On the dimension of the "cohits" space $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} H^*((\mathbb{R}P(\infty))^{\times t}, \mathbb{Z}_2)$ and some applications

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Abstract: We denote by $\mathbb{Z}_2$ the prime field of two elements and by $P_t = \mathbb{Z}_2[x_1, \dots, x_t]$ the polynomial algebra of t generators $x_1, \dots, x_t$ with the degree of each x_i being one. Let $\mathcal{A}_2$ be the Steenrod algebra over $\mathbb{Z}_2$. A central problem of homotopy theory is to determine a minimal set of generators for the "cohits" space $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t$. This problem, which is called *the "hit" problem* for Steenrod algebra, has been systematically studied for $t \leq 4$. The present paper is devoted to the investigation of the structure of $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t$ in some certain "generic" degrees. More specifically, we explicitly determine a monomial basis of $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_5$ in degree $n_s = 5(2^s - 1) + 42 \cdot 2^s$ for every non-negative integer s . As a result, it confirms Sum's conjecture [14] for a relation between the minimal sets of $\mathcal{A}_2$ -generators of the algebras P_{t-1} and P_t in the case $t = 5$ and degree n_s . As applications, we obtain the dimension of $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_6$ in the generic degree $5(2^{s+5} - 1) + n_0 \cdot 2^{s+5}$ for all $s \geq 0$, and show that the Singer's cohomological transfer [11] is an isomorphism in bidegree $(5, 5 + n_s)$.

Key words: Primary cohomology operations; Steenrod algebra; Peterson hit problem; Actions of groups on commutative rings; Algebraic transfer.

1. Introduction Recall that if R is a commutative ring, then there is an E_∞ -ring spectrum HR which represents cohomology with coefficients in R . Consider $R = \mathbb{Z}_2$, the prime field of two elements, then as well known, $H^*(H\mathbb{Z}_2, \mathbb{Z}_2) = \mathcal{A}_2$ is the Steenrod algebra of $\mathbb{Z}_2$ -cohomology operations. This is the graded homotopy of the $H\mathbb{Z}_2$ -module $\text{Map}(H\mathbb{Z}_2, H\mathbb{Z}_2)$, with (noncommutative) multiplication coming from a composition of cohomology operations. The algebra $\mathcal{A}_2$ is known to be a Hopf algebra (i.e., it is a bialgebra over the field $\mathbb{Z}_2$ that has an associative multiplication and a coassociative comultiplication and is equipped with a counit, a unit and an antipode). As an algebra over $\mathbb{Z}_2$, $\mathcal{A}_2$ is generated by the Steenrod squaring operations Sq^i of degree i for $i \geq 0$. These squares act on $H^*(X, \mathbb{Z}_2)$ for any spectrum X . Let $X = \mathbb{R}P(\infty)$ be the infinite real projective space. Then, it has been shown that $H^*(\mathbb{R}P(\infty), \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ and that the (left) action of $\mathcal{A}_2$ on $\mathbb{Z}_2[x]$ can be described by the rule $Sq^i(x^a) = \binom{a}{i} x^{a+i}$. This action is deduced by induction from the case $a = 1$ by using the comultiplication on $\mathcal{A}_2$. Also based on the comultiplication, one can explicitly depict the action of $\mathcal{A}_2$ on $H^*((\mathbb{R}P(\infty))^{\times t}, \mathbb{Z}_2) \cong P_t := \mathbb{Z}_2[x_1, \dots, x_t]$. So,

as well known, P_t is an unstable left $\mathcal{A}_2$ -module. It should be noted that $\mathcal{A}_2$ is not an unstable module over itself in general since $Sq^1 Sq^0 = Sq^1 \neq 0$. The $\mathbb{Z}_2$ -algebra P_t is a connected $\mathbb{Z}$ -graded algebra. That is, $P_t = \mathbb{Z}_2 \oplus (\bigoplus_{n>0} (P_t)_n)$, where $(P_t)_n$ is the vector space of homogeneous polynomials of degree n . It is natural to ask: What is a minimal set of generators for P_t as an $\mathcal{A}_2$ -module? When we fix the degrees n and the number of variables t , answering that question is the same as the problem of elementary linear algebra of finding the dimension of the quotient space, $(QP_t)_n := (\mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t)_n \cong (P_t)_n / \sum_{i>0} \text{Im}(Sq^i)$. Here $\sum_{i>0} \text{Im}(Sq^i) = (P_t)_n \cap \text{Ker}(\epsilon)P_t$, in which ϵ denotes the counit $\mathcal{A}_2 \rightarrow \mathbb{Z}_2$, and the Steenrod squaring operations $Sq^i : (P_t)_{n-i} \rightarrow (P_t)_n$ are defined, and their basic properties, such as Cartan's formula, are established. People have also shown that $(QP_t)_n$ has a module structure over the group algebra $\mathbb{Z}_2 GL_t$ of the usual general linear group GL_t . The above problem, which has first appeared in the work of Peterson [5], is usually called the *hit problem* for the Steenrod algebra. It is currently famously difficult and is of great interest to many authors. (See Kameko [2], Mothebe-Uys [4], the

present author [7, 8, 9], Singer [12], Sum [13, 14], Wood [16], and others.)

