Logical Operator Estimation

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Abstract

This report proposes an estimation method to find the unknown boolean input variable of a logical operation (AND, OR, XOR). The estimators might help to assess if the association between an input and output signal is the result of simple logical operations.

Nomenclature

Symbol	Sets	Description
N	$\mathbb{N}_{>0}$	number of training examples
k	$\mathbb{N}_{>0}$	the k -th training example, $k = \{1, 2, \dots, N\}$
y_k	$\mathbb{B}$	the k -th known boolean output signal
x_k	$\mathbb{B}$	the k -th known boolean input signal
w	$\mathbb{B}$	the unknown common boolean variable
q	$\{a \in \mathbb{R} 1 \geq a \geq 0\}$	probability of w being true $\Pr(w = 1)$
$M = 2^e$	$\mathbb{N}_{>0}$	vector dimension (or embedding dimension)
$\mathbf{y}_k$	$\mathbb{B}^{2^e}$	the k -th known output vector of booleans
$\mathbf{x}_k$	$\mathbb{B}^{2^e}$	the k -th known input vector of booleans
$\mathbf{w}$	$\mathbb{B}^{2^e}$	the unknown common vector of booleans
$\mathbf{q}$	$\{a \in \mathbb{R} 1 \geq a \geq 0\}^{2^e}$	vector of the probabilities $\Pr(w_j = 1)$

Table 1: *Used symbols and notations used in this paper.*

1 Derivation of the Estimators

1.1 AND operator estimation

The AND operator has four possible combinations of inputs and outputs as shown in Tab. 2. Assume the variable $\mathbf{w}$ is unknown (Tab. 3). We can *certainly* infer $w_2 = 1$ because $x_2 = 1 \wedge y_2 = 0$, and $w_4 = 1$ because $x_4 = 1 \wedge y_4 = 1$ but the two cases with $x_1 = x_3 = 0$ are *ambiguous* because $x_i = 0$ will always result in $y_i = 0$ no matter the value w_i . There is a 50% chance that $w_i = 0$ or $w_i = 1$. Given $k = \{1, 2, \dots, N\}$ training examples with inputs $x_k \in \mathbb{B}$ and outputs $y_k \in \mathbb{B}$, the probability of w being true is

$$q = \Pr(w = 1) = \frac{1}{N} \sum_{k=1}^N \overbrace{1_{x_k=1 \wedge y_k=1}}^{\text{certain}} + \overbrace{0.5_{x_k=0}}^{\text{ambiguous}} \quad (1)$$

After estimating q , for a given input x , the output can be inferred as $\hat{y}$

$$\hat{y} = (q > 0.5) \wedge x \quad (2)$$

w	x	y
0	0	0
0	1	0
1	0	0
1	1	1

Table 2: Truth table for $w \wedge x = y$.

w_i	x_i	y_i
w_1	0	0
w_2	1	0
w_3	0	0
w_4	1	1

Table 3: Unknown w_i for AND operations.

1.2 OR operator estimation

The OR operator has *certain* solutions for w_i when $x_i \neq y_i$ and $x_i = y_i = 0$, but is *ambiguous* when $x_i = y_i = 1$ (Tab. 5). The estimator for $\Pr(w = 1)$ can be formulated as follows:

$$q = \Pr(w = 1) = \frac{1}{N} \sum_{k=1}^N 1_{x_k=0 \wedge y_k=1} + 0.5_{x_k=1} \quad (3)$$

The output $\hat{y}$ can be inferred with the input signal x and the estimated probability p

$$\hat{y} = (q > 0.5) \vee x \quad (4)$$

w	x	y
0	0	0
0	1	1
1	0	1
1	1	1

Table 4: Truth table for $w \vee x = y$.

w_i	x_i	y_i
w_1	0	0
w_2	1	1
w_3	0	1
w_4	1	1

Table 5: Unknown w_i for OR operations.

1.3 XOR operator estimation

The beauty of XOR is that there is deterministic, *certain* solution $w = x \oplus y$. As a result the estimator is pure counting:

$$q = \Pr(w = 1) = \frac{1}{n} \sum_{k=1}^n 1_{x_k=0 \wedge y_k=1} + 1_{x_k=1 \wedge y_k=0} \quad (5)$$

Given the input signal x , and estimated probability $\Pr(w = 1)$ of the unknown boolean w , the output $\hat{y}$ can be inferred as follows:

$$\hat{y} = (q > 0.5) \oplus x \quad (6)$$

w	x	y
0	0	0
0	1	1
1	0	1
1	1	0

Table 6: Truth table for $w \oplus x = y$.

w_i	x_i	y_i
w_1	0	0
w_2	1	1
w_3	0	1
w_4	1	0

Table 7: Unknown w_i for XOR operations.

2 Probability of Error

The estimation of q can be treated as two-tailed hypothesis test whether q is clearly true or false.

$$H_0 : q = 0.5 \quad H_a : q \neq 0.5 \quad (7)$$

The probability of error (p-value) can be computed using the binomial test or the Pearson’s χ^2 test.

3 Serialize as Integer

A vector of $M = 2^e$ boolean variables $\mathbf{w}, \mathbf{x}_k, \mathbf{y}_k \in \mathbb{B}^M$ can be serialized as e -bit integer variable $\mathbf{N}$. The estimators are vectorizable, i.e. the probability $p_j = \Pr(w_j = 1)$ can be estimated for each association $(x_j, y_j) \forall j = 1, 2, \dots, M$ in parallel.

