The hit problem of five variables in the generic degree $5(2^s - 1) + 42 \cdot 2^s$ and its application

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Abstract: We denote by $\mathbb{Z}_2$ the prime field of two elements and by $P_t = \mathbb{Z}_2[x_1, \dots, x_t]$ the polynomial algebra of t generators $x_1, \dots, x_t$ with $\deg(x_j) = 1$. Let $\mathcal{A}_2$ be the Steenrod algebra over $\mathbb{Z}_2$. A central problem of homotopy theory is to determine a minimal set of generators for the $\mathbb{Z}_2$ -graded vector space $\{(\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t)_n\}_{n \geq 0}$. It is called the "hit" problem for Steenrod algebra and has been completely solved for $t \leq 4$.

In this article, we explicitly solve the hit problem of five variables in the "generic" degree $n = 5(2^s - 1) + 42 \cdot 2^s$ for every non-negative integer s . The result confirms Sum's conjecture [15] for the relation between the minimal sets of $\mathcal{A}_2$ -generators of the algebras P_{t-1} and P_t in the case $t = 5$ and degree n above. An efficient approach for surveying the hit problem of five variables has been presented. As an application, we obtain the dimension of $(\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t)_n$ for $t = 6$ and degree $5(2^{s+5} - 1) + 42 \cdot 2^{s+5}$ for all $s \geq 0$. At the same time, we show that the Singer transfer homomorphism is an isomorphism in bidegree $(5, 5 + n)$.

Key words: Primary cohomology operations; Steenrod algebra; Peterson hit problem; Actions of groups on commutative rings; Algebraic transfer.

1. Introduction Recall that if R is a commutative ring, then there is an E_∞ -ring spectrum HR which represents cohomology with coefficients in R . Let $\mathcal{A}_2 := H^*(H\mathbb{Z}_2, \mathbb{Z}_2)$ denote the Steenrod algebra of $\mathbb{Z}_2$ -cohomology operations. This is the graded homotopy of the $H\mathbb{Z}_2$ -module $\text{Map}(H\mathbb{Z}_2, H\mathbb{Z}_2)$, with (noncommutative) multiplication coming from composition of cohomology operations. This algebra is generated by Steenrod squares Sq^i of degree i for $i \geq 0$. These squares act on $H^*(X, \mathbb{Z}_2)$ for any spectrum X . It is well known that $X = B\mathbb{Z}_2 \cong \mathbb{R}P(\infty)$, and $H^*(\mathbb{R}P(\infty), \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ and the action of $\mathcal{A}_2$ can be described explicitly: $Sq^i(x^a) = \binom{a}{i} x^{a+i}$. This is deduced by induction from the case $a = 1$ by using the coproduct on $\mathcal{A}_2$. Here $\mathbb{R}P(\infty)$ denotes the infinite real projective space. By using the comultiplication, the action of $\mathcal{A}_2$ on $H^*(B\mathbb{Z}_2^t, \mathbb{Z}_2) = P_t := \mathbb{Z}_2[x_1, \dots, x_t]$ is determined by

$$Sq^i(x_j) = \begin{cases} x_j & \text{if } i = 0, \\ x_j^2 & \text{if } i = 1, \\ 0 & \text{if } i > 1, \end{cases}$$

and the usual Cartan formula. Then, P_t becomes a left $\mathcal{A}_2$ -module. It is natural to ask: What is a minimal set of generators for P_t as an $\mathcal{A}_2$ -module? This

is same as the problem of determining a basis for the graded vector space

$$QP_t := \mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t \cong P_t / \underline{\mathcal{A}}_2 P_t$$

as a representation of the general linear group $GL_t := GL(t, \mathbb{Z}_2)$. Here $\underline{\mathcal{A}}_2$ is an ideal of $\mathcal{A}_2$ generated by all Steenrod squares of positive degrees. This question is offered by Peterson [5] in 1987 and it is called the "hit" problem for Steenrod algebra. It has been surveyed by many authors. (See Kameko [2], Mothebe-Uys [4], the present author [8, 9, 10, 11], Singer [13], Sum [14, 15], Wood [17], and others.)