A qualitative approach to the hit problem is that one gives conditions on elements of $(P_t)_n$ to show that they go to zero in $(QP_t)_n$, i.e., belong to $\sum_{i>0} \text{Im}(Sq^i)$. By this, let denote $\alpha(n)$ the number of 1's in dyadic expansion of a non-negative integer n . It is important to consider the arithmetic function $\mu : \mathbb{N} \rightarrow \mathbb{N}$, which is defined by $\mu(0) = 0$ and $\mu(n) = \min\{\bullet \in \mathbb{N} : \alpha(n + \bullet) \leq \bullet\}$. Then, Peterson's conjecture [5], which has been established by Wood [16], points out that $(QP_t)_n = 0$ when $\mu(n) > t$. We are interested in a relatively efficient tool often used in studying hit problems. It is the $\mathbb{Z}_2GL_t$ -modules homomorphism $\widetilde{Sq}_*^0 : (QP_t)_{2n+t} \rightarrow (QP_t)_n$, determined by $\widetilde{Sq}_*^0([f]) = g$ if $f = x_1 \dots x_t g^2$, and $\widetilde{Sq}_*^0([f]) = 0$ otherwise. This linear map, given by the Kameko thesis [2], is related to the above μ -function through the following technical result.

Theorem 1.1 (see Kameko [2]). *If $\mu(2n+t) = t$, then $\widetilde{Sq}_*^0$ is an isomorphism.*

By straightforward calculations using this theorem and the Peterson conjecture, it follows that the structure of $(QP_t)_n$ can be determined in each degree n of the so-called the "generic" form:

$$(1) \quad n = r(2^s - 1) + m \cdot 2^s,$$

whenever $0 \leq \mu(m) < r < t$, and $s \geq 0$. The cases where $t \leq 4$ are known by Peterson [5], Kameko [2] and Sum [13]. For $t = 5$, it has been partially studied by us [6, 7, 8, 9, 14] and others. The problem is not yet solved in general. The present Note is motivated by the previous results and we here continue this line of investigation by considering $(QP_t)_n$ in degrees n of the form (1). More precisely, when $t = 5$, we explicitly determine the structure of $(QP_5)_n$ for $n_s := n = 5(2^s - 1) + 42 \cdot 2^s$. As a result, it confirms Sum's conjecture [14] for the relationship between the minimal sets of $\mathcal{A}_2$ -generators of the polynomial rings P_{t-1} and P_t in the case $t = 5$ and degree n_s . An efficient approach to solve the hit problem of five variables had been provided. As applications, we obtain the dimension of $(QP_6)_n$ in degree $5(2^{s+5} - 1) + n_0 \cdot 2^{s+5}$ for all $s \geq 0$, and show that Singer's conjecture [11] for the fifth cohomological transfer holds in degrees n_s .

This paper is an announcement whose details will appear elsewhere.

2. The Necessary Preliminaries In order to formulate the main results of this text and for the convenience of the reader, let us describe some preliminary material.

Definition 2.1 (Weight vector and exponent vector). We say that a sequence of non-negative integers $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ is a *weight vector*, if $\omega_i = 0$, for $i \gg 0$. Then, one defines $\deg(\omega) = \sum_{i \geq 1} 2^{i-1} \omega_i$. With a natural number n , let denote $\alpha_j(n)$ the j -th coefficients in dyadic expansion of n , then $\alpha(n) = \sum_{j \geq 0} \alpha_j(n)$, and $n = \sum_{j \geq 0} \alpha_j(n) 2^j$, where $\alpha_j(n) \in \{0, 1\}$ for all $j \geq 0$. For a monomial $x = x_1^{a_1} x_2^{a_2} \dots x_t^{a_t} \in P_t$, let us consider two sequences $\omega(x) := (\omega_1(x), \dots, \omega_i(x), \dots)$ and $(a_1, a_2, \dots, a_t)$, where $\omega_i(x) = \sum_{1 \leq j \leq t} \alpha_{i-1}(a_j)$, for every i . They are respectively called the *weight vector* and the *exponent vector* of x . By convention, the sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Definition 2.2 (Linear order on P_t). Let $x = x_1^{a_1} x_2^{a_2} \dots x_t^{a_t}$ and $y = x_1^{b_1} x_2^{b_2} \dots x_t^{b_t}$ be the monomials in $(P_t)_n$. We write a, b for the exponent vectors of x and y , respectively. We say that $x < y$ if and only if one of the following holds:

- (i) $\omega(x) < \omega(y)$;
- (ii) $\omega(x) = \omega(y)$ and $a < b$.

