Using ODEs to Model Projectile Motion

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Abstract

Projectile motion, the study of the movement of objects that are thrown or shot through the air, has long been an important area of investigation in classical physics. The development of equations to predict the position of projectiles was one of the first practical applications of calculus and has had numerous practical applications, including in the field of artillery. While modern artillery personnel use computers to calculate more complex versions of these equations, the fundamental principles underlying projectile motion are still taught at military academies worldwide. In recent years, the development of "smart" munitions that can guide themselves towards their targets has led to new questions about the relative performance of these advanced weapons and traditional "dumb" munitions. In this paper, we will provide a detailed overview of the history and principles of projectile motion and explore the ongoing efforts to model and compare the performance of smart and dumb munitions in order to determine the optimal mix for different countries' needs.

1 Fundamentals

In this study, we will analyze the trajectory of a projectile under certain simplifying assumptions. Specifically, we will assume that the projectile's movement in the lateral direction is negligible, allowing us to consider its movement in two dimensions only. The x-axis will represent the distance from the projectile's starting point to its target. Additionally, we will assume that the effects of air resistance and wind are negligible, and that the curvature and rotation of the earth can be ignored. Finally, we will assume that the force of gravity is constant and always directed downward.

To maintain brevity, we will consider the projectile to be launched at time $t = 0$ from the coordinates $(0, 0)$. The initial speed of the projectile will be denoted as v_0, and the angle of launch will be denoted as θ. At the moment of launch, the projectile will have a vertical velocity of $v_0 \sin\theta$ and a horizontal velocity of $v_0 \cos\theta$. As we have assumed that gravity only acts in the vertical direction, the horizontal velocity of the projectile will remain constant. This can be expressed mathematically as follows:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0 \tag{1}$$

We can multiply both sides by $\mathrm{d}t$ and integrate both sides:

$$\begin{aligned}\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\ \mathrm{d}t &= \int 0\ \mathrm{d}t \\ \frac{\mathrm{d}x}{\mathrm{d}t} + c_1 &= c_2 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= c_2 - c_1\end{aligned} \tag{2}$$

Since a constant minus a constant equals a constant, we can refer to $c_2 - c_1$ as c_3.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_3 \tag{3}$$

We know that

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = v_0 \cos\theta \tag{4}$$

due to the initial conditions we defined. Combining (3) and (4) gives us $c_3 = v_0 \cos\theta$.

We integrate (5) in the same way as (2):

$$\begin{aligned}\int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \int v_0 \cos\theta \mathrm{d}t \\ x(t) &= t v_0 \cos\theta + c_5\end{aligned} \tag{5}$$

We can add a constant of integration to just one side because constants on two sides can be compressed into one as we saw in (3) to (5). Since due to our initial assumptions, $x(0) = 0$, we can solve for c_5 by evaluating (8) at $t = 0$:

$$\begin{aligned} 0 &= 0 + c_5 \\ c_5 &= 0 \end{aligned} \tag{6}$$

Therefore, our solution for $x(t)$ is

$$x(t) = tv_0 \cos\theta \tag{7}$$

Since the only force acting on the projectile in the y direction is gravity,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -g \tag{8}$$

We can follow the exact same process we applied for the x direction:

$$\begin{aligned} \int \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\ \mathrm{d}t &= \int -g\ \mathrm{d}t \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -gt + c_1 \\ c_1 &= v_0 \sin\theta \\ \int \frac{\mathrm{d}y}{\mathrm{d}t}\ \mathrm{d}t &= \int -gt + v_0 \sin\theta \mathrm{d}t \\ y(t) &= -\frac{1}{2}gt^2 + tv_0 \sin\theta \end{aligned} \tag{9}$$

We can combine $x(t)$ and $y(t)$ into $y(x)$ by solving $x(t)$ for t and substituting into $y(t)$:

$$t = \frac{x}{v_0 \cos\theta}$$

$$\begin{aligned} y(x) &= -\frac{1}{2}g\frac{x^2}{v_0^2 \cos^2\theta} + \frac{x}{v_0 \cos\theta} v_0 \sin\theta \\ &= -\frac{gx^2}{2v_0^2 \cos^2\theta} + x\tan\theta \end{aligned} \tag{10}$$

2 Air resistance - low velocity

(5) At low velocities, an appropriate approximation for air resistance is $F_{air} = kv$, where k is an empirical constant dependent upon the projectile and the atmosphere and F acts opposite to the direction of motion. This assumes that air pressure and density are near-constant throughout the atmosphere, and that the effect of wind is negligible. We also assume that the projectile is close enough to being spherical that its rotation during flight does not appreciably change the strength of air resistance. We now have $\sum F = ma = F_{air} + F_g$.