For a non-negative integer n , let $(P_t)_n$ be the $\mathbb{Z}_2 GL_t$ -submodule of P_t consisting of all the homogeneous polynomials of degree n in P_t and let $(QP_t)_n$ be the subspace of QP_t consisting of all the classes represented by the elements in $(P_t)_n$. One of the useful tools for solving the hit problem is $\mathbb{Z}_2 GL_t$ -modules homomorphism of Kameko [2]:

$$\widetilde{Sq}_*^0 : (QP_t)_n \rightarrow (QP_t)_{(n-t)/2},$$

which is determined by $\widetilde{Sq}_*^0([x]) = [y]$, if $x = \prod_{1 \leq j \leq t} x_j y^2$, while $\widetilde{Sq}_*^0([x]) = 0$, if $x \neq \prod_{1 \leq j \leq t} x_j y^2$, for any $[x] \in (QP_t)_n$. Denote by $\alpha(n)$ the number of 1's in dyadic expansion of

n . The following arithmetic function has been also used in the study of the hit problem:

$$\mu(n) = \min\{\bullet \in \mathbb{N} : \alpha(n + \bullet) \leq \bullet\}.$$

Theorem 1.1 (see Peterson [5], Wood [17]). *If $\mu(n) > t$, then $(QP_t)_n = 0$.*

This theorem was originally posed by a conjecture of Peterson [5]. It was proved by Peterson himself for $t = 1, 2$. In the general case, the conjecture was proved by Wood [17].

Theorem 1.2 (see Kameko [2]). *If $\mu(n) = t$, then $\widehat{Sq}_*^0$ is an isomorphism of $\mathbb{Z}_2GL_t$ -modules.*

From Theorems 1.1 and 1.2, we need only to study the space $(QP_t)_n$ in the "generic" degree n of the "special" form (see also [9]):

$$(1) \quad n = r(2^s - 1) + m \cdot 2^s,$$

where $0 \leq \mu(m) < r < t$, and $s \geq 0$. The structure of QP_t was explicitly described by Peterson [5] for $t = 1, 2$, by Kameko [2] for $t = 3$, and by Sum [14] for $t = 4$. When $t = 5$ and in some generic degrees of the form (1), it has been surveyed by us [6, 7, 8, 9, 10, 11, 15], and Tím [16]. The problem is not yet determined in general, even with the help of the computer.

In the present study, we investigate the hit problem of five variables in degree n of the form (1) for $r = 5$, $m = 42$ and s an arbitrary non-negative integer (i.e., $n = 5(2^s - 1) + 42 \cdot 2^s$). The result confirms Sum's conjecture [15] for the relation between the minimal sets of $\mathcal{A}_2$ -generators of the algebras P_{t-1} and P_t in the case $t = 5$ and the above generic degree. An efficient approach for solving the hit problem of five variables in this case has been given. As an application, we obtained the dimension of $(QP_t)_n$ when $t = 6$ and degree $n = 5(2^{s+5} - 1) + 42 \cdot 2^{s+5}$ with $s \geq 0$. At the same time, we show that Singer's conjecture [12] that the fifth algebraic transfer is a monomorphism is true in degree above.

This paper is an announcement whose details will appear elsewhere.

2. Preliminaries To motivate the statement of our main result, we provide some notions and related results (see also [2], [6]).

Definition 2.1 (Weight vector and exponent vector). In what follows: For a natural number n , denote by $\alpha_j(n)$ the j -th coefficients in dyadic expansion of n . This means $\alpha(n) = \sum_{j \geq 0} \alpha_j(n)$. Further, n can be represented as follows: $n = \sum_{j \geq 0} \alpha_j(n) 2^j$, where

$\alpha_j(n) \in \{0, 1\}$ for $j = 0, 1, \dots$. Consider the monomial $x = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_t^{\alpha_t} \in P_t$, we define two sequences associated with x by

$$\omega(x) := (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots)$$

and $(a_1, a_2, \dots, a_t)$, where $\omega_i(x) = \sum_{1 \leq j \leq t} \alpha_{i-1}(a_j)$, for $i \geq 1$. They are called the *weight vector* and the *exponent vector* of x , respectively.

Suppose that $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ is a sequence of non-negative integers. Then, it is called the *weight vector* if $\omega_i = 0$ for $i \gg 0$. We define $\deg(\omega) = \sum_{i \geq 1} 2^{i-1} \omega_i$. The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Definition 2.2 (Linear order on P_t). Assume that $x = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_t^{\alpha_t}$ and $y = x_1^{b_1} x_2^{b_2} \dots x_t^{b_t}$ are the monomials in $(P_t)_n$. We write a, b for the exponent vectors of x and y , respectively. We say that $a < b$ if there is a positive integer d such that $a_j = b_j$ for all $j < d$ and $a_d < b_d$, and that $x < y$ if and only if one of the following holds:

- (i) $\omega(x) < \omega(y)$;
- (ii) $\omega(x) = \omega(y)$ and $a < b$.

Definition 2.3 (Binary relations on P_t).