Definition 2.3 (Equivalence relations on P_t). For a weight vector ω of degree n , let us denote two subspaces associated with ω by

$$(P_t)_n^\omega = \{x \in (P_t)_n \mid \deg(x) = \deg(\omega), \omega(x) \leq \omega\},$$

$$(P_t)_n^{<\omega} = \{x \in (P_t)_n \mid \deg(x) = \deg(\omega), \omega(x) < \omega\}.$$

Let f and g be two homogeneous polynomials in $(P_t)_n$, we define the binary relations " $\sim$ " and " $\sim_\omega$ " on $(P_t)_n$:

- (i) $f \sim g$ if and only if $(f - g) \in \sum_{i>0} \text{Im}(Sq^i)$. (Note that since working mod 2, $f - g = f + g$.)
- (ii) $f \sim_\omega g$ if and only if $f, g \in (P_t)_n^\omega$ and $(f - g) \in (((P_t)_n^\omega \cap \sum_{i>0} \text{Im}(Sq^i)) + (P_t)_n^{<\omega})$.

In particular, if $f \sim 0$ (resp. $f \sim_\omega 0$), then we say that f is *hit* (resp. ω -*hit*). It is straightforward to see that the above binary relations are equivalence ones. So, one has a quotient space

$$(Q_n^{\otimes t})^\omega := (P_t)_n^\omega / (((P_t)_n^\omega \cap \sum_{i>0} \text{Im}(Sq^i)) + (P_t)_n^{<\omega}),$$

from which it is not difficult to check the isomorphism $(QP_t)_n \cong \bigoplus_{\deg(\omega)=n} (Q_n^{\otimes t})^\omega$ (see [7]).

Definition 2.4 (Admissible monomial and inadmissible monomial). A monomial x in

$(P_t)_n$ is said to be *inadmissible* if there exist monomials $y_1, y_2, \dots, y_k$ in $(P_t)_n$ such that $y_j < x$ for all j , $1 \leq j \leq k$, and $x \sim (y_1 + y_2 + \dots + y_k)$. Then, we say that x is *admissible*, if it is not inadmissible.

The following important theorem, which is due to Kameko [2], can be viewed as a criterion in the calculation of the inadmissible monomials.

Theorem 2.5. *Let x and z be monomials in $(P_t)_n$. For an integer $r > 0$, assume that there exists an index $i > r$ such that $\omega_i(x) = 0$, from which if z is inadmissible, then xz^{2^r} is, too.*

From the above data, we have seen that $(QP_t)_n$ is an $\mathbb{Z}_2$ -vector space with a basis consisting of all the classes represent by the admissible monomials in $(P_t)_n$.

Definition 2.6 (Spike monomial). A monomial $z = \prod_{1 \leq j \leq t} x_j^{b_j}$ in $(P_t)_n$ is called a *spike* if every exponent b_j is of the form $2^{\beta_j} - 1$. In particular, if the exponents β_j can be arranged to satisfy $\beta_1 > \beta_2 > \dots > \beta_{r-1} \geq \beta_r \geq 1$, where only the last two smallest exponents can be equal, and $\beta_j = 0$ for $r < j \leq t$, then z is called a *minimal spike*.

Theorem 2.7 (see Phc-Sum [6]). *All the spikes in $(P_t)_n$ are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ of degree n is weakly decreasing and $\omega_1 \leq t$, then there is a spike $z \in (P_t)_n$ such that $\omega(z) = \omega$.*

Theorem 2.8 (see Singer [12]). *Suppose that $x \in (P_t)_n$ and $\mu(n) \leq t$. Let z be the minimal spike in $(P_t)_n$. If $\omega(x) < \omega(z)$, then $x \sim 0$.*

For $1 \leq l \leq t$, let $q_{(l,t)} : P_{t-1} \rightarrow P_t$ be an $\mathcal{A}_2$ -homomorphism, which is depicted by

$$q_{(l,t)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq l-1, \\ x_{j+1} & \text{if } l \leq j \leq t-1. \end{cases}$$

We assume throughout this text that $\mathcal{C}_n^{\otimes t}$ is the set of all admissible monomials in $(P_t)_n$. The following technicality is very useful for our calculations in the next section.

Theorem 2.9 (see Mothebe-Uys [4]). *Let l, d be positive integers such that $1 \leq l \leq t$. If $x \in \mathcal{C}_n^{\otimes(t-1)}$, then $x_l^{2^d-1} q_{(l,t)}(x) \in \mathcal{C}_{n+2^d-1}^{\otimes t}$.*

We set $\mathcal{N}_t := \{(l, \mathcal{L}) \mid \mathcal{L} = (l_1, l_2, \dots, l_r)\}$, where $1 \leq l < l_1 < l_2 < \dots < l_r \leq t$ and $0 \leq r \leq t-1$. By convention, $\mathcal{L} = \emptyset$ if $r = 0$. Let $r = \ell(\mathcal{L})$ be the length of $\mathcal{L}$. For each the pair $(l, \mathcal{L}) \in \mathcal{N}_t$, and $1 \leq u < r$, let us denote by

