# A Fuzzy Set Extension of Schelling's Spatial Segregation Model

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**Abstract**

This study explores a possible segregation mechanism assuming fuzzy group membership. We construct a fuzzy set extension of Schelling's spatial segregation model. In the fuzzy Schelling model, each agent is assumed to have fuzzy membership in groups, which is typically assumed to represent the strength of the agent's group identity. The degree of membership is represented by the value of the membership function. The model assumes that agents want to be with agents with the same or stronger (less fuzzy) group identity than themselves. Agents decide whether to stay or move depending on whether their neighborhood satisfies their desires. Analyzing a series of simulations reveals that: First, the fuzzy Schelling model can reproduce segregation at the macro level; here, segregation is formed by the accumulation of agents' modest desires and actions. This is the most important property of the Schelling model. Second, agents' behavior and situation differ depending on the fuzziness of their membership. Notably, agents with less fuzzy membership play an important role in the system's equilibrium. Third, the tendency to reach equilibrium differs depending on the density of the space, required similarity level in the neighborhood, and initial distribution of membership values. Finally, we discuss the implications of the results.

**Keywords:** segregation, Schelling model, fuzzy set, group identity

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# 1 Introduction

Since Schelling introduced a checkerboard model of spatial segregation [16, 17], this model has attracted extensive research attention. It has not only inspired studies directly related to empirical residential segregation phenomena [2] but also those trying to theoretically understand the model as evolutionary game [25, 27, 28] or pure physical processes [22].

The Schelling model demonstrates by simulation that members of two initially randomly scattered groups become distinctly segregated at the macro level as they repeatedly move. This is driven by their modest desire to be with at least some neighbors who are of the same group as themselves.

The Schelling model derives macro-level emergent properties (i.e., residential segregation) by applying simple assumptions to agents' utility and action choices. In this sense, the model is a pioneering and good example of an agent-based model. Based on the Schelling model, various extended models with certain characteristics have been proposed in several fields, and especially in social and computer sciences [1, 4, 5, 7, 8, 12, 18, 21, 24, 29].

Here, we extend the Schelling model by incorporating a new feature: fuzzy membership in groups, represented by fuzzy sets.

The simplest Schelling model assumes two distinct groups (e.g., blacks and whites; and surfers and bathers). The membership in each group can be mathematically represented by a characteristic function for a conventional set (crisp set). For example, for two sets $A$ and $B$ where all people are members of one of them and there is no overlap, membership in $A$ and $B$ can be represented by the characteristic functions $\varphi_A$ and $\varphi_B$, respectively as follows:

$$\varphi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad \varphi_B(x) = 1 - \varphi_A(x). \tag{1}$$

However, in reality, there are limited cases in which all individuals have clearly defined memberships at all times. Rather, the definitions or boundaries of groups are typically ambiguous, or the criteria for group membership vary from individual to individual. For example, membership in a racial or ethnic group leaves a gray area in terms of both the individual's self-perception and others' perception depending on the race and ethnicity of one's parents; gradations of appearance such as skin, eye, or hair color; and differences in language and cultural background. This is true even for seemingly clear and distinct groups, such as the Japanese [10, 23].

As a general model that allows for such ambiguous group membership, this study attempts to extend the Schelling model by adopting the fuzzy set framework, first proposed by Zadeh [26]; in particular, we consider the idea of a membership function. In this fuzzy set framework, the membership function of fuzzy set A, denoted as $\mu_A(x)$, is defined as a function that maps an element $x$ in the universe $X$ to a real value in the interval from 0 to 1; that is, $\mu_A : X \to [0, 1]$, representing the degree of membership of $x$ in set $A$, where 1 is full membership, 0 is complete non-membership, and 0.5 represents the midpoint that signifies neither full nor a non-membership. Thus, by extending the Schelling model with fuzzy membership functions, we can construct a general residential segregation model that allows for fuzzy membership.

This extension allows us to understand how segregation can occur when people have different group identities. This can help us in understanding some of the mechanisms of historical events, such as those in Croatia and Rwanda, where ethnic boundaries that were originally not clearly perceived were brought to the fore by agitation and collective fear, leading to violent ethnic conflict [6, 9].

Hereafter, for simplicity, we call the fuzzy set extension of the Schelling model as the "fuzzy Schelling model." The remainder of this article proceeds as follows. We first describe the assumptions of the fuzzy Schelling model in Section 2, and then present the results of a series of model simulations in Section 3. Finally, we discuss the implications of our results in Section 4.