For a weight vector ω of degree n , we denote two subspaces associated with ω :

$$(P_t)_n^\omega = \{x \in (P_t)_n \mid \deg(x) = \deg(\omega), \omega(x) \leq \omega\},$$

$$(P_t)_n^{<\omega} = \{x \in (P_t)_n \mid \deg(x) = \deg(\omega), \omega(x) < \omega\}.$$

For homogeneous polynomials f and g in $(P_t)_n$. We define the following binary relations " $\equiv$ " and " $\equiv_\omega$ " on $(P_t)_n$:

- (i) $f \equiv g$ if and only if $(f - g) \in \mathcal{A}_2(P_t)_n$.
- (ii) $f \equiv_\omega g$ if and only if $f, g \in (P_t)_n^\omega$ and

$$(f - g) \in ((\mathcal{A}_2(P_t)_n \cap (P_t)_n^\omega) + (P_t)_n^{<\omega}).$$

Here

$$\mathcal{A}_2(P_t)_n = \{f \in (P_t)_{n+i} : f = \sum_{j \geq 0} Sq^i(f_j)\},$$

where the map $Sq^i : (P_t)_n \rightarrow (P_t)_{n+i}$ is the Steenrod square of degree $i \geq 0$. In particular, if $f \equiv 0$, then f is called *hit*. If $f \equiv_\omega 0$, then f is called ω -*hit*. By a simple computation, we see that the above binary relations are equivalence ones. From the equivalence relation " $\equiv_\omega$ ", we have the $\mathbb{Z}_2$ -quotient space:

$$(Q_n^{\otimes t})^\omega := (P_t)_n^\omega / ((\mathcal{A}_2(P_t)_n \cap (P_t)_n^\omega) + (P_t)_n^{<\omega}).$$

Definition 2.4 (Admissible monomial and inadmissible monomial). A monomial $x \in (P_t)_n$ is said to be *inadmissible* if there exist monomials $y_1, y_2, \dots, y_k$ in $(P_t)_n$ such that $y_j < x$ for $1 \leq j \leq k$ and $x \equiv \sum_{1 \leq j \leq k} y_j$. Then, x is said to be *admissible* if it is not inadmissible.

Theorem 2.5 (see Kameko [2], Sum [14]). *Let x, y and z be monomials in $(P_t)_n$ such that $\omega_i(x) = 0$ for $i > r > 0, \omega_j(z) \neq 0$ and $\omega_i(z) = 0$ for $i > q > 0$. Then, if z is inadmissible, then xz^{2^r} is also inadmissible. Furthermore, if z is inadmissible, then zy^{2^q} is also inadmissible.*

Obviously, the set of all the admissible monomials in $(P_t)_n$ is a minimal set of $\mathcal{A}_2$ -generators for $(P_t)_n$. Therefore, $(Q P_t)_n$ is an $\mathbb{Z}_2$ -vector space with a basis consisting of all the classes represent by the admissible monomials in $(P_t)_n$.

Definition 2.6 (Spike monomial). A monomial $z = \prod_{1 \leq j \leq t} x_j^{u_j}$ in $(P_t)_n$ is called a *spike* if the powers u_j can be written as $2^{v_j} - 1$ for $v_j \in \mathbb{Z}, 1 \leq j \leq t$. If z is a spike with $u_1 > u_2 > \dots > u_{s-1} \geq u_s \geq 1$ and $u_j = 0$ for $j \geq s + 1$, then it is called a *minimal spike*.

Theorem 2.7 (see Phúc-Sum [6]). *All the spikes in $(P_t)_n$ are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ is weakly decreasing and $\omega_1 \leq t$, then there is a spike $z \in (P_t)_n$ such that $\omega(z) = \omega$.*

Theorem 2.8 (see Singer [13]). *Suppose that $x \in (P_t)_n$ and $\mu(n) \leq t$. Let z be the minimal spike in $(P_t)_n$. Then, if $\omega(x) < \omega(z)$, then $x \in \mathcal{A}_2(P_t)_n$. In other words, x is a hit monomial.*

For $1 \leq l \leq t$, let $\mathbf{q}_{(l,t)} : (P_{t-1})_n \rightarrow (P_t)_n$ be the $\mathcal{A}_2$ -homomorphism, which is determined by

$$\mathbf{q}_{(l,t)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq l-1, \\ x_{j+1} & \text{if } l \leq j \leq t-1. \end{cases}$$

From now on, we denote by $\mathcal{C}_n^{\otimes t}$ the set of all admissible monomials in $(P_t)_n$. Then, we have the following.

