[microreview]

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Groups: a concise approach

Open Mathematics Collaboration^{*†}

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Abstract

Some definitions and results of groups are presented within the fewest words and symbols as possible.

keywords: group theory, abstract algebra, groups

The most updated version of this paper is available at

<https://osf.io/bfyhm/download>

Introduction

1. The purpose of this *microreview* is to provide some basic results of **groups** using the fewest words and symbols as possible, without losing information.
2. We believe this *concise pedagogical approach* can be a very useful resource while one is studying and trying to solve pure mathematical problems.
3. These papers [1–3] can be read jointly.
4. [4–6]

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Overview

5. [7, 8]

Notation

6. G = **group** or a **set**, depending on the context.
7. $H \leq G$ means H is a *subgroup* of the **group** G .
8. $e = 1$ is the *identity* of a (sub)group.
9. 0 = *identity* of an **additive group**
10. a^{-1} = *inverse* of a
11. $-a$ = *additive inverse* of a
12. $A^\times$ = set of all nonzero elements
13. $\mathbb{Z}^+$ = set of positive integers
14. $a, b, c, g, g_i, x, y \in G$ for $i \in \{1, 2, 3, \dots\}$, unless stated otherwise.
15. $m, n \in \mathbb{Z}; \quad k, t = \{1, 2, 3, \dots\}$
16. p = prime
17. $\exists_{2n+1}$ *the number of elements is odd*

Definitions

18. Binary operation $\ast : G \times G \rightarrow G$.
19. Juxtaposition: $a \ast b = ab$.
20. Matrix A , $(\det(A) \neq 0) \rightarrow (A = \text{nonsingular})$

Group

21. **Group** $G = (G, *)$, $\forall a, b, c \in G$
- (a) associative, $a * (b * c) = (a * b) * c$
 - (b) identity, $e * a = a$
 - (c) inverse, $\exists a^{-1} \in G : a^{-1} * a = e$
22. $(ab = ba) \rightarrow (G = \text{abelian})$
23. $(aa = a) \rightarrow (a = e)$
24. **Identity** *commutes* and it is *unique*, $\forall a : ea = ae = a$.
25. *Cancellation law*, $((ga = gb) \vee (ag = bg)) \rightarrow (a = b)$.
26. $G = (G, +)$ = **additive group**, identity element $0 \in G$ such that $\forall a, b, c \in G$
- (a) $a + (b + c) = (a + b) + c$
 - (b) $0 + a = a$
 - (c) $\exists (-a) \in G : (-a) + a = 0$

Examples

27. Additive abelian groups: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
28. **Circle group** (circle S^1 of radius 1, center in the origin):
 $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, the operation is multiplication.
29. $\forall n \in \mathbb{Z}^+$, $\Gamma_n = \{\xi^k : 0 \leq k < n\}$ is the set of all the n th roots of unity, where $\xi = e^{2\pi i/n} = \cos(2\pi/n) + i \sin(2\pi/n)$.
30. Multiplicative abelian groups: $\mathbb{Q}^\times, \mathbb{R}^\times, \mathbb{C}^\times, S^1, \Gamma_n$.
31. The plane $\mathbb{R} \times \mathbb{R}$ is an *additive abelian group* (vector addition).

32. $\mathcal{P} = \{P, +\}$, $P = \{\text{even}, \text{odd}\} = \textit{abelian parity group}$
- (a) even + even = even = odd + odd
 - (b) even + odd = odd = odd + even
33. $A, B \subseteq X$, *symmetric difference* $A + B = (A - B) \cup (B - A)$;
 $\mathcal{B}(X) = \mathbf{Boolean\ group}$ = family of all subsets of X (operation = *symmetric difference*)
- (a) $A + B = B + A$
 - (b) $\emptyset = \text{identity}$
 - (c) $A^{-1} = A$, $A + A = \emptyset$
34. $\mathcal{B}(X) = \text{abelian}$
35. $\text{GL}(2, \mathbb{R}) = \mathbf{real\ general\ linear\ group}$ = all *nonsingular* matrices
(matrix multiplication, nonabelian group)
36. $\text{GL}(2, \mathbb{Q})$, $\text{GL}(2, \mathbb{C})$, $\text{GL}(2, \mathbb{Z})$
37. $\text{GL}(n, \mathbb{R})$
38. (a) $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \textit{special orthogonal matrices}$
- (b) $\text{SO}(2, \mathbb{R}) = 2 \times 2 \mathbf{special\ orthogonal\ group}$
39. $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}^k = \begin{bmatrix} \cos k\alpha & -\sin k\alpha \\ \sin k\alpha & \cos k\alpha \end{bmatrix}$
40. (a) $f_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$, $f_{a,b} = ax + b = \textit{affine map}$, $a, b \in \mathbb{R}$, $a \neq 0$
- (b) $\text{Aff}(1, \mathbb{R}) = \mathbf{affine\ group}$ = all *affine* maps
 - (c) group under composition