$x_{(\mathcal{L}, u)} := x_{l_u}^{2^{r-1}+2^{r-2}+\dots+2^{r-u}} \prod_{u < d \leq r} x_{l_d}^{2^{r-d}}$, where $x_{(\emptyset, 1)} = 1$. Sum [13] defines an $\mathbb{Z}_2$ -linear function $\psi_{(l, \mathcal{L})} : P_{t-1} \rightarrow P_t$, which is determined by

$$\begin{aligned} \psi_{(l, \mathcal{L})}(\prod_{1 \leq j \leq t-1} x_j^{a_j}) \\ = (x_l^{2^r-1} q_{(l,t)}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) / x_{(\mathcal{L}, u)}, \end{aligned}$$

if there exists u such that:

$$\begin{aligned} (2) \quad & a_{l_1-1} + 1 = \dots = a_{l_{(u-1)}-1} + 1 = 2^r, \\ & a_{l_u-1} + 1 > 2^r, \\ & \alpha_{r-d}(a_{l_u-1}) - 1 = 0, \quad 1 \leq d \leq u, \\ & \alpha_{r-d}(a_{l_d-1}) - 1 = 0, \quad u+1 \leq d \leq r, \end{aligned}$$

and $\psi_{(l, \mathcal{L})}(\prod_{1 \leq j \leq t-1} x_j^{a_j}) = 0$ otherwise. One has the following observation.

Remark 2.10. If $\mathcal{L} = \emptyset$, then $\psi_{(l, \mathcal{L})} = q_{(l,t)}$ for $1 \leq l \leq t$. It is in fact not hard to show that if $\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s}) \neq 0$, then $\omega(\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) = \omega(\prod_{1 \leq s \leq t-1} x_s^{a_s})$. Alternatively, $\psi_{(l, \mathcal{L})}$ is not the homomorphism of $\mathcal{A}_2$ -modules in general. As an illustrated example, we consider $t = 4$, $\mathcal{L} = (2, 3, 4) \neq \emptyset$, and the monomial $x_1^{12} x_2^6 x_3^9 \in (P_3)_{27}$, then, straightforward calculations show that

$$\begin{aligned} \psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9) &= \frac{x_1^{2^3-1} q_{(1,4)}(x_1^{12} x_2^6 x_3^9)}{x_{(\mathcal{L}, 1)}} \\ &= \frac{x_1^7 x_2^{12} x_3^6 x_4^9}{x_2^4 x_3^2 x_4} = x_1^7 x_2^8 x_3^4 x_4^8 \in (P_4)_{27}, \end{aligned}$$

and $\omega(x_1^{12} x_2^6 x_3^9) = (1, 1, 2, 2) = \omega(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9))$. So, using the Steenrod squares Sq^2 , one gets

$$\begin{aligned} Sq^2(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9)) &= x_1^9 x_2^8 x_3^4 x_4^8 \\ &\neq x_1^7 x_2^8 x_3^6 x_4^8 = \psi_{(1, \mathcal{L})}(Sq^2(x_1^{12} x_2^6 x_3^9)). \end{aligned}$$

For a subset $\mathcal{C} \subset (P_{t-1})_n$, let us consider the sets:

$$\begin{aligned} \widehat{\Phi}^0(\mathcal{C}) &= \bigcup_{1 \leq l \leq t} \psi_{(l, \emptyset)}(\mathcal{C}) = \bigcup_{1 \leq l \leq t} q_{(l,t)}(\mathcal{C}), \\ \widetilde{\Phi}^{>0}(\mathcal{C}) &= \bigcup_{(l, \mathcal{L}) \in \mathcal{N}_t, 1 \leq \ell(\mathcal{L}) \leq t-1} (\psi_{(l, \mathcal{L})}(\mathcal{C}) \setminus (P_t)_n^0), \end{aligned}$$

where

$$(P_t)_n^0 = \langle \{ \prod_{1 \leq i \leq t} x_i^{a_i} \in (P_t)_n : \prod_{1 \leq i \leq t} a_i = 0 \} \rangle$$

is an $\mathbb{Z}_2$ -subspace of $(P_t)_n$. For later convenience, let us denote by $\widetilde{\Phi}_*(\mathcal{C}) := \widehat{\Phi}^0(\mathcal{C}) \cup \widetilde{\Phi}^{>0}(\mathcal{C})$. Let ω be a weight vector of degree n and let $(\mathcal{C}_n^{\otimes t})^\omega := \mathcal{C}_n^{\otimes t} \cap (P_t)_n^\omega$. In [14], Sum sets up the following conjecture, which plays a crucial role in studying the minimal set of $\mathcal{A}_2$ -generators for P_t in each positive degree.