Theorem 2.9 (see Mothebe-Uys [4]). *Let l, d be positive integers such that $1 \leq l \leq t$. If $x \in \mathcal{C}_n^{\otimes(t-1)}$, then $x_l^{2^d-1} \mathbf{q}_{(l,t)}(x) \in \mathcal{C}_{n+2^d-1}^{\otimes t}$.*

We put $\mathcal{N}_t := \{(l, \mathcal{L}) \mid \mathcal{L} = (l_1, l_2, \dots, l_r)\}$, where $1 \leq l < l_1 < l_2 < \dots < l_r \leq t$ and $0 \leq r \leq t-1$, and by convention, $\mathcal{L} = \emptyset$ if $r = 0$. Denote by $r = \ell(\mathcal{L})$ the length of $\mathcal{L}$. We consider $(l, \mathcal{L}) \in \mathcal{N}_t, 1 \leq r < t$ and

$$x_{(\mathcal{L}, u)} := x_{l_u}^{2^{r-1}+2^{r-2}+\dots+2^{r-u}} \prod_{u < d \leq r} x_{l_d}^{2^{r-d}}$$

for $1 \leq u \leq r, x_{(\emptyset, 1)} = 1$. In [14], Sum constructed an interesting $\mathbb{Z}_2$ -linear map $\psi_{(l, \mathcal{L})} : (P_{t-1})_n \rightarrow (P_t)_n$, given by

$$\begin{aligned} & \psi_{(l, \mathcal{L})}(\prod_{1 \leq j \leq t-1} x_j^{a_j}) \\ &= (x_l^{2^r-1} \mathbf{q}_{(l,t)}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) / x_{(\mathcal{L}, u)} \end{aligned}$$

if there exists u such that:

$$\begin{aligned} (2) \quad & a_{l_1-1} + 1 = \dots = a_{l_{(u-1)}-1} + 1 = 2^r, \\ & a_{l_u-1} + 1 > 2^r, \\ & \alpha_{r-d}(a_{l_u-1}) - 1 = 0, 1 \leq d \leq u, \\ & \alpha_{r-d}(a_{l_d-1}) - 1 = 0, u+1 \leq d \leq r, \end{aligned}$$

while $\psi_{(l, \mathcal{L})}(\prod_{1 \leq j \leq t-1} x_j^{a_j}) = 0$, if there is no value of u , which satisfies (2), for any the monomial $\prod_{1 \leq j \leq t-1} x_j^{a_j} \in (P_{t-1})_n$. Then, we have following remarks.

Remark 2.10. If $\mathcal{L} = \emptyset$, then $\psi_{(l, \mathcal{L})} = \mathbf{q}_{(l,t)}$ for all t , and $1 \leq l \leq t$. Further if $\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s}) \neq 0$, then $\omega(\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) = \omega(\prod_{1 \leq s \leq t-1} x_s^{a_s})$. In addition, $\psi_{(l, \mathcal{L})}$ is not the homomorphism of $\mathcal{A}_2$ -modules in general. Indeed, for instance, we consider $t = 4, \mathcal{L} = (2, 3, 4) \neq \emptyset$, and the monomial $x_1^{12} x_2^6 x_3^9 \in (P_3)_{27}$. Then, we have

$$\begin{aligned} \psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9) &= \frac{x_1^{2^3-1} \mathbf{q}_{(1,4)}(x_1^{12} x_2^6 x_3^9)}{x_{(\mathcal{L}, 1)}} \\ &= \frac{x_1^7 x_2^{12} x_3^6 x_4^9}{x_2^4 x_3^2 x_4} = x_1^7 x_2^8 x_3^4 x_4^8 \in (P_4)_{27}, \end{aligned}$$

and $\omega(x_1^{12} x_2^6 x_3^9) = (1, 1, 2, 2) = \omega(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9))$. By a simple computation, we deduce that

$$\begin{aligned} Sq^2(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9)) &= x_1^9 x_2^8 x_3^4 x_4^8 \\ &\neq x_1^7 x_2^8 x_3^6 x_4^8 = \psi_{(1, \mathcal{L})}(Sq^2(x_1^{12} x_2^6 x_3^9)). \end{aligned}$$

For a subset $\mathcal{C} \subset (P_{t-1})_n$, let us consider the sets:

$$\begin{aligned} \widetilde{\Phi}^0(\mathcal{C}) &= \bigcup_{1 \leq l \leq t} \psi_{(l, \emptyset)}(\mathcal{C}) = \bigcup_{1 \leq l \leq t} \mathbf{q}_{(l,t)}(\mathcal{C}), \\ \widetilde{\Phi}^{>0}(\mathcal{C}) &= \bigcup_{(l, \mathcal{L}) \in \mathcal{N}_t, 1 \leq \ell(\mathcal{L}) \leq t-1} (\psi_{(l, \mathcal{L})}(\mathcal{C}) \setminus (P_t)_n^0), \end{aligned}$$

where

$$(P_t)_n^0 = \{ \prod_{1 \leq i \leq t} x_i^{a_i} \in (P_t)_n : \prod_{1 \leq i \leq t} a_i = 0 \}$$

is an $\mathcal{A}_2$ -submodule of P_t .

