Explicit formulas for the Euler's phi function and the prime counting function

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Abstract

An explicit formula which characterizes the pairs of integers that are relatively prime is obtained here. It doesn't require the knowledge of the prime factors of the arguments. Then other formulas are deduced for the Euler's totient function and the prime counting function .

1. Introduction

We shall deduce in the first place a characteristic function for pairs of relatively prime numbers (without common prime factors) which will be useful for instance to obtain exact formulas for the 'Euler's phi' or the 'prime counting' function, as we do later.

For the first one, also called totient function, that gives the amount of positive integers which are -relatively- prime with some given integer greater than them, it is well known the 'Euler's formula'

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

meaning that (the primes) p must be divisors of n , but obtaining these divisors is a different task. It is possible also to write φ as the real part of the discrete Fourier transform of the GCD -greatest common divisor- evaluated in 1 (as in [7]) but that formula requires all the GCD of the argument with the previous numbers:

$$\varphi(n) = \sum_{k=1}^n \text{GCD}(k,n) \cos\left(\frac{2\pi k}{n}\right)$$

Regarding the asymptotic behaviour of the function there are many articles but it is not so easy to find exact expressions like these. We deduce here two formulas with the advantage that they don't need the prime factors of the argument nor other algorithms, so that they allow a direct evaluation by substitution (at least theoretically: while the roundings to certain decimal order don't affect the exact value of the integer parts involved).

In relation to the prime counting function, $\pi(x)$, which gives the amount of primes not greater than a given number, asymptotic formulas have also been found and some other exact ones with mainly a theoretical interest, because they don't provide efficient methods for big numbers due to speed and precision. One quite known, also using trigonometric functions, is Willans' ([10]), based directly on the characterization of the primes by the Wilson's theorem and its inverse, although it includes factorials in the arguments, or gamma function in Connes' version [1], with the inconvenient of its so fast growth. Another elementary formulas are some proposed by Zinn [11], somewhat similar to the ones we comment next for π but deduced in another way, from a characteristic function for one number's divisors; the expressions given here are maybe simpler in the sense that they don't mix products with subtractions.

2. Characterizing relative primes

Definition 2.1. For every integer $i > 1$, define the real (i-th) function *rp-wave* (of a real variable x) in the next way:

$$(1) \quad v_i(x) \equiv \prod_{k=1}^{F(i/2)} \left| \sin^{-1} \left(\frac{k \pi}{i} \right) \sin \left(\frac{k \pi x}{i} \right) \right| = \alpha_i \prod_{k=1}^{F(i/2)} \left| \sin \left(\frac{k \pi x}{i} \right) \right|$$

$$(\text{where } \alpha_i \equiv \left(\prod_{k=1}^{F(i/2)} \sin \left(\frac{k \pi}{i} \right) \right)^{-1})$$

Here $F(\cdot)$ stands for floor function and the bars, for absolute value. The negative exponents mean numerical inverses in both lines (not arcsines in the first one). Each factor $|\sin(k \pi x / i)|$ will be referred to as the *k component* of the i-th rp-wave.

We are interested in positive integer -natural- values of x . Next we prove that, at every integer $x = j > 0$, the i-th rp-wave takes only the values 1 -when i and j are relative primes- and 0 -when that's not the case-. This gives us a way of characterizing the pairs of integers without prime factors in common: it will suffice to take also i as a variable in (1), giving x only natural values.

Proposition 2.1. For $1 < i \in \mathbb{Z}$ and $0 < j \in \mathbb{Z}$:

$$(2.1) \quad v_i(j) = 0 \Leftrightarrow \text{GCD}(i, j) > 1$$

Proof. Remind that the LCM -least common multiple- equals the product of two integers if and only if their GCD is 1. We assume that $x = j$ is a positive integer.

From left to right, if the product of sines vanish in (1), it means that at least one of the fractions $k \cdot j / i$ in the arguments has an integer value, so that the product of j by the

corresponding k -not greater than $i/2$ by definition- is a multiple of i less than $j \cdot i$, so that $\text{GCD}(i, j)$ exceeds 1. Inversely, when $\text{GCD}(i, j)$ is greater than 1, then $\text{LCM}(i, j)$ is less than $j \cdot i$, what means that the product of j by some integer k not greater than $i/2$ is a multiple of i and the fraction $k \cdot j / i$ with such value of k in (1) is an integer and the corresponding sine vanishes. $\square$

Proposition 2.2. For $1 < i \in \mathbb{Z}$ and $0 < j \in \mathbb{Z}$:

$$(2.2) \quad v_i(j) = 1 \Leftrightarrow \text{GCD}(i, j) = 1$$

Proof. Using proposition 1 it is enough to prove the implication from right to left, but this will require several steps; we make first some observations relating these functions.

