QUANTUM INFORMATION, ENTANGLEMENT AND ENTROPY

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Abstract

In this review I had given an introduction to axiomatic Quantum Mechanics, Quantum Information and its measure entropy. Also, I had given an introduction to Entanglement and its measure as the minimum of relative quantum entropy of a density matrix of a Hilbert space with respect to all separable density matrices of the same Hilbert space. I also discussed geodesic in the space of density matrices, their orthogonality and Pythagorean theorem of density matrix.

Postulates of Quantum Mechanics

The basic postulate of quantum mechanics is about the Hilbert space formalism:

- **Postulates 0:** To each quantum mechanical system, has a complex Hilbert space $\mathbb{H}$ associated to it:

Set of all pure state, $\mathbb{H}/\sim := \{[f] | [f] := \{h: h \sim f\}\}$ and $h \sim f \Leftrightarrow \exists z \in \mathbb{C}$ such that $h = z f$ where z is called phase

- **Postulates 1:** The physical states of a quantum mechanical system are described by statistical operators acting on the Hilbert space.

A **density matrix** or **statistical operator**, ρ is a positive operator of trace 1 on the Hilbert space. Where ρ is called positive if $\langle x, \rho x \rangle \geq 0$ for all $x \in \mathbb{H}$ and $\langle *, * \rangle$.

- **Postulates 2:** The observables of a quantum mechanical system are described by self-adjoint operators acting on the Hilbert space.

A self-adjoint operator A on a Hilbert space $\mathbb{H}$ is a linear operator $A: \mathbb{H} \rightarrow \mathbb{H}$ which satisfies

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

for $x, y \in \mathbb{H}$.

Lemma: The density matrices acting on a Hilbert space form a convex set whose extreme points are the pure states.

Proof. Denote by Σ the set of density matrices. It is obvious that a convex combination of density matrices is positive and of trace one. Therefore, Σ is a convex set.

Lemma: Let

$$\rho = \sum_{e \in \mathbb{B}} |x_e\rangle\langle x_e| = \sum_{h \in \mathbb{B}} |y_h\rangle\langle y_h|$$

be decompositions of a density matrix and where $\mathbb{B}$ is the basis of Hilbert space, $\mathbb{H}$. Then there exists a unitary matrix $(U_{eh})_{eh \in \mathbb{B}}$ such that

$$\sum_{e \in \mathbb{B}} U_{eh} |x_e\rangle = |y_h\rangle$$

Where,

$$U_{eh} := \frac{\langle x_e | y_h \rangle}{\sqrt{\langle x_e | x_h \rangle}}$$

- **Postulates 3:** Let $\mathcal{X}$ be a finite set and for $x \in \mathcal{X}$ an operator $O_x \in \mathcal{B}(\mathbb{H})$ be given such that

$$\sum_{x \in \mathcal{X}} O_x^* O_x = id_{\mathbb{H}}$$

where O_x^* is conjugate of O_x . Such an indexed family of operators is a model of a measurement with values in $\mathcal{X}$. If the measurement is performed in a state ρ , then the outcome $x \in \mathcal{X}$ appears with probability $\text{tr}(O_x^* \rho O_x)$ and after the measurement the state of the system is

$$\frac{O_x^* \rho O_x}{\text{tr}(O_x^* \rho O_x)}$$

If $\varphi : \mathcal{B}(\mathbb{H}) \rightarrow \mathbb{C}$ is a linear functional such that $\varphi(A) \geq 0$ if A is positive and $\varphi(id) = 1$, Then there exists a density matrix ρ_φ such that,

$$\varphi(A) = \text{tr}(\rho_\varphi A)$$

The functional φ associates the expectation value to the observables A .

The density matrices ρ and ρ' are called **orthogonal** if any eigenvector of ρ is orthogonal to any eigenvector of ρ' .

Let $\mathbb{B}$ be an orthonormal basis in a Hilbert space $\mathbb{H}$. The unit vector $\xi \in \mathbb{H}$ is **complementary** to the given basis if

$$|\langle e, \xi \rangle| = \frac{1}{\sqrt{\dim \mathbb{H}}}$$

For all $e \in \mathbb{B}$, where $\dim \mathbb{H}$ is dimension of $\mathbb{H}$.

Two orthonormal bases are called complementary if all vectors in the first basis are complementary to the other basis.