# 2 Model

## 2.1 Model assumptions

In the fuzzy Schelling model, as in the conventional Schelling model, each cell of the grid in the two-dimensional space is assumed to be a residence. A finite number of agents are assumed to be distributed in the space and occupy one of the cells. In the actual simulation, the space is assumed to be a torus, which is a space with periodic boundary conditions.

Agents are assumed to identify with two groups, $A$ and $B$. The degree of identification of agent $x$ with group $A$ is represented by membership function $\mu_A(x) \in [0, 1]$. As a baseline assumption, the degree of identification with the group $B$ is represented by $\mu_B(x) = 1 - \mu_A(x)$, which means that $B$ is the complement of $A$ in a fuzzy theory framework [26].

Regarding agents' desires, it is assumed that agents want to be with agents with the same or stronger (less fuzzy) group identity than themselves. This can be understood as a way of implementing people's tendency to conform to the opinions of prototype group members, as noted in self-categorization theory [13, 14]. Agents decide whether to stay or move depending on whether their neighborhood satisfies their desires.

Let $N(x)$ be the set of neighborhoods of $x$; that is, the set of agents residing in the cells adjacent to the cell of $x$. As a baseline assumption, we use the Moore neighborhood, which is the set of agents in the eight cells surrounding cell $x$.

Let $p \in [0, 1]$ be the required similarity rate, which is an exogenous parameter that determines the level of neighborhood similarity required by agents. Agents stay in neighborhoods where the actual similarity rate for a group aligning with their identity is equal to or greater than $p$; otherwise, they move randomly to an open space.

Specifically, for $x$ who has an identity in the direction of group $A$, or $x$ such that $\mu_A(x) > 0.5$, if

$$\frac{\#\{y \in N(x) | \mu_A(y) \geq \mu_A(x)\}}{\#N(x)} \geq p \tag{2}$$

holds, then $x$ stays in the current position; otherwise $x$ moves, where $\#S$ is the number of elements in set $S$. Conversely, for $x$ such that $\mu_A(x) < 0.5$, or $x$ who has an identity in the direction of group $B$, if

$$\frac{\#\{y \in N(x) | \mu_A(y) \leq \mu_A(x)\}}{\#N(x)} \geq p \tag{3}$$

holds, then $x$ stays; otherwise $x$ moves. For $x$ such that $\mu_A(x) = 0.5$, or $x$ who has indifferent identity between groups $A$ and $B$, $x$ stays as long as there is at least one agent in the neighborhood. That is, $\#N(x) \geq 1$. For all types of agents, $x$ is assumed to move if there is no one in the neighborhood; that is, $\#N(x) = 0$.

Figure 1 shows an example of the moving decision for $x$ such that $\mu_A(x) = 0.6$ with the required similarity rate $p = 0.5$. In the left case, $x$ stays in the current position because the actual similarity rate in the neighborhood is 0.5, which is equal to $p$. In the right case, $x$ moves to the open space because the actual similarity rate in the neighborhood is 0.33, which is less than $p$.

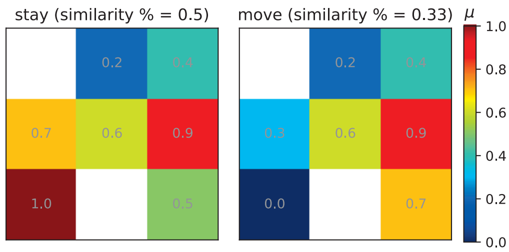


**Fig. 1** Moving decision for $x$ such that $\mu_A(x) = 0.6$ and $p = 0.5$.

## 2.2 Simulation conditions

In the simulation, the exogenous conditional variables are the size of the space, density of agents in the space, denoted by $d$, and required similarity rate $p$. Given the size and density of the space, the actual number of agents is determined.

In addition, the results of the simulation depend on the distribution of the values of the membership function for group $A$ (or $B$) of the agents. We assume that the distribution follows a beta distribution. That is, $\mu_A \sim \mathrm{Beta}(\alpha, \beta)$, where the two parameters of the beta distribution $\alpha$ and $\beta$ are used to represent differences in the shape of the distribution. Here, we consider the following three as typical cases (Figure 2):

- uniform: $\alpha = 1, \beta = 1$
- unimodal: $\alpha = 2, \beta = 2$
- bimodal: $\alpha = 0.5, \beta = 0.5$

The uniform distribution is a case in which agents are randomly distributed with respect to their group identity; this is the baseline and benchmark assumption. In the