We can apply this formula to each component:

$$\begin{aligned} m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -k\frac{\mathrm{d}y}{\mathrm{d}t} - mg \\ m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -k\frac{\mathrm{d}x}{\mathrm{d}t} + 0 \end{aligned} \tag{11}$$

Solving the second equation is simpler. Since this is a linear differential equation containing no constants, the solution will be of the form $ae^{rt} + c$.
This gives us:

$$mar^2e^{rt} = -kare^{rt} \tag{12}$$

Since e^{rt} can never equal 0 ,

$$mar^2 = -kar \tag{13}$$

Since our solution clearly cannot have a or r equal to 0 ,

$$\begin{aligned} mr &= -k \\ r &= -\frac{k}{m} \end{aligned} \tag{14}$$

We solve for a and c by substitution of the initial conditions.

$$\begin{aligned} v_0 \cos\theta &= rae^0 \\ v_0 \cos\theta &= -\frac{k}{m}a \\ a &= -\frac{m}{k}v_0 \cos\theta \\ 0 &= -\frac{m}{k}v_0 \cos\theta e^0 + c \\ c &= \frac{m}{k}v_0 \cos\theta \end{aligned} \tag{15}$$

Thus we end up with:

$$\begin{aligned} x(t) &= -\frac{m}{k}v_0 \cos\theta e^{-\frac{kt}{m}} + \frac{m}{k}v_0 \cos\theta \\ &= \frac{m}{k}v_0 \cos\theta \left(-e^{-\frac{kt}{m}} + 1\right) \end{aligned} \tag{16}$$

Since the first equation (12) contains the constant $-mg$, the solution will be of the form $ae^{rt} + bt + c$. Substituting:

$$\begin{aligned} mar^2e^{rt} &= -k\left(are^{rt} + b\right) - mg \\ mar^2e^{rt} &= -kare^{rt} - kb - mg \\ mar^2e^{rt} + kare^{rt} &= -kb - mg \end{aligned} \tag{17}$$

We will arrive at a solution if we assume that both sides are equal to zero.
Solving for b :

$$
\begin{aligned}
0 &= -kb - mg \\
kb &= -mg \\
b &= -\frac{mg}{k}
\end{aligned} \tag{18}
$$

Solving for r :

$$
\begin{aligned}
\mathrm{mar}^2 e^{rt} + kare^{rt} &= 0 \\
mr + k &= 0 \\
r &= -\frac{k}{m}
\end{aligned} \tag{19}
$$

Solving for a :

$$
\begin{aligned}
v_0 \sin\theta &= are^0 + b \\
v_0 \sin\theta &= -a\frac{k}{m} - \frac{mg}{k} \\
a &= -\frac{m}{k}\left(v_0 \sin\theta + \frac{mg}{k}\right)
\end{aligned} \tag{20}
$$

Solving for c :

$$
\begin{aligned}
0 &= ae^{rt} + bt + c \\
0 &= -\frac{m}{k}\left(v_0 \sin\theta + \frac{mg}{k}\right) e^0 + c \\
c &= \frac{m}{k}\left(v_0 \sin\theta + \frac{mg}{k}\right)
\end{aligned} \tag{21}
$$

Thus we have:

$$
\begin{aligned}
y(t) &= -\frac{m}{k}\left(v_0 \sin\theta + \frac{mg}{k}\right) e^{-\frac{kt}{m}} - \frac{mgt}{k} + \frac{m}{k}\left(v_0 \sin\theta + \frac{mg}{k}\right) \\
&= \frac{m}{k}\left(\left(v_0 \sin\theta + \frac{mg}{k}\right)\left(-e^{-\frac{kt}{m}} + 1\right) - gt\right)
\end{aligned} \tag{22}
$$