We put

$$\widetilde{\Phi}_*(\mathcal{C}) := \widetilde{\Phi}^0(\mathcal{C}) \bigcup \widetilde{\Phi}^{>0}(\mathcal{C}).$$

Since $\mathbf{q}_{(l,t)}$ is a homomorphism of $\mathcal{A}_2$ -modules, if $\mathcal{C}$ is a minimal set of generators for the $\mathcal{A}_2$ -module $(P_{t-1})_n$, then $\widetilde{\Phi}^0(\mathcal{C})$ is also a minimal set of generators for the $\mathcal{A}_2$ -module $(P_t)_n^0$. Assume that ω be the weight vector of degree n , from now on, we denote by

$$(\mathcal{C}_n^{\otimes t})^\omega := \mathcal{C}_n^{\otimes t} \cap (P_t)_n^\omega.$$

In [15], Sum made the following conjecture, which plays an important role in studying the minimal set of $\mathcal{A}_2$ -generators for P_t in certain generic degrees.

Conjecture 2.11. *Under the above notations, for each $(l, \mathcal{L}) \in \mathcal{N}_t$, and $1 \leq r = \ell(\mathcal{L}) \leq t-1$, if $x = \prod_{1 \leq j \leq t-1} x_j^{a_j} \in \mathcal{C}_n^{\otimes(t-1)}$, and there exist u , $1 \leq u \leq r$, which satisfies (2), then $\psi_{(l, \mathcal{L})}(x) \in \mathcal{C}_n^{\otimes t}$. Moreover, if ω is a weight vector of degree n , then $\widetilde{\Phi}_*((\mathcal{C}_n^{\otimes(t-1)})^\omega) \subseteq (\mathcal{C}_n^{\otimes t})^\omega$.*

To end this subsection, we recall a result in Sum [14] on the inductive formula for the dimension of the vector space $(QP_t)_n$.

Theorem 2.12 (see Sum [14]). *Consider the degree n of the form (1) with $r = t-1$, and s, m positive integers such that $1 \leq t-3 \leq \mu(m) \leq t-2$, and $\mu(m) = \alpha(m + \mu(m))$. Then*

$$\dim(QP_t)_n = (2^t - 1) \dim(QP_{t-1})_m,$$

for all $s \geq t-1$.

3. Statement of results We now survey our main results in this paper. Let us consider the generic degree of the form (1) with $r = t = 5$, and $m = 42$. Then, we have $n_s := 5(2^s - 1) + 42 \cdot 2^s$ with $s \geq 0$. To solve the hit problem of five variables in this degree, our main idea is to combine the homomorphisms $\psi_{(l, \mathcal{L})}$ and Theorem 2.9 above. This approach is quite effective and it helps us to reduce remarkably in solving the hit problem of five variables. This is shown specifically in Theorem 3.3 below.

Notation 3.1. We will use the following notations.

$$(P_t)_{n_s}^{>0} = \langle \{ \prod_{1 \leq i \leq t} x_i^{a_i} \in (P_t)_{n_s} : \prod_{1 \leq i \leq t} a_i > 0 \} \rangle$$

$$(QP_t)_{n_s}^0 = \mathbb{Z}_2 \otimes_{\mathcal{A}_2} (P_t)_{n_s}^0, (QP_t)_{n_s}^{>0} = \mathbb{Z}_2 \otimes_{\mathcal{A}_2} (P_t)_{n_s}^{>0}.$$

Then, we see that $(P_t)_{n_s}^{>0}$ is an $\mathcal{A}_2$ -submodule of P_t , and $(QP_t)_{n_s} \cong (QP_t)_{n_s}^0 \oplus (QP_t)_{n_s}^{>0}$.

For a polynomial $f \in (P_t)_{n_s}$, we denote by $[f]$ the classes in $(QP_t)_{n_s}$ represented by f . If ω

is a weight vector of degree n_s , and $f \in (P_t)_{n_s}^\omega$, then denote by $[f]_\omega$ the classes in $(QP_t)_{n_s}^\omega$ represented by f . For a subset $\mathcal{C} \subset (P_t)_{n_s}$, denote by $[\mathcal{C}] = \{[f] : f \in \mathcal{C}\}$. If $\mathcal{C} \subset (P_t)_{n_s}^\omega$, then denote by $[\mathcal{C}]_\omega = \{[f]_\omega : f \in \mathcal{C}\}$. Also note that if ω is a weight vector of minimal spike, then $[\mathcal{C}]_\omega = [\mathcal{C}]$. We put

$$(\mathcal{C}_{n_s}^{\otimes t})^{\omega^0} := (\mathcal{C}_{n_s}^{\otimes t})^\omega \cap (P_t)_{n_s}^0,$$

$$(\mathcal{C}_{n_s}^{\otimes t})^{\omega^{>0}} := (\mathcal{C}_{n_s}^{\otimes t})^\omega \cap (P_t)_{n_s}^{>0},$$

$$(Q_{n_s}^{\otimes t})^{\omega^0} := (QP_t)_{n_s}^\omega \cap (QP_t)_{n_s}^0,$$

$$(Q_{n_s}^{\otimes t})^{\omega^{>0}} := (QP_t)_{n_s}^\omega \cap (QP_t)_{n_s}^{>0}.$$

