Factorial of Sum of Nonnegative Integers for Computing and Algorithms

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Abstract: This paper presents a theorem for factorials with the sum of nonnegative integers. The results of theorems can be used as application in computing science and cryptography to develop algorithms like RSA algorithm and Elliptic Curve Cryptography.

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1. Introduction

The factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

For example, 7!=1×2×3 ×4 ×5×6×7 = 5040. Note that zero factorial is always one, that is, 0! =1.

2. Factorial of Sum of Nonnegative Integers

The theorem [1-7] states that the factorial of sum of any k nonnegative integers is equal to multiple of the product of factorials of the k nonnegative integers.

Theorem: For any k nonnegative integers $n_1, n_2, n_3, \cdots$ *and* n_k,

$$(n_1 + n_2 + n_3 + \cdots + n_k)! = (a_1 \times a_2 \times a_3 \times \cdots \times a_{k-1}) \times n_1! \times n_2! \times n_3! \times \cdots \times n_k!,$$

$$\text{that is,} \left(\sum_{i=1}^{k} n_i \right)! = A \prod_{i=1}^{k} n_i\, !,$$

where $\mathrm{A} = a_1 \times a_2 \times a_3 \times \cdots \times a_{k-1}$ and $A, a_1, a_2, a_3, \cdots, a_{k-1}$ are nonnegative integers.

$Proof.$ Let $x = n_2 + n_3 + n_4 + \cdots + n_k$.
Then, $(n_1 \times x)! = n_1! \times (n_1 + 1)(n_1 + 2)(n_1 + 3) \cdots (n_1 + x) = a_1 \times n_1! \times x!$ can be proved by mathematical induction.

Basis. Let us show that $(n_1 + 3)! = n_1! \times (n_1 + 1)(n_1 + 2)(n_1 + 3) = n_1! \times c \times 3!$ is true.
If $n_1 = 2, (2 + 3)! = 2! \times (2 + 1)(2 + 2)(2 + 3) = 120 = 2! \times 20 \times 3!,$ where $c = 20$.

Inductive hypothesis. Let us assume that the result is true for $(x - n_k)$ such that
$(n_1 \times (x - n_k))! = n_1! \times (n_1 + 1)(n_1 + 2)(n_1 + 3) \cdots (n_1 + x - a_k) = c \times n_1! \times (x - n_{k)}!$.

Inductive Step. We must show that the inductive hypothesis is true for $(x - n_k) + n_k$.

$(n_1 + x - n_k + n_k)! = n_1! \times (n_1 + 1)(n_1 + 2) \cdots (n_1 + x) = k \times n_1 \ \times (x - n_k + n_k)!.$
that is, $(n_1 + x)! = n_1! \times (n_1 + 1)(n_1 + 2) \cdots (n_1 + x) = a_1 \times n_1 \ \times x!, \qquad (k = a_1).$

Hence, $(n_1 + (n_2 + n_3 + n_4 + \cdots + n_k))! = a_1 \times n_1! \times (n_2 + n_3 + n_4 + \cdots + n_k)!$ is proved by mathematical induction.

Similarly, we can apply the same induction method to prove each of the sums such that $(n_2 + n_3 + n_4 + \cdots + n_k)! = a_2 \times n_2! \times (n_3 + n_4 + \cdots + n_k)!$; $(n_3 + n_4 + n_5 + \cdots + n_k)! = a_3 \times n_3! \times (n_4 + n_5 + \cdots + n_k)!$; $(n_4 + n_5 + n_6 + \cdots + n_k) = a_4 \times n_4! \times (n_5 + n_6 + \cdots + n_k)!$; $, \cdots, and\ (n_{k-1} + n_k) == a_{k-1} \times n_{k-1}! \times n_k.$

It is understood that, by mathematical induction, we proved theorem such that
$(n_1 + n_2 + n_3 + \cdots + n_k)! = (a_1 \times a_2 \times a_3 \times \cdots \times a_{k-1}) \times n_1! \times n_2! \times n_3! \times \cdots \times n_k!,$

$$\text{that is,} \left(\sum_{i=1}^{k} n_i \right)! = A \prod_{i=1}^{k} n_i\,!, \qquad \text{where } A = a_1 \times a_2 \times a_3 \times \cdots \times a_{k-1}.$$

How to find out the coefficient *A* in an alternative way?
A can be found out by using the followng result: $A = \dfrac{(n_1 + n_2 + n_3 + \cdots + n_k)!}{n_1! \times n_2! \times n_3! \times \cdots \times n_k!}.$
For our convenience, we can rearrange the nonnegative integer A as follows.
Let $A = 64$. $A = 1 \times 64$; $A = 2 \times 32$; $A = 2 \times 2 \times 16$; $A = 2 \times 4 \times 8$, etc.
Similarly, if $A = 21$, Then, $A = 1 \times 21$ *or* $A = 3 \times 7$.

Corollary: If $n_1 = n_2 = n_3 = \cdots = n_k = n$. Then, $(n_1 + n_2 + n_3 + \cdots + n_k)! = (k \times n)!$.
For instance,
If $n_1 = n_2 = n_3 = \cdots = n_k = 0$. Then, $(n_1 + n_2 + n_3 + \cdots + n_k)! = (k \times 0)! = 0! = 1$.
If $n_1 = n_2 = n_3 = \cdots = n_k = 1$. Then, $(n_1 + n_2 + n_3 + \cdots + n_k)! = (k \times 1)! = k!$.
If $n_1 = n_2 = n_3 = \cdots = n_k = 2$. Then, $(n_1 + n_2 + n_3 + \cdots + n_k)! = (k \times 2)! = (2k)!$.
If $n_1 = n_2 = n_3 = \cdots = n_k = k$. Then, $(n_1 + n_2 + n_3 + \cdots + n_k)! = (k \times k)! = k^2!$.

This idea can help to the researchers working in computer science, computational science, and cryptography for developing algorithms and software to solve the real-world problems.

3. Conclusion

In this article, an innovative combinatorial technique and theorem are introduced and the theorem states that the factorial of sum of any k nonnegative integers is equal to multiple of the product of factorials of the k nonnegative integers. This methodological advance can enable the researchers working in computer science, computational science, and cryptography for developing algorithms and software to solve the most real life problems and meet today's challenges [8].

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