- **Postulates 4**: The composite system is described by the tensor product Hilbert space

$$\bigotimes_{\vartheta} \mathbb{H}_{\vartheta}$$

Given a density matrix ρ on $\bigotimes_{\vartheta} \mathbb{H}_{\vartheta}$ there are density matrices $\rho_{\vartheta} \in \mathcal{B}(\mathbb{H}_{\vartheta})$ such that

$$\text{tr} \left(\rho \bigotimes_{\gamma} \mathcal{A}_{\gamma}(\vartheta) \right) = \text{tr}(\rho_{\vartheta} A_{\vartheta})$$

Where,

$$\mathcal{A}_{\gamma}(\vartheta) := \begin{cases} A_{\vartheta}, \gamma = \vartheta \\ id_{\mathbb{H}_{\vartheta}}, \gamma \neq \vartheta \end{cases}$$

and $A_{\vartheta} \in \mathcal{B}(\mathbb{H}_{\vartheta})$, here ρ_{ϑ} is called **reduced density matrices**.

Let,

$$\rho = \bigotimes_{\vartheta} \rho_{\vartheta}$$

Then we define $\text{tr}_{\gamma}(\rho)$ as follows:

$$\text{tr}_{\gamma}(\rho) := \text{tr}(\rho_{\gamma}) \bigotimes_{\vartheta \neq \gamma} \rho_{\vartheta}$$

Information and its Measures

- **Shannon entropy**: In his revolutionary paper Shannon proposed a statistical approach and he posed the problem in the following way: “Suppose we have a set of possible events whose probabilities of occurrence are $p_1, p_2, \dots, p_n$. These probabilities are known but that is all we know concerning which event will occur. Can we find a measure of how much “choice” is involved in the selection of the event or how uncertain we are of the outcome?” Denoting such a measure by $H(p_1, p_2, \dots, p_n)$, he listed three very reasonable requirements which should be satisfied, those three postulates are as follows:
 - a. **Continuity**: $H(p, 1 - p)$ is a continuous function of p .
 - b. **Symmetry**: $H(p_1, p_2, \dots, p_n)$, is a symmetric function of its variables.

- c. **Recursion:** For every $0 \leq \lambda < 1$ the recursion $H(p_1, \dots, \lambda p_r, \dots, (1 - \lambda)p_r, \dots, p_n) = H(p_1, p_2, \dots, p_r, \dots, p_n) + p_r H(\lambda, 1 - \lambda)$, holds

According to him there is only one function satisfying all this postulate is:

$$H(p_1, p_2, \dots, p_r, \dots, p_n) = -\kappa \sum_{i=1}^n p_i \ln(p_i)$$

Let $(X_i)_{i=1}^n$ be random variables with values in the set $\mathcal{X} := \prod_{i=1}^n \mathcal{X}_i$. The following notation will be used.

- $p(x_i)_{i=1}^n$ is probability of $(X_i)_{i=1}^n$ at $(x_i)_{i=1}^n$
- $p(x_i | x_j)$ is probability of X_i at x_i knowing the probability of X_j at x_j

Then,

$$H(X_i)_{i=1}^n := - \sum_{(x_i)_{i=1}^n \in \mathcal{X}} p(x_i)_{i=1}^n \ln p(x_i)_{i=1}^n$$

Where if $n = 2$

$$H(X_i, X_j) := - \sum_{x_i \in \mathcal{X}_i} \sum_{x_j \in \mathcal{X}_j} p(x_i, x_j) \ln p(x_i, x_j)$$

and

$$p(x_i)_{i=1}^n = p(x_2)p(x_1 | x_2, x_3, \dots, x_n)$$

Which leads to:

$$p(x_i)_{i=1}^n = \frac{p\left(\left(\left((x_1 | x_2) | x_3\right) | \dots\right) | x_n\right)}{p(x_1)} \prod_{i=1}^n p(x_2)$$

Theorem: If $(X_i)_{i=1}^n$ are random variables of finite range, then

$$H(X_i)_{i=1}^n \leq \sum_{i=1}^n H(X_i)$$

Proof:

$$H(X_i)_{i=1}^n := - \sum_{(x_i)_{i=1}^n \in \mathcal{X}} p(x_i)_{i=1}^n \ln p(x_i)_{i=1}^n$$

And

$$\sum_{i=1}^n H(X_i) := - \sum_{(x_i)_{i=1}^n \in \mathcal{X}} \prod_{i=1}^n p(x_i) \ln \prod_{i=1}^n p(x_i)$$