Then, $[(\mathcal{C}_{n_s}^{\otimes t})^\omega]_\omega$, $[(\mathcal{C}_{n_s}^{\otimes t})^{\omega^0}]_\omega$ and $[(\mathcal{C}_{n_s}^{\otimes t})^{\omega^{>0}}]_\omega$ are respectively the bases of the vector spaces $(Q_{n_s}^{\otimes t})^\omega$, $(Q_{n_s}^{\otimes t})^{\omega^0}$ and $(Q_{n_s}^{\otimes t})^{\omega^{>0}}$.

We now have the following remark.

Remark 3.2. It is easy to see that $\mu(n_s) = 5$ for all $s > 0$. So, by Theorem 1.2, the iterated Kameko homomorphism

$$(3) \quad (\widetilde{Sq}_*)^s : (QP_5)_{n_s} \rightarrow (QP_5)_{n_0}$$

is an isomorphism of $\mathbb{Z}_2$ -vector spaces for any $s \geq 0$. Hence, we need only to compute $(QP_5)_{n_0}$. We have $n_0 = 5(2^0 - 1) + 42 \cdot 2^0 = 42$. Then, if $x \in \mathcal{C}_{42}^{\otimes 5}$, then the weight vector $\bar{\omega} := \omega(x)$ is one of the following sequences: $\bar{\omega}_{(1)} = (4, 3, 2, 1, 1)$, $\bar{\omega}_{(2)} = (4, 3, 2, 3)$, and $\bar{\omega}_{(3)} = (4, 3, 4, 2)$. Indeed, we note that $z = x_1^{31} x_2^7 x_3^3 x_4 \in (P_5)_{42}$ is minimal spike. So, by Theorem 2.7, $z \in \mathcal{C}_{42}^{\otimes 5}$. Further because $\omega(z) = (4, 3, 2, 1, 1)$, by Theorem 2.8, if $x \in \mathcal{C}_{42}^{\otimes 5}$, then $\omega_1(x) = 4$. This means that x has the form $x = x_i x_j x_k x_l y^2$ with $y \in (P_5)_{19}$ and $1 \leq i < j < k < l \leq 5$. Since $x \in \mathcal{C}_{42}^{\otimes 5}$, according to Theorem 2.5, $y \in \mathcal{C}_{19}^{\otimes 5}$. Then, by Tín [16], we have $\omega(y) \in \{(3, 2, 1, 1), (3, 2, 3), (3, 4, 2)\}$. This implies that $\omega(x)$ is one of the sequences $\bar{\omega}_{(j)}$, $1 \leq j \leq 3$.

From the above remark, we have an isomorphism:

$$(4) \quad (QP_5)_{42} \cong (QP_5)_{42}^0 \bigoplus \left(\bigoplus_{1 \leq j \leq 3} (Q_{42}^{\otimes 5})^{\bar{\omega}_{(j)}^{>0}} \right).$$

Since $\mu(42) = 4$, $(QP_4)_{42} \cong (QP_4)_{19}$. Then, by a simple computation, we see that

$$(QP_4)_{42} = (QP_4)_{42}^{>0} = (Q_{42}^{\otimes 4})^{\bar{\omega}_{(1)}^{>0}} = (Q_{42}^{\otimes 4})^{\bar{\omega}_{(1)}}$$

and $\dim(QP_4)_{42} = \dim(QP_4)_{19} = 140$. On the other hand, in [11], we have shown that

$$\dim(QP_5)^0 = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim(QP_s)^{>0},$$

for all $n \geq 0$. So, by Theorems 1.1 and 1.2, we get

$$(5) \quad \begin{aligned} \dim(QP_5)^0_{42} &= \dim(Q_{42}^{\otimes 5})^{\bar{\omega}(1)} \\ &= 140 \cdot \binom{5}{4} = 700, \\ (\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}(1)} &= \widetilde{\Phi}^0((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}(1)})^{\bar{\omega}(1)}) = \widetilde{\Phi}^0(\mathcal{C}_{42}^{\otimes 4}). \end{aligned}$$

For $1 \leq l \leq 5$, we consider the following sets:

$$\begin{aligned} \mathcal{C}_1 &:= \{x_l \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{41}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_2 &:= \{x_l^3 \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{39}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_3 &:= \{x_l^7 \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{35}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_4 &:= \{x_l^{31} \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{11}^{\otimes 4})^{>0}\}. \end{aligned}$$

Then, according to Theorem 2.9,

$$\mathcal{C}^{>0}(\bar{\omega}(1)) := \bigcup_{1 \leq i \leq 4} \mathcal{C}_i \subseteq (\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}(1)}.$$

We put

$$E := \widetilde{\Phi}^{>0}((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}(1)}) \setminus (\widetilde{\Phi}^{>0}((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}(1)}) \cap \mathcal{C}^{>0}(\bar{\omega}(1))).$$

Then, we have following theorem, which is our first main result.