4 Batch training

Given $1 > r > 0$ the learning rate, $\mathbf{q}^{(l)}$ the previous probabilities, and $\mathbf{q}^{\text{new}}$ the probabilities estimated based on the l -th batch, the probabilities $\mathbf{q}$ are updated as follows:

$$\mathbf{q}^{(l+1)} = (1 - r)\mathbf{q}^{(l)} + r\mathbf{q}^{\text{new}} \quad (8)$$

The learning rate can be set to $r = N_B/N$ with N_B the batch size and N the size of the training set.

5 Conclusion and further research

The report proposed estimators to identify patterns of logical connections between boolean inputs and outputs. The next steps are to find applications for the proposed estimators, e.g. encode batches of boolean input and output vectors into weights $q \in \mathbb{R}^d$ for neural networks, train masks for sparse matrices, find associations between integer data and binary data.

References

- [VD09] Guido Van Rossum and Fred L. Drake. *Python 3 Reference Manual*. Scotts Valley, CA: CreateSpace, 2009. ISBN: 1441412697.
- [Har+20] Charles R. Harris et al. “Array programming with NumPy”. en. In: *Nature* 585.7825 (Sept. 2020), pp. 357–362. ISSN: 1476-4687. DOI: 10.1038/s41586-020-2649-2.

[Vir+20] Pauli Virtanen et al. “SciPy 1.0: fundamental algorithms for scientific computing in Python”. In: *Nature Methods* 17.3 (Mar. 2020), pp. 261–272. ISSN: 1548-7105. DOI: 10.1038/s41592-019-0686-2.

A Demo Implementation

The demo code is written in Python3 [VD09] and uses the packages NumPy [Har+20] and SciPy [Vir+20]. Generate toy data

```

1 # load packages
2 import numpy as np
3 import scipy.stats
4 np.random.seed(42)
5
6 # number of training examples
7 n_examples = 100
8
9 # create demo data
10 x = np.random.randint(low=0, high=256, size=(n_examples, 1), dtype=np.uint8)
11 x = np.unpackbits(x, axis=1) # deserialize integers to boolean vectors
12 print(x.shape) # (n_examples, n_bits)
13
14 y = np.random.randint(low=0, high=256, size=(n_examples, 1), dtype=np.uint8)
15 y = np.unpackbits(y, axis=1) # deserialize integers to boolean vectors
16 print(y.shape) # (n_examples, n_bits)

```

AND operator estimation

```

1 def and_train(x, y):
2     assert x.shape == y.shape
3     n_obs, n_bits = x.shape
4     p = np.empty((n_bits,))
5     for j in range(n_bits):
6         p[j] = np.mean(
7             np.logical_and((x[:, j] == 1), (y[:, j] == 1))
8             + 0.5 * (x[:, j] == 0))
9     return p
10
11 def and_predict(x, p):
12     assert x.shape[1] == len(p)
13     return np.logical_and((p > 0.5), x)
14
15 def get_pvalues(p, n_obs):
16     return 2.0 * scipy.stats.chisquare(
17         f_obs=[p * n_obs, (1 - p) * n_obs], f_exp=[n_obs/2.], ddof=0)[1]
18
19 p = and_train(x, y)
20 print(p.round(2)) # array with 8 probabilities of Pr(w>0.5)
21
22 pvalues = get_pvalues(p, n_examples)
23 print(pvalues.round(2)) # with random data, the p-values will be big
24
25 xnew = np.array([[0,1,0,1,0,1,0,1], [0,0,0,0,0,0,0,0], [1,1,1,1,1,1,1,1]])
26 yhat = and_predict(xnew, p)
27 print(yhat.astype(int))

```

OR operator estimation

```

1 def or_train(x, y):
2     assert x.shape == y.shape
3     n_obs, n_bits = x.shape
4     p = np.empty((n_bits,))
5     for j in range(n_bits):

```

```

6         p[j] = np.mean(
7             np.logical_and((x[:, j] == 0), (y[:, j] == 1))
8             + 0.5 * (x[:, j] == 1))
9     return p
10
11 def or_predict(x, p):
12     assert x.shape[1] == len(p)
13     return np.logical_or((p > 0.5), x)
14
15 p = or_train(x, y)
16 print(p.round(2)) # array with 8 probabilities of Pr(w>0.5)
17
18 xnew = np.array([[0,1,0,1,0,1,0,1], [0,0,0,0,0,0,0,0], [1,1,1,1,1,1,1,1]])
19 yhat = or_predict(xnew, p)
20 print(yhat.astype(int))

```

XOR operator estimation

```

1 def xor_train(x, y):
2     assert x.shape == y.shape
3     n_obs, n_bits = x.shape
4     p = np.empty((n_bits,))
5     for j in range(n_bits):
6         p[j] = np.mean(
7             np.logical_and((x[:, j] == 0), (y[:, j] == 1))
8             + np.logical_and((x[:, j] == 1), (y[:, j] == 0)))
9     return p
10
11 def xor_predict(x, p):
12     assert x.shape[1] == len(p)
13     return np.logical_xor((p > 0.5), x)
14
15 p = xor_train(x, y)
16 print(p.round(2)) # array with 8 probabilities of Pr(w>0.5)
17
18 xnew = np.array([[0,1,0,1,0,1,0,1], [0,0,0,0,0,0,0,0], [1,1,1,1,1,1,1,1]])
19 yhat = xor_predict(xnew, p)
20 print(yhat.astype(int))

```