Conjecture 2.11. *Under the notations chosen, for each $(l, \mathcal{L}) \in \mathcal{N}_t$, and $1 \leq r = \ell(\mathcal{L}) \leq t-1$, if $x = \prod_{1 \leq j \leq t-1} x_j^{a_j} \in \mathcal{C}_n^{\otimes(t-1)}$, and there exist u , $1 \leq u \leq r$, which satisfies (2), then $\psi_{(l, \mathcal{L})}(x) \in \mathcal{C}_n^{\otimes t}$. Moreover, if ω is a weight vector of degree n , then $\widetilde{\Phi}_*((\mathcal{C}_n^{\otimes(t-1)})^\omega) \subseteq (\mathcal{C}_n^{\otimes t})^\omega$.*

Let us consider the set

$$\mathcal{U}(t, n) := \{\rho \in \mathbb{N} : \alpha(n - (2^\rho - 1) + t - 1) \leq t - 1\}.$$

For each $\rho \in \mathcal{U}(t, n)$, suppose that the monomial $\prod_{1 \leq j \leq t-1} x_j^{a_j} \in \mathcal{C}_{n-(2^\rho-1)}^{\otimes(t-1)}$. We denote

$$\mathcal{C}(t, n) := \bigcup_{\rho \in \mathcal{U}(t, n)} \langle \{x_1^{a_1} \dots x_{j-1}^{a_{j-1}} x_j^{2^\rho-1} x_{j+1}^{a_{j+1}} \dots x_t^{a_t} \in (P_t)_n : 1 \leq j \leq t\} \rangle \text{ and write}$$

$$(\mathcal{C}(t, n))^\omega = \{x \in \mathcal{C}(t, n) : \omega(x) = \omega\}.$$

Combining Conjecture 2.11 and Theorem 2.9, we have the following prediction.

Conjecture 2.12. *For each $(l, \mathcal{L}) \in \mathcal{N}_t$, if $x \in (\mathcal{C}_n^{\otimes(t-1)})^\omega$ and*

$$\max \left\{ r = \ell(\mathcal{L}) \leq t-1 : \psi_{(l, \mathcal{L})}(x) \neq 0 \right\} = |\mathcal{U}(t, n)|,$$

then

$$(\widetilde{\Phi}_*((\mathcal{C}_n^{\otimes(t-1)})^\omega) \cup (\mathcal{C}(t, n))^\omega) = (\mathcal{C}_n^{\otimes t})^\omega.$$

To close this section, we recall a result of Sum [13] on the inductive formula for the dimension of $(QP_t)_n$.

Theorem 2.13 (see Sum [13]). *Consider the degree n of the form (1) with $r = t-1$, and s, m positive integers such that $1 \leq t-3 \leq \mu(m) \leq t-2$, and $\mu(m) = \alpha(m + \mu(m))$. Then for each $s \geq t-1$, we have $\dim(QP_t)_n = (2^t - 1) \dim(QP_{t-1})_m$.*

By using iteration of the Kameko homomorphism $\widetilde{Sq}_*$ and the proof of Theorem 2.13, one has an equivalent statement of Theorem 2.13 that will be applied in the sequel.

Theorem 2.14. *Let us consider the generic degrees of the form (1): $n_s := (t-1)(2^s - 1) + m \cdot 2^s$. Suppose there is a integer ζ such that $0 \leq \zeta < s$ and $1 \leq t-3 \leq \mu(n_\zeta) = \alpha(n_\zeta + \mu(n_\zeta)) \leq t-2$. Then, for each $s \geq \zeta$, we have*

$$\dim(QP_t)_{(t-1)(2^{s-\zeta+t-1}-1)+n_\zeta 2^{s-\zeta+t-1}} = (2^t - 1) \dim(QP_{t-1})_{n_s}.$$

3. Statement of results We now survey our main results in this Note. Let us consider the generic degree of the form (1) with $r = t = 5$, and

$m = 42$. Then, we have $n_s := 5(2^s - 1) + 42 \cdot 2^s$ with $s \geq 0$. To solve the hit problem of five variables in this degree, our main idea is to combine the homomorphisms $\psi_{(l, \mathcal{L})}$ and Theorem 2.9 above. This approach, which is quite effective, helps us to reduce many computations. This is shown specifically in Theorem 3.3 below.

Notation 3.1. In what follows:

$$(P_t)_{n_s}^{>0} = \langle \{ \prod_{1 \leq i \leq t} x_i^{a_i} \in (P_t)_{n_s} : \prod_{1 \leq i \leq t} a_i > 0 \} \rangle,$$

$$(QP_t)_{n_s}^0 = \langle \{ [x] \in (QP_t)_{n_s} : x \in (P_t)_{n_s}^0 \} \rangle,$$

$$(QP_t)_{n_s}^{>0} = \langle \{ [x] \in (QP_t)_{n_s} : x \in (P_t)_{n_s}^{>0} \} \rangle.$$

Clearly, $(P_t)_{n_s}^{>0}$ is a $\mathbb{Z}_2$ -subspace of $(P_t)_n$, and $(QP_t)_{n_s} \cong (QP_t)_{n_s}^0 \oplus (QP_t)_{n_s}^{>0}$.