Theorem 3.3. *Under the above notations, the following statements are true:*

- i) $(Q_{42}^{\otimes 5})^{\bar{\omega}(j)} = (Q_{42}^{\otimes 5})^{\bar{\omega}(j)} = 0$ for $j = 2, 3$.
- ii) $(Q_{42}^{\otimes 5})^{\bar{\omega}(1)} \cong \langle [E] \rangle \oplus \langle [F] \rangle \oplus \langle [\mathcal{C}^{>0}(\bar{\omega}(1))] \rangle$,
where $\dim \langle [E] \rangle = \dim \langle [E]_{\bar{\omega}(1)} \rangle = 542$,
 $\dim \langle [F] \rangle = \dim \langle [F]_{\bar{\omega}(1)} \rangle = 248$, and
 $\dim \langle [\mathcal{C}^{>0}(\bar{\omega}(1))] \rangle = \dim \langle [\mathcal{C}^{>0}(\bar{\omega}(1))]_{\bar{\omega}(1)} \rangle = 1030$.

Consequently, $(Q_{42}^{\otimes 5})^{\bar{\omega}(1)}$ is 1820-dimensional.

From (3), (5), and Theorem 3.3, we conclude that Conjecture 2.11 holds for $t = 5$ and the generic degree n_s for any $s \geq 0$. As an immediate corollary from (3), (4), (5) and Theorem 3.3, we get

Corollary 3.4. *There exist exactly 2520 admissible monomials of degree n_s in P_5 for all $s \geq 0$. Consequently, $(QP_5)_{n_s}$ is 2520-dimensional.*

The above computations confirm the dimension of $(QP_5)_{n_s}$, which is noted in [11] by using the MAGMA computer algebra system. Based upon Theorems 1.2 and 2.12, we have the following, which is our second main result.

Theorem 3.5. *Let us consider the generic degrees of the form (1): $n_s := (t-1)(2^s - 1) + m \cdot 2^s$. Suppose there is a integer ζ such that: $0 \leq \zeta < s$ and*

$1 \leq t-3 \leq \mu(n_\zeta) = \alpha(n_\zeta + \mu(n_\zeta)) \leq t-2$. Then, for each $s \geq \zeta$, we have

$$\begin{aligned} \dim(QP_t)_{(t-1)(2^s - \zeta + t - 1) + n_\zeta 2^s - \zeta + t - 1} \\ = (2^t - 1) \dim(QP_{t-1})_{n_s}. \end{aligned}$$

As an application of Corollary 3.4 and Theorem 3.5, we have immediately the following.

Corollary 3.6. *With degree n_s as in Theorem 3.5, for $t = 6$, $m = 42$, and $s \geq 0$, we have $\dim(QP_6)_{n_{s+5}} = (2^6 - 1) \dim(QP_5)_{n_s} = 158760$.*

We knew that the group GL_t acts regularly on P_t by matrix substitution. Further, the two actions of $\mathcal{A}_2$ and GL_t upon P_t commute with each other; hence there is an inherited action of GL_t on QP_t . From this event, solving the hit problem becomes important in studying the representation of GL_t . This is of interest since the Singer cohomological "transfer" relates QP_t to $\text{Ext}_{\mathcal{A}_2}^t(\mathbb{Z}_2, \mathbb{Z}_2)$, the $\mathbb{Z}_2$ -cohomology of $\mathcal{A}_2$, and thus to the stable homotopy groups of spheres. Singer's transfer is established as follows: Firstly, let us recall that if $\mathcal{M}$ the category of graded left $\mathcal{A}_2$ -modules and degree zero of $\mathcal{A}_2$ -linear maps, then for each integer t , the supension functor $S^t : \mathcal{M} \rightarrow \mathcal{M}$ is defined by $(S^t U)^n := U^{n-t}$ where $n \in \mathbb{Z}$. The action of $\mathcal{A}_2$ on $S^t U$ is given by $\theta(S^t u) = S^t(\theta u)$, for all $u \in U$ and $\theta \in \mathcal{A}_2$. Next, consider the polynomial ring $P_1 = \mathbb{Z}_2[x]$ with $\deg(x) = 1$, the canonical $\mathcal{A}_2$ -action on P_1 is extended to an $\mathcal{A}_2$ -action on the ring of finite Laurent series $\mathbb{Z}_2[x_1, x_1^{-1}]$. Then, we have an $\mathcal{A}_2$ -submodule $\bar{P} = \langle \{x_1^t \mid t \geq -1\} \rangle$ of $\mathbb{Z}_2[x_1, x_1^{-1}]$. One has a short-exact sequence $0 \rightarrow P_1 \xrightarrow{q} \bar{P} \xrightarrow{\pi} S^{-1}\mathbb{Z}_2$, where q is the inclusion and π is given by $\pi(x_1^t) = 0$ if $t \neq -1$ and $\pi(x_1^{-1}) = 1$. We now denote by e_1 the corresponding element in $\text{Ext}_{\mathcal{A}_2}^1(S^{-1}\mathbb{Z}_2, P_1)$. Based on the cross, the Yoneda and the cap products in $\text{co}(\text{homology})$ with $\mathbb{Z}_2$ -coefficients, we have a homomorphism

