectively. Then we have*

$$\zeta(s) = \frac{1}{2^s - 2} \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \binom{2k+1}{k+1} \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s} \quad (\text{where } s \neq 1 + \frac{2\pi i n}{\ln 2}), \quad (2)$$

and

$$\beta(s) = \frac{1}{4^s} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\left(\frac{3}{4} \right)^{(k+1)} - \left(\frac{1}{4} \right)^{(k+1)} \right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s}, \quad (3)$$

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where $x^{(n)} = (x)(x+1)\cdots(x+n-1)$ denotes the rising factorial.

Proof. We begin with the following result

$$\int_0^\infty x^{n-1} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \frac{\pi}{\sin n\pi} (\zeta(s) - \zeta(s, 1-n)), \quad (4)$$

where $\mathbf{Li}_s(-x)$ denotes the polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s, 1-n)$ is the Hurwitz zeta function. The above integral is valid for all $s \in \mathbb{C} \setminus \{1\}$, and $0 < n < 1$. This integral can be obtained from formula 3.2.1.6 in the book [3].

Letting $n = 1/2$ in equation (4) gives us

$$\zeta(s) = \frac{1}{\pi(2-2^s)} \int_0^\infty \frac{x^{-1/2} \mathbf{Li}_s(-x)}{1+x} dx \quad (\text{where } s \neq 1). \quad (5)$$

The *polylogarithm function*, $\mathbf{Li}_s(z)$, has the following series representation [4, Theorem 2.1]

$$\mathbf{Li}_s(z) = \sum_{k=0}^\infty \left(\frac{-z}{1-z} \right)^{k+1} \sum_{m=0}^k \binom{k}{m} \frac{(-1)^{m+1}}{(m+1)^s}, \quad (6)$$

for all $s, z \in \mathbb{C}$ with $\Re(z) < 1/2$. We can now derive equation (2) using equations (5), (6), and the fact that

$$\begin{aligned} \int_0^\infty \frac{x^{k+1/2}}{(1+x)^{k+2}} dx &= \frac{\sqrt{\pi}(2k+1)}{2(k+1)!} \Gamma(k+1/2) \\ &= \frac{\pi(2k+1)}{2(k+1)!} 2^{1-2k} \frac{\Gamma(2k)}{\Gamma(k)} \\ &= \frac{\pi(2k+1)}{(k+1)!} 2^{-2k} \frac{(2k-1)!}{(k-1)!} \\ &= \frac{\pi}{2^{2k+1}} \binom{2k+1}{k+1}. \end{aligned}$$

We now proceed to prove equation (3). Using (4) we also have

$$\int_0^\infty \frac{\mathbf{Li}_s(-x)}{1+x} (x^{-3/4} - x^{-1/4}) dx = \sqrt{2}\pi(\zeta(s, 1/4) - \zeta(s, 3/4)) = \sqrt{2}\pi 4^s \beta(s). \quad (7)$$

We can now derive equation (3) using equations (7), (6), and the fact that

$$\begin{aligned} \int_0^\infty \frac{(x^{k+1/4} - x^{k+3/4})}{(1+x)^{k+2}} dx &= \frac{1}{(k+1)!} (\Gamma(3/4)\Gamma(k+5/4) - \Gamma(1/4)\Gamma(k+7/4)), \\ &= \frac{\Gamma(1/4)\Gamma(3/4)}{(k+1)!} \left(\left(\frac{1}{4} \right)^{(k+1)} - \left(\frac{3}{4} \right)^{(k+1)} \right) \\ &= \frac{\sqrt{2}\pi}{(k+1)!} \left(\left(\frac{1}{4} \right)^{(k+1)} - \left(\frac{3}{4} \right)^{(k+1)} \right). \end{aligned}$$

2. HASSE'S SERIES REPRESENTATION

Letting $n \rightarrow 0$ on both sides of the equation (4) we get

$$\zeta(s) = \frac{-1}{s-1} \int_0^{\infty} \frac{\text{Li}_{s-1}(-x)}{(x)(1+x)} dx.$$

This along with equation (6) gives us Hasse's result (1). □

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