Leaving apart the constant α_i (a normalizing one, as we see later), the i -th rp-wave is a product of components which are absolute values of sine functions that, due to their arguments, have periods that are all fractions of i . So their product is another periodic non negative function with period i -the same of the component $k = 1$ - that vanishes at least at the extremes of the -we could say basic- interval $[0, i]$. It is also true that each component is a symmetric function on that interval with respect to its central abscissa $i/2$ (independently of the parities of i and k) and in the same way the rp-wave is also symmetric, with same images at same distances of that centre.

Besides, the rp-wave is injective at most on intervals of amplitude $i/2$ as $[0, i/2]$, due to the combination of symmetry and periodicity. So it can at most take $F(i/2) + 1$ different values for integer abscissas in general because on $[0, i/2]$ there are only $F(i/2) + 1$ different integer values for x . Therefore, as α_i is the inverse of the product of the components at $x = 1$, it is enough to check that the i -th rp-wave has the same value at any other positive integer that, as 1, is relatively prime with i (what means that it is normalized at natural abscissas because at the ones not prime with i it must vanish, according to the first proposition).

For $x = 1$ the arguments of the sines of the components are the successive multiples of the quotient π / i , until $F(i/2) \cdot \pi / i$, so that the sines take positive increasing values, all different. That π / i is the minimum increment for the arguments in general, so that at each other positive integer x every component must vanish or equal one of those same $F(i/2)$ values of the sine function, thanks to the symmetry and periodicity of its absolute value. As a result, in order to check what is said at the end of the previous paragraph, it suffices to show that all the components take different values at the integers $x (\equiv j)$ which are prime with the index i of the considered rp-wave: according to the first proposition, none of them can vanish there, so that the number of the possible values for them at those x is the same that the number of components, that is, $F(i/2)$. The disorder of factors won't affect the total product if they are all distinct.

And none of the components can equal the value 1 for integer x if i is odd because all k are then less than $i/2$. But 1 equals the last of the components, the one with $k = i/2$, when the index i of the pr-wave is even for an integer $x = j$ prime with i . This is obvious not only for $x = 1$, and no other component can equal 1 at those abscissas (odd, in particular) because $i/2$ in these cases is also prime with j , so that there is no other integer less than $i/2$ which gives a multiple of $i/2$ when multiplied by that j . We will remind this later and focus for the moment on the other components, those with $k < i/2$ independently of the parity of i (and so with sine's arguments not multiples of $\pi/2$ for any integer x prime with the index i , what we assume until the last paragraph).

In relation with the fact that the minimum possible increment for the arguments of the sines for integer x is the quotient π/i , we will use the 'cancellation property' in modular arithmetic: for $i > 1$ prime with j , $a \cdot j = b \cdot j \pmod{i}$ implies $a = b \pmod{i}$, where all numbers are supposed to be integers. We distinguish two cases for each natural x : the components k with sine's arguments in odd angular quadrants -arguments between 0 and $\pi/2$ except for maybe a difference of a multiple of π , that is, with even value of $F(2kx/i)$ - and the components with sine's arguments in even angular quadrants -with odd value of $F(2kx/i)$ -, being the first quadrant $(0, \pi/2)$, as usual.

Now, it is clear that two different components with sine's arguments $(k' \pi j/i)$ and $(k'' \pi j/i)$ both in odd quadrants take different values, not only when $j = 1$, because their respective k are distinct numbers which must give distinct results $\pmod{i}$ when multiplied by that j according to the cited property under the assumptions, and this means different arguments also after any difference of a multiple of π in one of them. As the absolute value of the sine is strictly increasing with the argument in those quadrants, also the values of the components must differ. In the same way we can deduce that two distinct components with arguments both in even quadrants have different values (now the absolute value of the sine decreases strictly with the argument).