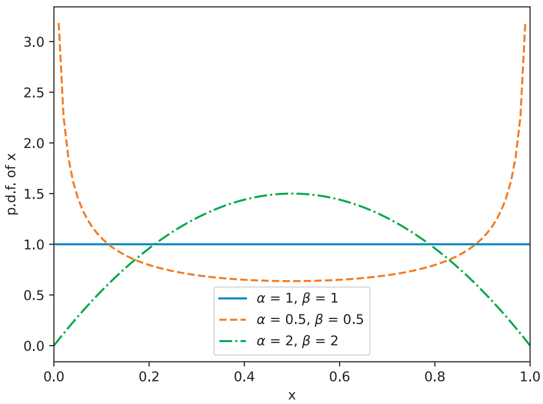


**Fig. 2** Three cases of beta distribution

bimodal distribution, agents are divided into two groups with strong identities. In the unimodal distribution, the majority of agents have an intermediate identity between $A$ and $B$, or weak identities.

In the actual simulation, the membership values vary from 0 to 1 in increments of 0.1 for simplicity.

## 2.3 Measurements

The following three measures are used to determine the current state of the agents during the simulation process.

First, the most basic measure is the percentage of agents who are not satisfied with the current situation in their neighborhood and are willing to move, simply called "percent unhappy." When the percent unhappy is zero, no one is moving and the situation is in equilibrium.

Second, the neighbor similarity of an agent $x$, denoted by $s(x)$, is defined as follows:

$$s(x) = 1 - \frac{1}{\#N(x)} \sum_{y \in N(x)} (\mu_A(y) - \mu_A(x))^2. \tag{4}$$

$s(x)$ ranges from 0 to 1, indicating that the similarity between $x$ and its neighbors is low (high) when $s(x)$ is close to 0 (1).

Third, the fuzziness of the group identity is considered. An agent $x$'s membership in group $A$ or $B$ is the most distinct when the membership value is 0 or 1, and is the most fuzzy when the membership value is 0.5. Therefore, the fuzziness of the group identity of an agent $x$ can be measured by the binary entropy of $\mu_A(x)$ (or equivalently $\mu_B(x) = 1 - \mu_A(x)$). The binary entropy of $p$, denoted by $h(p)$, is defined as follows:

$$h(p) = -p \log_2 p - (1-p) \log_2 (1-p). \tag{5}$$

The fuzziness of a set can be measured using the mean of the binary entropy of the elements in the universe [3, 19]. Then, the fuzziness of the group identity among $x$'s neighbors, denoted by $\phi(x)$, is defined by the following:

$$\phi(x) = \frac{1}{\#N(x)+1} \sum_{z \in N(x)\cup\{x\}} h(\mu_A(z)), \tag{6}$$

ranging from 0 as the most distinct to 1 as the most fuzzy.

# 3 Results

## 3.1 A uniform distribution

Here, the result of the simulation with a uniform distribution is used as an example. The size of the space is assumed to be $50 \times 50$, density of agents $d$ is assumed to be 0.7, and required similarity rate $p$ is assumed to be 0.3. The initial distribution of the agents' membership value for $A$ is assumed to be uniform, where $\alpha = 1$ and $\beta = 1$. The simulation is run for up to 200 time steps or until equilibrium is reached.

In total, the simulation actually ends in 147 steps. Figure 3 shows the initial and final distribution of membership value $\mu_A(x)$. Each different color represents a different membership value for group $A$. The animation of the simulation is provided as a supplement file.

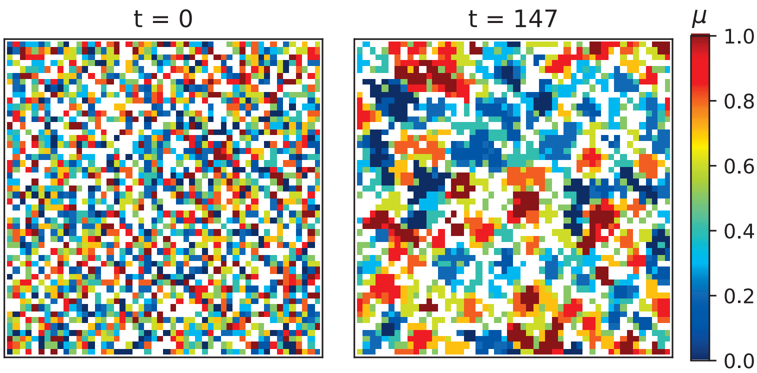


**Fig. 3** Initial and final distribution of $\mu$ for the simulation with uniform distribution ($\alpha = 1, \beta = 1, d = 0.7$, and $p = 0.3$)

In the initial state, the agents are randomly distributed. However, in the final state, the agents are clustered by color because of their modest preference, which requires at least 30% of neighbors to have a similar or stronger group identity compared to themselves.