For a polynomial $f \in (P_t)_{n_s}$, we denote by $[f]$ the classes in $(QP_t)_{n_s}$ represented by f . If ω is a weight vector of degree n_s , and $f \in (P_t)_{n_s}$, then denote by $[f]_\omega$ the classes in $(QP_t)_{n_s}^\omega$ represented by f . For a subset $\mathcal{C} \subset (P_t)_{n_s}$, we write $|\mathcal{C}|$ for the cardinality of $\mathcal{C}$, and put $[\mathcal{C}] = \{[f] : f \in \mathcal{C}\}$. If $\mathcal{C} \subset (P_t)_{n_s}^\omega$, then put $[\mathcal{C}]_\omega = \{[f]_\omega : f \in \mathcal{C}\}$. It should be noted that if ω is a weight vector of minimal spike, then $[\mathcal{C}]_\omega = [\mathcal{C}]$. We set

$$(\mathcal{C}_{n_s}^{\otimes t})^{\omega^0} := (\mathcal{C}_{n_s}^{\otimes t})^\omega \cap (P_t)_{n_s}^0,$$

$$(\mathcal{C}_{n_s}^{\otimes t})^{\omega^{>0}} := (\mathcal{C}_{n_s}^{\otimes t})^\omega \cap (P_t)_{n_s}^{>0},$$

$$(Q_{n_s}^{\otimes t})^{\omega^0} := (Q_{n_s}^{\otimes t})^\omega \cap (QP_t)_{n_s}^0,$$

$$(Q_{n_s}^{\otimes t})^{\omega^{>0}} := (Q_{n_s}^{\otimes t})^\omega \cap (QP_t)_{n_s}^{>0}.$$

Then, $[(\mathcal{C}_{n_s}^{\otimes t})^\omega]_\omega$, $[(\mathcal{C}_{n_s}^{\otimes t})^{\omega^0}]_\omega$ and $[(\mathcal{C}_{n_s}^{\otimes t})^{\omega^{>0}}]_\omega$ are respectively the bases of the vector spaces $(Q_{n_s}^{\otimes t})^\omega$, $(Q_{n_s}^{\otimes t})^{\omega^0}$, and $(Q_{n_s}^{\otimes t})^{\omega^{>0}}$.

For a start, we have the following important remark.

Remark 3.2. Noticing that $\mu(n_s) = 5$ for all $s > 0$. This, together with Theorem 1.1, implies that the map

$$(3) \quad (\widetilde{Sq}_*)^s : (QP_5)_{n_s} \rightarrow (QP_5)_{n_0}$$

is an isomorphism of $\mathbb{Z}_2$ -vector spaces for any $s \geq 0$, from which we only need to compute $(QP_5)_{n_0}$. We have $n_0 = 5(2^0 - 1) + 42 \cdot 2^0 = 42$. Then, if $x \in \mathcal{C}_{42}^{\otimes 5}$, then the weight vector $\bar{\omega} := \omega(x)$ is one of the following sequences: $\bar{\omega}_{(1)} = (4, 3, 2, 1, 1)$, $\bar{\omega}_{(2)} = (4, 3, 2, 3)$, and $\bar{\omega}_{(3)} = (4, 3, 4, 2)$. Indeed, it should be noted that $z = x_1^{31} x_2^7 x_3^3 x_4 \in (P_5)_{42}$ is minimal spike, and so, by Theorem 2.7, z belongs to $\mathcal{C}_{42}^{\otimes 5}$. Moreover, because $\omega(z) = (4, 3, 2, 1, 1)$, based upon Theorem 2.8, we see that if $x \in \mathcal{C}_{42}^{\otimes 5}$, then $\omega_1(x) = 4$. This means that x is of the form $x_i x_j x_k x_l y^2$ with $y \in (P_5)_{19}$

and $1 \leq i < j < k < l \leq 5$. Since $x \in \mathcal{C}_{42}^{\otimes 5}$, following Theorem 2.5, we must have $y \in \mathcal{C}_{19}^{\otimes 5}$. So, the desired conclusion now follows from a result in [15] that $\omega(y)$ belongs to the set $\{(3, 2, 1, 1), (3, 2, 3), (3, 4, 2)\}$.

