A FORMULA FOR THE NUMBER OF PARTITIONS OF n IN TERMS OF THE PARTIAL BELL POLYNOMIALS

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ABSTRACT. We derive a formula for $p(n)$ (the number of partitions of n) in terms of the partial Bell polynomials using Faà di Bruno's formula and Euler's pentagonal number theorem.

1. MAIN RESULT

Recall the classical partition function, denoted by $p(n)$, gives the number of ways of writing the integer n as a sum of positive integers, where the order of summands is not considered significant. For example, $p(4) = 5$, since there are 5 ways of ways to represent 4 as sum of positive integers, namely, $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

We also recall another classical statistic, the (n, k) th partial Bell polynomial in the variables $x_1, x_2, \dots, x_{n-k+1}$, denoted by $B_{n,k} \equiv B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ ([1, p. 134], [5, Ch. 12]), defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}.$$

Cvijović [3] gives the following formula for calculating these polynomials

$$\begin{aligned} B_{n,k+1} &= \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \overbrace{\binom{n}{\alpha_1} \binom{\alpha_1}{\alpha_2} \cdots \binom{\alpha_{k-1}}{\alpha_k}}^k \times \cdots \\ (1) \quad &\times x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \cdots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k} \quad (n \geq k+1, k = 1, 2, \dots) \end{aligned}$$

We prove the following here.

Theorem 1. *We have*

$$(2) \quad p(n) = \frac{1}{n!} \sum_{k=0}^n (-1)^k k! B_{n,k}(\lambda_1, \lambda_2, \dots, \lambda_{n-k+1})$$

where

$$(3) \quad \lambda_m = \begin{cases} (-1)^{\frac{1+\sqrt{1+24m}}{6}} m! & \text{if } \frac{1+\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ (-1)^{\frac{1-\sqrt{1+24m}}{6}} m! & \text{if } \frac{1-\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We begin by the following generating function [2, Equation 22.13]

$$(4) \quad \sum_{n \geq 0} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}.$$

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We recall the Euler's pentagonal number theorem [2, Equation 7.8]

$$(5) \quad E(q) := \prod_{j=1}^{\infty} (1 - q^j) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} \\ = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \dots$$

Let $f(q) = 1/q$. Using Faà di Bruno's formula ([1, p. 137], [5, Ch. 12]) we have

$$(6) \quad \frac{d^n}{dq^n} f(E(q)) = \sum_{k=0}^n f^{(k)}(E(q)) \cdot B_{n,k} \left(E'(q), E''(q), \dots, E^{(n-k+1)}(q) \right).$$

Since $f^{(k)}(q) = \frac{(-1)^k k!}{q^{k+1}}$ and $E(0) = 1$, letting $q \rightarrow 0$ in the above equation gives

$$p(n) n! = \sum_{k=0}^n (-1)^k k! B_{n,k} \left(E'(0), E''(0), \dots, E^{(n-k+1)}(0) \right).$$

The Euler's pentagonal number theorem (5) gives us

$$E^{(m)}(0) = \lambda_m$$

where λ_m is as defined in (3). □

Combining equations (1) and (2) we can conclude that

$$p(n) = -\theta_n + \underbrace{\sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \dots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \theta_{n-\alpha_1} \theta_{\alpha_1-\alpha_2} \dots \theta_{\alpha_{k-1}-\alpha_k} \theta_{\alpha_k}$$

where

$$\theta_m = \begin{cases} (-1)^{\frac{1+\sqrt{1+24m}}{6}} & \text{if } \frac{1+\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ (-1)^{\frac{1-\sqrt{1+24m}}{6}} & \text{if } \frac{1-\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1. Let $E(q)^r := \prod_{j=1}^{\infty} (1 - q^j)^r = \sum_{n=0}^{\infty} p_r(n) q^n$ with $p_r(0) = 1$ (see [4]). Then

$$p(n) = \sum_{r=0}^n (-1)^r \binom{n+1}{r+1} p_r(n),$$

where by the virtue of the Faà di Bruno's formula (6) with $f(q) = q^l$ we have

$$p_l(n) = \frac{1}{n!} \sum_{k=0}^l \binom{l}{k} k! B_{n,k}(\lambda_1, \dots, \lambda_{n-k+1}).$$

Proof. We start with the generating function for the partial Bell polynomials [1, Equation (3a') on p. 133] as follows

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k}(\lambda_1, \dots, \lambda_{n-k+1}) \frac{q^n}{n!} &= \frac{1}{k!} \left(\sum_{j=1}^{\infty} \lambda_j \frac{q^j}{j!} \right)^k \\ &= \frac{1}{k!} (E(q) - 1)^k \\ &= \frac{1}{k!} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} E(q)^r \\ &= \frac{1}{k!} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} \sum_{n=0}^{\infty} p_r(n) q^n \end{aligned}$$

to conclude that

$$B_{n,k}(\lambda_1, \dots, \lambda_{n-k+1}) = \frac{n!}{k!} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} p_r(n).$$

The above equation, together with the formula (2), gives us

$$\begin{aligned} p(n) &= \sum_{k=0}^n \sum_{r=0}^k (-1)^r \binom{k}{r} p_r(n) \\ &= \sum_{r=0}^n (-1)^r p_r(n) \sum_{k=r}^n \binom{k}{r} \\ &= \sum_{r=0}^n (-1)^r \binom{n+1}{r+1} p_r(n). \end{aligned}$$

□

Remark 1. Note that similar ideas are used in [6] with relation to partition zeta functions.

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