For the moment we have used the fact that the first positive integers until $F(i/2)$ are part of a complete residue system modulo i , but the absolute value of the sine is injective at most in intervals of amplitude $\pi/2$. In the case of two components with arguments $(k' \pi j/i)$ and $(k'' \pi j/i)$ in quadrants with different parities, we make use of the known formula $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, reasoning by reductio ad absurdum. We suppose that two such components are equal for that $x = j$ prime with i ; then there would be two possibilities for the pair of arguments, because the corresponding sines -without absolute value- would be either equal or opposite numbers. But both of them lead to a contradiction: being $k' < k''$ ($< i/2$ as we still exclude the value 1 for the components) the two possibilities are:

- a) If $\sin(k' \pi j / i) \equiv \sin A = \sin(k'' \pi j / i) \equiv \sin B$ then, due to the opposite parities of the quadrants it is also true that $\cos B = -\cos A$ and $\sin(A + B) = \sin A \cdot (-\cos A + \cos A) = 0$, so that the argument $A + B = (k' j + k'' j) \cdot \pi / i$ must be a multiple of π .
- b) If $\sin(k' \pi j / i) \equiv \sin A = -\sin(k'' \pi j / i) \equiv -\sin B$, then $\cos B = \cos A$ and $\sin(A + B) = \sin A \cdot (\cos A - \cos A) = 0$, obtaining the same conclusion.

Both cases are impossible because if $(k'j + k''j) / i$ is integer then $(k' + k'') \cdot j = q \cdot i$, with q also an integer, but j is supposed to be prime with i , so that all the prime factors of i (including possible repetitions) should be included in the factor $(k' + k'')$ and this is false because it is smaller than i due to the assumption $k' < k'' < i / 2$.

After this, we only have to remind that for an even i the component $k = i / 2$ is the only one which can equal 1 for integer x , only when it is odd. So, including also this component it is clear that at any x prime with i , where the $F(i / 2)$ components can't vanish, their values must be all different and form necessarily the same set as for $x = 1$. Hence according to the previous reasoning, the rp-wave equals 1 there. $\square$

3. Formulas for the Euler's φ function

Lemma 3.1. Let i, j be integers greater than 1. Then $v_i(j) = v_j(i)$.

Proof. Straightforward from the previous propositions and the symmetry of the GCD with respect to its arguments. $\square$

Using (1) and this symmetry we obtain two distinct expressions for the 'totient' function, which counts the positive integers relatively prime with another greater one.

Theorem 3.1. For $2 < n \in \mathbb{Z}$:

$$(3.1) \quad \varphi(n) = 1 + \sum_{i=2}^{n-1} \prod_{k=1}^{F(i/2)} \left| \sin^{-1} \left(\frac{k \pi}{i} \right) \sin \left(\frac{n k \pi}{i} \right) \right|$$

Proof. Straightforward from the previous propositions and the definition of the function φ (with the sum over the values at that n of the successive rp-waves, added 1 because they are defined from index 2). $\square$

Examples. We can check easily some values:

$$\varphi(3) = 1 + \left| \sin^{-1} \left(\frac{\pi}{2} \right) \sin \left(\frac{3\pi}{2} \right) \right| = 1 + 1 = 2$$

$$\varphi(4) = 1 + \left| \sin^{-1}\left(\frac{\pi}{2}\right) \sin\left(\frac{4\pi}{2}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{3}\right) \sin\left(\frac{4\pi}{3}\right) \right| = 1 + 0 + 1 = 2$$

$$\varphi(5) = 1 + \left| \sin^{-1}\left(\frac{\pi}{2}\right) \sin\left(\frac{5\pi}{2}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{3}\right) \sin\left(\frac{5\pi}{3}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{4}\right) \sin\left(\frac{5\pi}{4}\right) \sin^{-1}\left(\frac{2\pi}{4}\right) \sin\left(\frac{10\pi}{4}\right) \right|$$

$$= 1 + 1 + 1 + 1 = 4$$

$$\varphi(6) = 1 + \left| \sin^{-1}\left(\frac{\pi}{2}\right) \sin\left(\frac{6\pi}{2}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{3}\right) \sin\left(\frac{6\pi}{3}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{4}\right) \sin\left(\frac{6\pi}{4}\right) \sin^{-1}\left(\frac{2\pi}{4}\right) \sin\left(\frac{12\pi}{4}\right) \right|$$

$$+ \left| \sin^{-1}\left(\frac{\pi}{5}\right) \sin\left(\frac{6\pi}{5}\right) \sin^{-1}\left(\frac{2\pi}{5}\right) \sin\left(\frac{12\pi}{5}\right) \right| = 1 + 0 + 0 + 0 + 1 = 2$$

....