From the above remarks, the following is immediate:

$$(4) \quad (QP_5)_{42} \cong (QP_5)_{42}^0 \bigoplus \left(\bigoplus_{1 \leq j \leq 3} (Q_{42}^{\otimes 5})^{\bar{\omega}_{(j)}^0} \right).$$

Since $\mu(42) = 4$, $(QP_4)_{42} \cong (QP_4)_{19}$. Then, one can verify that

$$(QP_4)_{42} = (QP_4)_{42}^{>0} = (Q_{42}^{\otimes 4})^{\bar{\omega}_{(1)}^0} = (Q_{42}^{\otimes 4})^{\bar{\omega}_{(1)}},$$

and so, $\dim(QP_4)_{42} = \dim(QP_4)_{19} = 140$. On the other side, it has been shown (see also [8]) that

$$\dim(QP_5)^0 = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim(QP_s)^{>0},$$

for all $n \geq 0$. Using this formula, together with Peterson's conjecture and Theorem 1.1, we are forced to conclude that

$$(5) \quad \begin{aligned} \dim(QP_5)_{42}^0 &= \dim(Q_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0} \\ &= 140 \cdot \binom{5}{4} = 700, \\ (\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0} &= \widetilde{\Phi}^0((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}_{(1)}^0}) = \widetilde{\Phi}^0(\mathcal{C}_{42}^{\otimes 4}). \end{aligned}$$

For $1 \leq l \leq 5$, we consider the following sets:

$$\begin{aligned} \mathcal{C}_{l,1} &:= \{x_l \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{41}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_{l,2} &:= \{x_l^3 \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{39}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_{l,3} &:= \{x_l^7 \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{35}^{\otimes 4})^{>0}\}, \\ \mathcal{C}_{l,4} &:= \{x_l^{31} \mathbf{q}_{(l,5)}(x) : x \in (\mathcal{C}_{11}^{\otimes 4})^{>0}\}. \end{aligned}$$

Then, using Theorem 2.9, it can be shown that

$$\mathcal{C}^{>0}(\bar{\omega}_{(1)}) := \bigcup_{1 \leq j \leq 4} \bigcup_{1 \leq l \leq 5} \mathcal{C}_{l,j} \subseteq (\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0}.$$

We put

$E := \widetilde{\Phi}^{>0}((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}_{(1)}}) \setminus (\widetilde{\Phi}^{>0}((\mathcal{C}_{42}^{\otimes 4})^{\bar{\omega}_{(1)}}) \cap \mathcal{C}^{>0}(\bar{\omega}_{(1)}))$. Then, we have the theorem below, which is our first main result.

Theorem 3.3. *The following statements hold:*

- i) $(\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(j)}^0} = (\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(j)}} = \emptyset$, for $j = 2, 3$.
- ii) $(\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0} = E \cup F \cup \mathcal{C}^{>0}(\bar{\omega}_{(1)})$,

where $|E| = 542$, $|F| = 248$, and $|\mathcal{C}^{>0}(\bar{\omega}_{(1)})| = 1030$. Consequently, $(Q_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0} = \langle [(\mathcal{C}_{42}^{\otimes 5})^{\bar{\omega}_{(1)}^0}]_{\bar{\omega}_{(1)}} \rangle$ is 1820-dimensional.

From (3), (5), and Theorem 3.3, it may be concluded that Conjecture 2.11 holds for $t = 5$ and the generic degree n_s for any $s \geq 0$.

As immediate consequences from (3), (4), (5) and Theorem 3.3, we may assert that

Corollary 3.4. *There exist exactly 2520 admissible monomials of degree n_s in P_5 for all $s \geq 0$. Consequently, $(QP_5)_{n_s}$ is 2520-dimensional.*

The above computations confirm the dimension of $(QP_5)_{n_s}$, which is informed in [8] by using the MAGMA computer algebra system.

Next, based on Theorem 2.14 and Corollary 3.4, we have immediately the following.

Corollary 3.5. *When degree n_s is as in Theorem 2.14, for $t = 6$, $m = 42$, and $s \geq 0$, we have $\dim(QP_6)_{n_s+5} = (2^6 - 1) \dim(QP_5)_{n_s} = 158760$.*