And

$$\prod_{i=1}^n p(x_i) \leq p(x_i)_{i=1}^n$$

As

$$p(x_1) \leq p\left(\left(\left((x_1|x_2)|x_3\right)|\dots\right)|x_n\right)$$

Hence,

$$H(X_i)_{i=1}^n \leq \sum_{i=1}^n H(X_i)$$

Fano's inequality: Let X and Y be random variables such that their range is in a set of cardinality d and let $p := \text{Prob}(X \neq Y)$. Then

$$H(X|Y) \leq p \log(d-1) + H(p, 1-p)$$

- **von Neumann Entropy:** Let ρ be density matrix of a quantum system then von Neumann Entropy of that quantum system:

$$S(\rho) := -\kappa \text{tr}(\rho \ln \rho)$$

Theorem: Let ρ and σ be densities on a d -dimensional Hilbert space and let

$$p := \frac{\|\rho - \sigma\|_1}{2}$$

where $\|\cdot\|_1: \mathcal{B}(\mathbb{H}) \rightarrow \mathbb{R}_0^+$, is the norm on operator space $\mathcal{B}(\mathbb{H})$ of Hilbert space $\mathbb{H}$. Then,

$$|S(\rho) - S(\sigma)| \leq p \log(d-1) + H(p, 1-p)$$

Holds.

- **Quantum Relative Entropy:**

The relative entropy, or I-divergence of the probability distributions $p(x)$ and $q(x)$, is defined as

$$D(p||q) := \int_{-\infty}^{\infty} p(x) \ln \frac{p(x)}{q(x)} dx$$

Assume that ρ and σ are density matrices on a Hilbert space $\mathbb{H}$, then

$$S(\rho||\sigma) := \begin{cases} \text{tr } \rho(\ln \rho - \ln \sigma) & \text{if } \text{supp}(\rho) \leq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}$$

Hilbert–Schmidt inner product

$$\Delta a = \rho a \sigma^{-1}$$

For all $a \in \mathcal{B}(\mathbb{H})$

$$\text{tr } \rho(\ln \rho - \ln \sigma) = -\langle \rho^{\frac{1}{2}}, (\ln \Delta) \rho^{\frac{1}{2}} \rangle$$

where, $\langle A, B \rangle := \text{tr } (A^* B)$ and A^* is conjugate of A

$$\text{tr } \rho(\ln \rho - \ln \sigma) = -\text{tr} \left(\left(\rho^{\frac{1}{2}} \right)^* \left(\rho^{\frac{1}{2}} \right) (\ln \Delta) \right)$$

Let $\rho \equiv e^H$, ω and σ be three invertible densities. The e – *geodesic* connecting ρ and σ is the curve

$$\gamma_e(t) := \frac{e^{H+tA}}{\text{tr}(e^{H+tA})}$$

for all $t \in [0,1]$, where $A := \ln \sigma - \ln \rho$. Then, $\gamma_e(0) = \rho$ and $\gamma_e(1) = \sigma$. The m – *geodesic* connecting ρ and ω is the curve.

$$\gamma_m(t) := \rho + tB$$

for all $t \in [0,1]$ where $B := \omega - \rho$. Then $\gamma_m(0) = \rho$ and $\gamma_m(1) = \omega$

Assume that the e – *geodesic* connecting ρ and σ is orthogonal to the m – *geodesic* connecting ρ and ω with respect to the inner product

$$\langle E, F \rangle_\rho := \int_0^\infty \text{tr} ((s\mathbb{I} + \rho)^{-1} E^* (s\mathbb{I} + \rho)^{-1} F) ds$$

A plain computation yields

$$S(\omega||\rho) + S(\rho||\sigma) - S(\omega||\sigma) = \text{tr } (AB) = \langle A, B \rangle_{HS}$$

$$T_\rho : X \mapsto \int_0^\infty (s\mathbb{I} + \rho)^{-1} X (s\mathbb{I} + \rho)^{-1} ds$$

Then According Dénes Petz

$$\langle X, Y \rangle_{HS} = \langle T_\rho(X), Y \rangle_\rho$$

And,

$$T_\rho(A) = \dot{\gamma}_e(0)$$

It is obvious that:

$$B = \dot{\gamma}_m(0)$$

Therefore

$$\langle A, B \rangle_{HS} = \langle T_\rho(A), B \rangle_\rho = \langle \dot{\gamma}_e(0), \dot{\gamma}_m(0) \rangle_\rho$$

and we can conclude that if our assumption holds then that implies:

$$\langle \dot{\gamma}_e(0), \dot{\gamma}_m(0) \rangle_\rho = 0 \Rightarrow S(\omega||\rho) + S(\rho||\sigma) = S(\omega||\sigma)$$

This is some time called Pythagorean theorem of density matrix.