$$\begin{aligned} Tr_t^{\mathcal{A}_2} : \text{Tor}_t^{\mathcal{A}_2}(\mathbb{Z}_2, S^{-t}\mathbb{Z}_2) &\longrightarrow \text{Tor}_0^{\mathcal{A}_2}(\mathbb{Z}_2, P_t) = QP_t \\ z &\longmapsto e_t \cap z, \end{aligned}$$

where $e_t = (e_1 \times P_{t-1}) \circ \cdots \circ (e_1 \times P_1) \circ e_1$. Its image is a submodule of the invariant space $(QP_t)^{GL_t}$. It is known, the supension S^{-t} induces an isomorphism

$$\text{Tor}_{t,\bullet}^{\mathcal{A}_2}(\mathbb{Z}_2, S^{-t}\mathbb{Z}_2) \cong H_{t,t+\bullet}(\mathcal{A}_2, \mathbb{Z}/2).$$

Hence, $Tr_t^{\mathcal{A}_2}$ induces the homomorphism

$$Tr_t^{\mathcal{A}_2} : H_{t,t+\bullet}(\mathcal{A}_2, \mathbb{Z}/2) \rightarrow (QP_t)_{\bullet}^{GL_t}.$$

Then, its dual

$$(Tr_t^{A_2})^* : ((QP_t)_{\bullet}^{GL_t})^* \rightarrow H^{t,t+\bullet}(A_2, \mathbb{Z}/2)$$

is called *the cohomological transfer of rank h* . Here, $((QP_t)_{\bullet}^{GL_t})^*$ is the space of GL_t -coinvariants of $(QP_t)_{\bullet}$, and $H^{t,t+\bullet}(A_2, \mathbb{Z}/2)$ is the $\mathbb{Z}/2$ -cohomology groups of A_2 . In [12], Singer proved that $(Tr_t^{A_2})^*$ is an isomorphism for $t = 1, 2$. Later, Boardman [1], with additional calculations by Kameko [2], showed that $(Tr_3^{A_2})^*$ is also an isomorphism. These facts indicate that the Singer transfer is highly non-trivial. It is currently expected to be one of the useful tools for studying the cohomology of Steenrod ring. Remarkably, when $t \in \{4, 5\}$, based on invariant theory, Singer also investigated in [12] that $(Tr_4^{A_2})^*$ is an isomorphism in some internal degrees and $(Tr_5^{A_2})^*$ is not an epimorphism in degree nine. Then, he gave the following prediction.

Conjecture 3.7. *For each positive integer t , the algebraic transfer $(Tr_t^{A_2})^*$ is a monomorphism.*

The conjecture was confirmed for $t \leq 3$ by Singer [12] and Boardman [1]. It can be verified for $t = 4$ by using the results on the hit problem of four variables in Sum [14]. It remains open for $t \geq 5$. Now, based upon an admissible monomial basis of QP_5 in degree $5(2^s - 1) + 42 \cdot 2^s$ (see Theorem 3.3), we verify Conjecture 3.7 in respective degree. The following theorem is our third main result.

Theorem 3.8. *The cohomological transfer*

$$(Tr_5^{A_2})^* : ((QP_5)_{5(2^s-1)+42 \cdot 2^s}^{GL_5})^* \rightarrow H^{5,47 \cdot 2^s}(A_2, \mathbb{Z}/2)$$

is a trivial isomorphism for all $s \geq 0$.

This result can be proved by explicitly computing the space of GL_5 -invariants $(QP_5)_{42}^{GL_5}$ combining the fact that $H^{5,47 \cdot 2^s}(A_2, \mathbb{Z}/2)$ is trivial (see Lin [3]). Thus, with these data in hand, the following is immediate:

Corollary 3.9. *Conjecture 2.11 holds for the rank 5 case and degree $5(2^s - 1) + 42 \cdot 2^s$ with $s \geq 0$.*

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