Theorem 3.2. For $2 < n \in \mathbb{Z}$:

$$(3.2) \quad \varphi(n) = 1 + \sum_{j=2}^{n-1} \prod_{k=1}^{F(n/2)} \left| \sin^{-1}\left(\frac{k\pi}{n}\right) \sin\left(\frac{j k \pi}{n}\right) \right|$$

Proof. Straightforward from the previous propositions and the definition of the function φ (now with the sum over the values at the successive natural numbers, from 2, of the n -th rp-wave, adding 1 because it is the value of any rp-wave at 1). $\square$

Examples. Now we will obtain the first two values:

$$\varphi(3) = 1 + \left| \sin^{-1}\left(\frac{\pi}{3}\right) \sin\left(\frac{2\pi}{3}\right) \right| = 1 + 1 = 2$$

$$\varphi(4) = 1 + \left| \sin^{-1}\left(\frac{\pi}{4}\right) \sin\left(\frac{2\pi}{4}\right) \sin^{-1}\left(\frac{2\pi}{4}\right) \sin\left(\frac{4\pi}{4}\right) \right| + \left| \sin^{-1}\left(\frac{\pi}{4}\right) \sin\left(\frac{3\pi}{4}\right) \sin^{-1}\left(\frac{2\pi}{4}\right) \sin\left(\frac{6\pi}{4}\right) \right|$$

$$= 1 + 0 + 1 = 2$$

...

A first observation is that for an even n we could skip the addends with smaller even indexes, but this would make the expressions less general. As we announced, they have mainly a conceptual interest because they are not quite practical in a computational sense. A bit more about this is said after the next section.

4. Formulas for the prime counting function

According to the usual notation it is $\pi(x) \equiv \#\{p \leq x\}$ for real abscissas in general, where p design prime numbers and $\#$ is cardinality.

Proposition 4.1. For $4 < n \in \mathbb{Z}$:

$$(4) \quad a) \quad \pi(n) = 2 + \sum_{j=5}^n \prod_{i=2}^{F(\sqrt{j})} v_i(j)$$

$$b) \quad \pi(n) = 2 + \sum_{j=5}^n \prod_{i=2}^{F(\sqrt{j})} v_j(i)$$

Proof. From the previous propositions (2.) and lemma, where i can be less or greater than j : in a) and b) there is an addend for each j greater than 4 until n , besides the number 2 which counts the two first primes (less than 4), and each term is a product of factors that equals 0 or 1 according with (2.), where those with value 1 stand for the relative primes with (the corresponding index) j . So an addend with a nonprime index j -that is, with some divisor which is different from 1 and j - must vanish. There is no need to exceed the square root because divisors distinct from that root always form couples with one of the numbers smaller than it. $\square$

Theorem 4.1. Let O be the set of odd integers and $11 \leq n \in O$:

$$(5) \quad a) \quad \pi(n) = 4 + \sum_{\substack{j=11 \\ j \in O}}^n \prod_{i \in O \cap [3, \sqrt{j}]} \prod_{k=1}^{F(i/2)} \left| \sin^{-1} \left(\frac{k \pi}{i} \right) \sin \left(\frac{k \pi j}{i} \right) \right|$$

$$b) \quad \pi(n) = 4 + \sum_{\substack{j=11 \\ j \in O}}^n \prod_{i \in O \cap [3, \sqrt{j}]} \prod_{k=1}^{F(j/2)} \left| \sin^{-1} \left(\frac{k \pi}{j} \right) \sin \left(\frac{k \pi i}{j} \right) \right|$$

Proof. It is clear using (4) if we remind that only odd integers are prime above 2 and that it is enough to check each GCD with smaller odd integers, from 3 until the square root, to confirm if a new odd number is prime. We have also substituted the rp-waves with (1) and the initial 4 counts the first primes until 7. $\square$

