Simple Example of a Compact NonHausdorff Space with Closed Singletons

Yu-Lin Chou*

Abstract

We give as a main message a simple but “finer” example of a compact non-Hausdorff space where every singleton is closed.

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Although every singleton, i.e. a set consisting of exactly one point, in a Hausdorff space is closed, not every space with closed singletons is Hausdorff.

We give in particular a simple example of a nonHausdorff space where every singleton is closed, with the space even being compact. What we are showing is more refined. A space X with at least two points is called *hypernonHausdorff* if and only if $x, y \in X$ being distinct and G_1, G_2 being neighborhoods of x and y respectively imply $G_1 \cap G_2 \neq \emptyset$.

We prove

Theorem 1. *There is some compact hypernonHausdorff space with all singletons being closed.*

Proof. Let X be an infinite set; let $\mathcal{T}_X := \{\emptyset\} \cup \{G \mid G^c \text{ is finite}\}$. Then $\mathcal{T}_X$ is a topology of X .

If $x, y \in X$ are distinct, and if $V_1, V_2 \in \mathcal{T}_X$ are neighborhoods of x and y respectively with empty intersection, then $V_1^c \cup V_2^c = X$, which is a finite set by construction and also an infinite set by assumption; so X is hypernonHausdorff.

Since $X \setminus \{x\} \in \mathcal{T}_X$ for all $x \in X$, every singleton in X is closed.

To see that X is compact, let $\{G_\theta\}_{\theta \in \Theta}$ be an open cover of X . Fix a finite subset Γ of Θ ; then $X \setminus \bigcup_{\theta \in \Gamma} G_\theta$ is finite. If $x_1, \dots, x_n$ are all the points of $X \setminus \bigcup_{\theta \in \Gamma} G_\theta$, choose $\theta_1, \dots, \theta_n \in \Theta \setminus \Gamma$ such that $x_i \in G_{\theta_i}$ for all $1 \leq i \leq n$. But then we have obtained a finite subcover $\{G_\theta\}_{\theta \in \Gamma \cup \{\theta_1, \dots, \theta_n\}}$ of $\{G_\theta\}_{\theta \in \Theta}$. $\square$

*Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: y.l.chou@gapp.nthu.edu.tw.

The proof of Theorem 1 immediately suggests

Proposition 1. *There is some second-countable compact hypernonHausdorff space with all singletons being closed.*

Proof. Let $X := \mathbb{N}$, and topologize X in the same way as in Theorem 1. Then X is by Theorem 1 a compact hypernonHausdorff space with all singletons being closed.

But X is evidently second-countable; for instance, we may consider the countable collection $\{X \setminus \{1, \dots, n\} \mid n \in \mathbb{N}\}$. □