Singer's cohomological transfer. The next contribution of the paper is to apply Theorem 3.3 into the study of the behavior of the fifth transfer homomorphism of W. Singer [11]. It may need to be recalled that the group GL_t acts regularly on P_t by matrix substitution. Further, the two actions of $\mathcal{A}_2$ and GL_t upon P_t commute with each other; hence there is an inherited action of GL_t on $QP_t = \mathbb{Z}_2 \otimes_{\mathcal{A}_2} P_t$. Thus, the hit problem becomes important in studying the modular representation of GL_t . This is of interest since the Singer cohomological "transfer" [11] relates $(QP_t)_n$ to $H^{t,t+n}(\mathcal{A}_2, \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}_2}^{t,t+n}(\mathbb{Z}_2, \mathbb{Z}_2)$, the $\mathbb{Z}_2$ -cohomology groups of $\mathcal{A}_2$, and thus to the stable homotopy groups of spheres. Singer's transfer is established as follows. Let us recall first that if $\mathcal{M}$ the category of graded left $\mathcal{A}_2$ -modules and degree zero of $\mathcal{A}_2$ -linear maps, then for each integer t , the supension functor $S^t : \mathcal{M} \rightarrow \mathcal{M}$ is defined by $(S^t U)^n := U^{n-t}$ where $n \in \mathbb{Z}$. The action of $\mathcal{A}_2$ on $S^t U$ is given by $\theta(S^t u) = S^t(\theta u)$, for all $u \in U$ and $\theta \in \mathcal{A}_2$. Next, consider the polynomial ring $P_1 = \mathbb{Z}_2[x]$ with $\deg(x) = 1$, the canonical $\mathcal{A}_2$ -action on P_1 is extended to an $\mathcal{A}_2$ -action on the ring of finite Laurent series $\mathbb{Z}_2[x_1, x_1^{-1}]$. Then, there exists an $\mathcal{A}_2$ -submodule $\bar{P} = \langle \{x_1^t \mid t \geq -1\} \rangle$ of $\mathbb{Z}_2[x_1, x_1^{-1}]$. One has a short-exact sequence $0 \rightarrow P_1 \xrightarrow{q} \bar{P} \xrightarrow{\pi} S^{-1}\mathbb{Z}_2$, where q is the inclusion and π is given by $\pi(x_1^t) = 0$ if $t \neq -1$ and $\pi(x_1^{-1}) = 1$. Writing e_1 for the corresponding element in $\text{Ext}_{\mathcal{A}_2}^1(S^{-1}\mathbb{Z}_2, P_1)$. Basing the cross, the Yoneda and the cap products in co(homology) with $\mathbb{Z}_2$ -coefficients, we have a homomorphism

$$\begin{aligned} Tr_t^{\mathcal{A}_2} : \text{Tor}_t^{\mathcal{A}_2}(\mathbb{Z}_2, S^{-t}\mathbb{Z}_2) &\longrightarrow \text{Tor}_0^{\mathcal{A}_2}(\mathbb{Z}_2, P_t) = QP_t \\ z &\longmapsto e_t \cap z, \end{aligned}$$

where $e_t = (e_1 \times P_{t-1}) \circ \cdots \circ (e_1 \times P_1) \circ e_1$. Its image is a submodule of the space of GL_t -invariants

$(QP_t)^{GL_t}$. It is known, the suspension S^{-t} induces an isomorphism

$$\mathrm{Tor}_{t,n}^{\mathcal{A}_2}(\mathbb{Z}_2, S^{-t}\mathbb{Z}_2) \cong H_{t,t+n}(\mathcal{A}_2, \mathbb{Z}_2)$$

and so $Tr_t^{\mathcal{A}_2}$ induces the homomorphism

$$Tr_t^{\mathcal{A}_2} : H_{t,t+n}(\mathcal{A}_2, \mathbb{Z}_2) \rightarrow (QP_t)_n^{GL_t}.$$

The dual $(Tr_t^{\mathcal{A}_2})^* : ((QP_t)_n^{GL_t})^* \rightarrow H^{t,t+n}(\mathcal{A}_2, \mathbb{Z}_2)$ of $Tr_t^{\mathcal{A}_2}$ is called *the t -th cohomological transfer*. In [11], Singer proves that $(Tr_t^{\mathcal{A}_2})^*$ is an isomorphism for $t = 1, 2$. Afterwards, based on Kameko's thesis [2], Boardman [1] states that $(Tr_3^{\mathcal{A}_2})^*$ is also an isomorphism. These works tell us that Singer's transfer map is highly non-trivial, and so, it serves as a reliable tool for depicting the cohomology of the Steenrod ring. In higher cohomological degrees, Singer [11] gives some calculations to show that the fourth transfer is an isomorphism up to a range of internal degrees, but the fifth transfer is not an epimorphism, from which he makes the following prediction.

Conjecture 3.6. *The cohomological transfer is a monomorphism.*

This was confirmed for $1 \leq t \leq 3$ by Singer himself [11] and Boardman [1]. Our recent work [10] shows that the conjecture is also true for $t = 4$. So far it remains open in general. Very little information is known when $t = 5$. Now, based upon an admissible monomial basis for QP_5 in degree $5(2^s - 1) + 42 \cdot 2^s$ (see Theorem 3.3), we verify Conjecture 3.6 for $t = 5$ and the respective degree. The following theorem is our second main result.

Theorem 3.7. *The transfer homomorphism $(Tr_5^{\mathcal{A}_2})^* : ((QP_5)_{5(2^s-1)+42 \cdot 2^s}^{GL_5})^* \rightarrow H^{5,47 \cdot 2^s}(\mathcal{A}_2, \mathbb{Z}_2)$ is a trivial isomorphism, for all $s \geq 0$.*

This result can be proved by explicitly computing the space of GL_5 -invariants $(QP_5)_{42}^{GL_5}$ combining the fact that $H^{5,47 \cdot 2^s}(\mathcal{A}_2, \mathbb{Z}_2)$ is trivial (see Lin [3]). Thus, with these data in hand, the reader can see that

Corollary 3.8. *Conjecture 2.11 holds for the rank 5 case and degree $5(2^s - 1) + 42 \cdot 2^s$ with $s \geq 0$.*

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