Permutations

41. S_X = set of all *permutations* of a set X
42. S_n = set of all *permutations* of $\{1, 2, \dots, n\}$
43. $\alpha \in S_n$; β_i = disjoint r_i -cycles
- (a) $(\alpha = r\text{-cycle}) \rightarrow (|\alpha| = r)$
 - (b) $(\alpha = \beta_1 \cdots \beta_t) \rightarrow (|\alpha| = m = \text{lcm}(r_1, \dots, r_t))$
 - (c) $(|\alpha| = p) \leftrightarrow ((\alpha = p\text{-cycle}) \vee (\alpha = \text{product disjoint } p\text{-cycles}))$

Power

44. **powers** g^n , $n \geq 1$, $g^1 = g$, $g^{n+1} = gg^n$; $g^0 = 1$; $g^{-n} = (g^{-1})^n$
45. **Inverse** *commutes* and it is *unique*, $\forall a : a^{-1}a = aa^{-1} = e$.
46. *Inverse* of the *inverse*, $(a^{-1})^{-1} = a$.
47. $(ab)^{-1} = b^{-1}a^{-1}$
48. $\forall n \geq 2 : (a_1a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1}a_1^{-1}$
49. $(g^{-1})^n = (g^n)^{-1}$
50. $g^{m+n} = g^m g^n$; $(g^m)^n = g^{mn}$
51. **Law of Exponents:**
- (a) $(ab = ba) \rightarrow ((ab)^n = a^n b^n)$
 - (b) $(a^n)^m = a^{mn}$
 - (c) $a^m a^n = a^{m+n}$
52. $(\forall g : g^2 = 1) \rightarrow (\text{abelian})$
53. $G = (G, \cdot)$; $(|G| = \text{odd}) \rightarrow (\forall g' : \exists! g : g^2 = g')$

Order

54. $|G| = \text{order} = \text{number of elements}$
55. (a) $|a| = \text{order of } a$
 (b) $(a^k = 1, k \geq 1) \rightarrow (|a| = \text{smallest } k)$
 (c) $(\nexists k) \rightarrow (|a| = \infty)$
56. (a) $|a| = k, (a^n = 1) \rightarrow (k \mid n)$
 (b) $\{n : a^n = 1\} = \text{set of all multiples of } k$
57. $|g| = m; (m = pt) \rightarrow (|g^t| = p)$
58. $|g| = pk; k \geq 1; (\exists x : x^p = g) \rightarrow (|x| = p^2k) \rightarrow (|x| > |g|)$
59. $(|G| = \text{even}) \rightarrow (\exists_{2n+1} g_i : |g_i| = 2)$
60. $(\forall G : |G| < 6) \rightarrow (ab = ba)$
61. $|ab| = |ba|$
62. $G = (G, \cdot); (|G| = \text{odd}) \rightarrow (\forall g' : \exists! g : g^2 = g')$

Center

63. (a) $Z(G) = \{z \in G : \forall g. zg = gz\}$ **center of G**
 (b) $(zg = gz) \equiv (z = gzg^{-1})$
64. $Z(G) = \{1\}$ **centerless**
65. $(gg' = g'g) \leftrightarrow (Z(G) = G)$
66. $Z(S_n) = \{1\}, \forall n \geq 3$
67. $Z(\text{GL}(2, \mathbb{R})) = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}, d \neq 0$

Open Invitation

Review, add content, and co-author this paper [9, 10].

Join the **Open Mathematics Collaboration**.

Send your contribution to mplobo@uft.edu.br.

Open Science

The **latex file** for this paper together with other *supplementary files* are available [8].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [11], please cite it accordingly [1]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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