Which, when consolidated with $x(t)$, gives us

$$
y(x) = \frac{m}{k}\left(\left(v_0 \sin\theta + \frac{mg}{k}\right)\left(\frac{kx}{mv_0 \cos\theta}\right) + \frac{mg}{k}\ln\left|1 - \frac{kx}{mv_0 \cos\theta}\right|\right) \tag{23}
$$

3 Air resistance - high velocity

At high velocities, it is more appropriate to assume that $F_{air} = kv^2$. We will keep all of our other assumptions.

$$
\begin{aligned}
m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -k\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - mg \\
m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= -k\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 0
\end{aligned} \tag{24}
$$

Again, we can solve the second equation first. Since this is a nonlinear differential equation, the conjectured solution we used last time won't work. Let's substitute u for $\frac{\mathrm{d}x}{\mathrm{d}t}$:

$$\begin{aligned} m\frac{\mathrm{d}u}{\mathrm{d}t} &= -ku^2 \\ \frac{1}{u^2}\ \mathrm{d}u &= -\frac{k}{m}\ \mathrm{d}t \\ \int \frac{1}{u^2}\ \mathrm{d}u &= \int -\frac{k}{m}\ \mathrm{d}t \\ -\frac{1}{u} &= -\frac{kt}{m} + c_1 \\ u &= \frac{1}{\frac{kt}{m} + c_1} \end{aligned} \tag{25}$$

Solving for c_1 :

$$\begin{aligned} v_0 \cos\theta &= \frac{1}{c_1} \\ c_1 &= \frac{1}{v_0 \cos\theta} \end{aligned} \tag{26}$$

We repeat the integration-substitution step:

$$\int u\ \mathrm{d}t = \int \frac{1}{\frac{kt}{m} + \frac{1}{v_0 \cos\theta}} \mathrm{d}t \tag{27}$$

Let's substitute w for $\frac{kt}{m} + \frac{1}{v_0 \cos\theta}$, where $\mathrm{d}w = \frac{k}{m}$ d.

$$\begin{aligned} x(t) &= \int \frac{1}{w}\left(\frac{m}{k}\ \mathrm{d}w\right) \\ &= \frac{m}{k}\int \frac{1}{w}\ \mathrm{d}w \\ &= \frac{m}{k}\ln|w| + c_2 \\ c_2 &= -\frac{m}{k}\ln\left|\frac{1}{v_0\cos\theta}\right| \end{aligned} \tag{28}$$

This gives us:

$$\begin{aligned} x(t) &= \frac{m}{k}\ln|w| + -\frac{m}{k}\ln\left|\frac{1}{v_0\cos\theta}\right| \\ &= \frac{m}{k}\left(\ln|w| - \ln\left|\frac{1}{v_0\cos\theta}\right|\right) \\ &= \frac{m}{k}\ln|wv_0\cos\theta| \\ &= \frac{m}{k}\ln\left|\frac{ktv_0\cos\theta}{m} + 1\right| \end{aligned} \tag{29}$$

$y(x)$ is slightly more difficult. We substitute u for $\frac{\mathrm{d}y}{\mathrm{d}t}$:

$$\begin{aligned}
m\frac{\mathrm{d}u}{\mathrm{d}t} &= -ku^2 - mg \\
\frac{m\frac{\mathrm{d}u}{\mathrm{d}t}}{-mg - ku^2} &= 1 \\
\int \frac{m\frac{\mathrm{d}u}{\mathrm{d}t}}{-mg - ku^2}\ \mathrm{d}t &= \int \mathrm{d}t \\
-\frac{1}{g}\int \frac{\frac{\mathrm{d}u}{\mathrm{d}t}}{1+\left(\frac{\sqrt{k}u}{\sqrt{gm}}\right)^2}\ \mathrm{d}t &= t + c_1
\end{aligned} \tag{30}$$