5. Some practical considerations

Regarding φ , without comparing with other methods, (3.2) seems slower than (3.1) due to the greater number of addends. Anyway, the arguments in the sines are quite similar and we could make a more detailed comparison, but the whole idea is clearly based on the arguments and allows in fact the development of a simple algorithm to evaluate φ that avoids the calculation of sines and the products. Using (3.1), the idea is to evaluate directly and successively the quotients $n \cdot k / i$ in the arguments for each i from 2 until $n - 1$ and k from $F(i / 2)$ until 1 so as to check if they are integer using for instance the floor function -as long as the computer is reliable-. So not all quotients

have to be evaluated, of course, because once one of them is integer we can skip the next ones (k) for the same i and begin with the ones for the next i , without adding 1 in that case to a total sum that begins from 1 (1 is prime with every other n).

This method can be optimized or combined with other elementary technics to be compared with others, beginning with the direct testing of common divisors with smaller integers, though it is in principle faster to combine the simple Euler's formula with the previous factorization of the number to obtain its prime divisors: even for a prime n we can check in such way that it is prime doing not more than $n - 2$ divisions, while the described method without modification requires at least one for each of the $n - 2$ smaller integers above 1 also when n is composite. And adaptations for the prime counting function can also be done from that algorithm and section 4, though the same speed inconvenient arises in the first place.

But algorithms in general and of course the sophisticated ones available today don't allow a short and direct display as that of the previous formulas, as claimed at the beginning. In this sense, maybe the more compact expressions that would arise by replacing (1) in (4) are also of theoretical or didactical interest.

6. Generalizations

We can express both functions in a more general way from the same basis. Instead of using the function $|\sin(\pi y)|$, with real variable $y = k x / i$, to define components of 'rp-waves' as before, it suffices to use a real function -which we call now $rpc(y)$ - as long as it is also periodic (with period 1 to make it simpler), symmetric (meaning $rpc(y) = rpc(1 - y)$) and non negative so that it vanishes at integer abscissas and only there. Then it is possible to proceed in an analogous way with the same $y = k x / i$ to obtain *components* (of a more general *rp-function*) corresponding to the different values of k :

$$rp_i(x) \equiv \beta_i \prod_{k=1}^{F(i/2)} rpc\left(\frac{k x}{i}\right) \quad \left(\beta_i \equiv \left(\prod_{k=1}^{F(i/2)} rpc\left(\frac{k}{i}\right) \right)^{-1} \right)$$

To check this using the proofs of the initial propositions there isn't much more to do: if k, i and $x = j$ are integers, in the set of the sine's arguments of the form $k \pi j / i$ a partition in equivalence classes can be done thanks to the fact that sine is injective in each angular quadrant of amplitude $\pi / 2$, so that each one of the $F(i/2) + 1$ possible distinct images of the sine's absolute value for those arguments define a different class, the one of the arguments with precisely that image (for instance, the value 0 for the images define the class of arguments that are multiples of π). But these classes univocally correspond to classes of arguments of the new function rpc if we divide each number by π , which means that proposition (2.2) reads the same for the new

components: when x is not 1 but another j that is relative prime with i , the $F(i/2)$ numbers k/i change in general to others $k/j/i$ also with exactly one number representing each class except the one of image 0. That is to say, if we substitute each component $|\sin(\pi y)|$ by the corresponding $rpc(y)$ with same k , the total product of them from $k = 1$ to $k = F(i/2)$ will be constant as well when changing x from 1 to another j prime with i . This is clearly guaranteed by the conditions imposed to $rpc(y)$, not being necessary that it is injective in half period or continuous -as it is $|\sin(\pi y)|$ - : a simple example is a function rpc that is positive and constant except at integer abscissas, where it vanishes.

Of course, according to this correspondence, if j is not prime with i -and only then- at least one of the rpc components will vanish due to an integer argument, according to (2.1). So we can add easily two pairs of formulas that generalize the previous ones with the conditions imposed above on the new functions rp_i :

$$\varphi(n) = 1 + \sum_{i=2}^{n-1} rp_i(n)$$

$$\varphi(n) = 1 + \sum_{j=2}^{n-1} rp_n(j)$$

$$\pi(n) = 2 + \sum_{j=5}^n \prod_{i=2}^{F(\sqrt{j})} rp_i(j)$$

$$\pi(n) = 2 + \sum_{j=5}^n \prod_{i=2}^{F(\sqrt{j})} rp_j(i)$$

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