- **Renyi Entropy, Quantum Renyi Entropy and Quantum Relative Renyi Entropy:**

The Renyi entropy of order $\alpha \neq 1$ of the probability distribution $(p_1, p_2, \dots, p_n)$ is defined by

$$H_\alpha(p_1, p_2, \dots, p_n) = \frac{1}{1-\alpha} \ln \sum_{k=1}^n p_k^\alpha$$

The Quantum Renyi entropy of order $\alpha \neq 1$ of the density matrix ρ is defined by

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \ln \text{tr} (\rho^\alpha)$$

The Quantum Renyi entropy of order $\alpha \neq 1$ of the density matrix ρ with respect σ to is defined by

$$S_\alpha(\rho||\sigma) = \frac{1}{\alpha-1} \ln \text{tr} (\rho^\alpha \sigma^{1-\alpha})$$

Entanglement

Let $\mathcal{B}(\mathbb{H}_A)$ and $\mathcal{B}(\mathbb{H}_B)$ be the algebras of bounded operators acting on the Hilbert spaces $\mathbb{H}_A$ and $\mathbb{H}_B$. The Hilbert space of the composite system is $\mathbb{H}_{AB} := \mathbb{H}_A \otimes \mathbb{H}_B$. The algebra of the operators acting on $\mathbb{H}_{AB}$ is $\mathcal{B}(\mathbb{H}_{AB}) := \mathcal{B}(\mathbb{H}_A) \otimes \mathcal{B}(\mathbb{H}_B)$.

In the vector space $\mathcal{B}(\mathbb{H})$ the standard positive cone is the set of all positive matrices. This cone induces the partial ordering.

$$A \leq B \Leftrightarrow \langle \eta, A\eta \rangle \leq \langle \eta, B\eta \rangle \text{ for every vector } \eta$$

In the product space $\mathcal{B}(\mathbb{H}_{AB}) := \mathcal{B}(\mathbb{H}_A) \otimes \mathcal{B}(\mathbb{H}_B)$, we have two natural positive cones, $\mathcal{B}(\mathbb{H}_{AB})^+$ consists of the positive matrices acting on $\mathbb{H}_{AB} := \mathbb{H}_A \otimes \mathbb{H}_B$, and the cone $\mathcal{S}$ consists of all operators of the form

$$\sum_i A_i \otimes B_i$$

Where $A_i \in \mathcal{B}(\mathbb{H}_A)^+$ and $B_i \in \mathcal{B}(\mathbb{H}_B)^+$. It is obvious that $\mathcal{S} \subset \mathcal{B}(\mathbb{H}_{AB})^+$. A state is called separable (or unentangled) if its density belongs to $\mathcal{S}$. And if not then they are called entangled states.

pure states are extreme points in the state space. If a pure state is a convex combination $\sum_i p_i P_i \otimes B_i$ of product pure states, then this convex combination must be trivial, that is, $P \otimes Q$.

Lemma: Any unit vector $\Psi \in \mathbb{H}_{AB}$ can be written in the form

$$\Psi \in \sum_k \sqrt{p_k} g_k \otimes h_k$$

where the vectors $g_k \in \mathbb{H}_A$ and $h_k \in \mathbb{H}_B$ are pairwise orthogonal and normalized; moreover (p_k) is a probability distribution. This expansion of Ψ is called Schmidt decomposition.

Let $\dim \mathbb{H}_A = \dim \mathbb{H}_B = n$. A pure state $|\Phi\rangle\langle\Phi|$ on the Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$ is called maximally entangled if the following equivalent conditions hold:

- The reduced densities are maximally mixed states.
- When the vector $|\Phi\rangle$ is written in the form, then $p_k = n^{-1}$ for every $1 \leq k \leq n$.
- There is a product basis such that $|\Phi\rangle$ is complementary to it.

The density matrix of a maximally entangled state on $\mathbb{C}^n \otimes \mathbb{C}^n$ is of the form

$$\rho = \frac{1}{\sqrt{n}} \sum_{\mu\nu} e_{\mu\nu} \otimes e_{\mu\nu}$$

- **Entanglement Measures:**

The degree of entanglement of a state ρ_{AB} on the bipartite Hilbert space $\mathbb{H}_{AB} := \mathbb{H}_A \otimes \mathbb{H}_B$ is its distance from the convex set of separable states. One possibility is to use the relative entropy as a distance function. In this way, we can arrive at the concept of the relative entropy of entanglement:

$$E_{RE}(\rho_{AB}) := \inf \{ S(\rho_{AB} || D) : D \in \mathcal{S} \}$$

where D runs over the set of separable states on $\mathbb{H}_{AB} := \mathbb{H}_A \otimes \mathbb{H}_B$. If ρ_{AB} is faithful, then the minimizer is unique and can be called “the best separable approximation of ρ_{AB} ”.

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