Substitute w for $\frac{\sqrt{k}u}{\sqrt{gm}}$, where $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\sqrt{k}}{\sqrt{mg}}\frac{\mathrm{d}u}{\mathrm{d}t}$:

$$\begin{aligned}
-\frac{1}{g}\int \frac{\frac{\sqrt{mg}}{\sqrt{k}}\frac{\mathrm{d}w}{\mathrm{d}t}}{1+w^2}\ \mathrm{d}t &= t + c_1 \\
-\frac{1}{g}\frac{\sqrt{mg}}{\sqrt{k}}\int \frac{1}{1+w^2}\ \mathrm{d}w &= t + c_1 \\
-\frac{\sqrt{m}}{\sqrt{gk}}\tan^{-1} w &= t + c_1 \\
w &= \tan\left(-\frac{t\sqrt{gk}}{\sqrt{m}} + c_1\right) \\
u &= \frac{\sqrt{mg}}{\sqrt{k}}\tan\left(-\frac{t\sqrt{gk}}{\sqrt{m}} + c_1\right) \\
c_1 &= \tan^{-1}\left(\frac{\sqrt{k}v_0\sin\theta}{\sqrt{mg}}\right)
\end{aligned} \tag{31}$$

Solving for $y(t)$:

$$\int u\ \mathrm{d}t = \int \frac{\sqrt{gm}}{\sqrt{k}}\tan\left(-\frac{t\sqrt{gk}}{\sqrt{m}} + \tan^{-1}\left(\frac{\sqrt{k}v_0\sin\theta}{\sqrt{mg}}\right)\right)\mathrm{d}t \tag{32}$$

Let's substitute q for $-\frac{t\sqrt{gk}}{\sqrt{m}} + \tan^{-1}\left(\frac{\sqrt{k}v_0\sin\theta}{\sqrt{mg}}\right)$, where $\mathrm{d}q = \frac{\sqrt{gk}}{\sqrt{m}}\ \mathrm{d}t$:

$$\begin{aligned}
y(t) &= \frac{\sqrt{mg}}{\sqrt{k}}\frac{\sqrt{m}}{\sqrt{gk}}\int \tan q\ \mathrm{d}q \\
&= \frac{m}{k}\ln|\cos q| + c_2 \\
c_2 &= -\frac{m}{k}\ln\left|\cos\left(\tan^{-1}\left(\frac{\sqrt{k}v_0\sin\theta}{\sqrt{mg}}\right)\right)\right|
\end{aligned} \tag{33}$$

We consolidate $x(t)$ and $y(t)$ to get:

$$y(x) = \frac{m}{k} \ln \left| \frac{\cos\left(-\frac{\sqrt{mgk}\left(e^{\frac{kx}{m}} - 1\right)}{kv_0 \cos\theta} + \tan^{-1}\left(\frac{\sqrt{k}v_0 \sin\theta}{\sqrt{mg}}\right)\right)}{\cos\left(\tan^{-1}\left(\frac{\sqrt{k}v_0 \sin\theta}{\sqrt{mg}}\right)\right)} \right| \tag{34}$$

Here, x must be restricted to the domain

$$x \leq \frac{m}{k} \ln \left| 1 - \frac{kv_0 \cos\theta}{\sqrt{mgk}} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{k}v_0 \sin\theta}{\sqrt{mg}}\right)\right)\right| \tag{35}$$

since values of x higher than this are impossible and will thus yield absurd values of y.

4 Creating a model

In conclusion, the use of differential equations in modeling projectile motion provides a powerful tool for understanding the dynamics of such motion. By utilizing analytic solutions, we can derive complex and accurate models that take into account factors such as the exponential decrease in air pressure with altitude. However, there are limitations to the precision of these solutions, and in some cases, no known analytic solutions exist. In these cases, it is necessary to rely on numerical methods, such as computer simulations, to evaluate the motion of the projectile. It is important to note that while these numerical solutions can be highly accurate, they are dependent on the size of the time iteration chosen.

It is also worth mentioning that while the mathematical models presented in this paper provide a valuable understanding of projectile motion, there are other areas of fluid dynamics that also require significant attention and research. One of the most important and challenging of these areas concerns the Navier-Stokes equations, which describe the motion of fluids and are fundamental to understanding a wide range of phenomena, such as turbulent flow and the behavior of gases and liquids under extreme conditions. These equations are highly nonlinear, and despite much research, no general analytic solutions have been found. Instead, numerical methods and computational simulations must be used to evaluate the motion of fluids. However, the development of more accurate and efficient numerical methods for solving the Navier-Stokes equations is an active area of research and has the potential to lead to many practical applications, including improved weather forecasting, aircraft and ship design, and the prediction of fluid flows in industrial processes.