A closer look at the equilibrium state at the final point reveals that the colors are scattered and locally concentrated rather than concentrated in one or two places. However, they form large clusters of warm or cold tonal colors that indicate the same identity direction. There are also cases where clusters that strongly identified with group $A$ (membership value is around 1) and group $B$ (membership value is around 0), respectively, were adjacent to each other.

Figure 4 shows the initial and final distributions of the fuzziness of the neighborhood $\phi(x)$ of the simulation. In the equilibrium state at the final time point, several hotspots of low fuzziness are observed, corresponding to proximity clusters with strong identities.

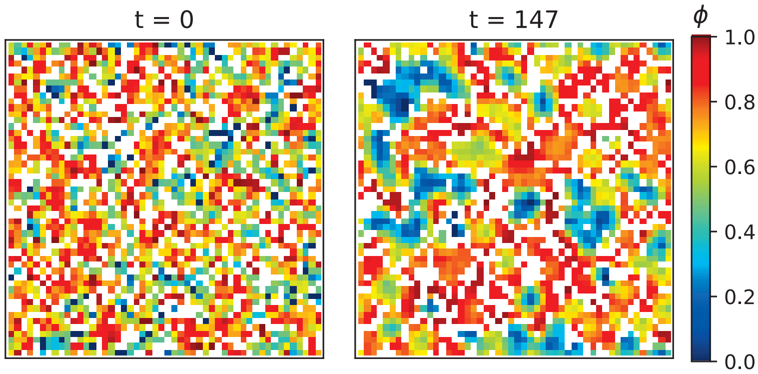


**Fig. 4** Initial and final distribution of fuzziness for the simulation with uniform distribution ($\alpha = 1, \beta = 1, d = 0.7$, and $p = 0.3$)

To analyze how the states of the agents change at each step and reach equilibrium, we classify the agents according to the entropy value of their membership values and observe the change in percent unhappy, average similarity, and average fuzziness. Here, we can assume that the lower the entropy value, the stronger the group identity. The correspondence between membership and entropy values is as follows: $h(0) = h(1) = 0.000, h(0.1) = h(0.9) = 0.469, h(0.2) = h(0.8) = 0.722, h(0.3) = h(0.7) = 0.881, h(0.4) = h(0.6) = 0.970$, and $h(0.5) = 1.000$.

First, we examine the change in percent unhappy (Figure 5). In the initial state, the lower the entropy value, the lower the percentage of unhappy agents and the faster it decreases afterwards. Even after agents with high entropy (i.e., neutral or not strong identity) settle down early, agents with low entropy (strong identity) tend to wander for longer in search of a place that meets their requirements.

Figure 6 shows the change in average similarity. The average similarity in the initial state varies depending on the entropy value. The group with the highest entropy with a membership value of 0.5 has the highest tolerance for heterogeneity and settles quickly so that the average similarity remains almost unchanged throughout. The other groups show increasing similarity toward equilibrium. However, the group with the lowest entropy values with membership values of 0 and 1 has smaller values than the other groups with intermediate entropy values. This suggests that agents with the strongest identities tend to live in areas with not very high purity of group identity. This finding is interesting in terms of its correspondence with the empirical evidence. In this extreme case, there are areas where agents with membership values of 0 and

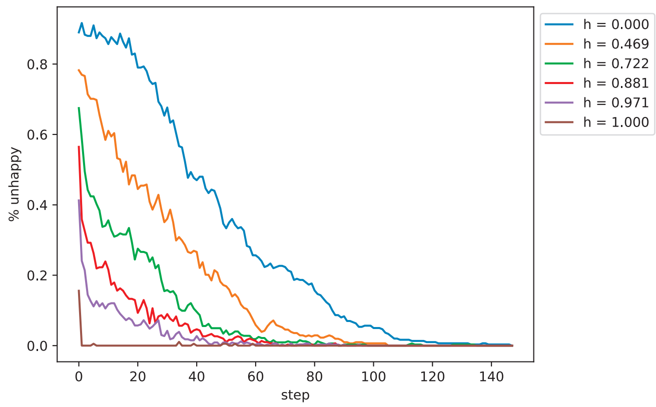


**Fig. 5** Change in percent unhappy for each entropy value of the simulation with uniform distribution ($\alpha = 1, \beta = 1, d = 0.7$, and $p = 0.3$)

1 live next to each other, as seen in Figure 3. This indicates that the micro-level desire of people to have neighbors with the same or stronger identities as themselves paradoxically results in neighbors with sharply different identities at the macro-level.

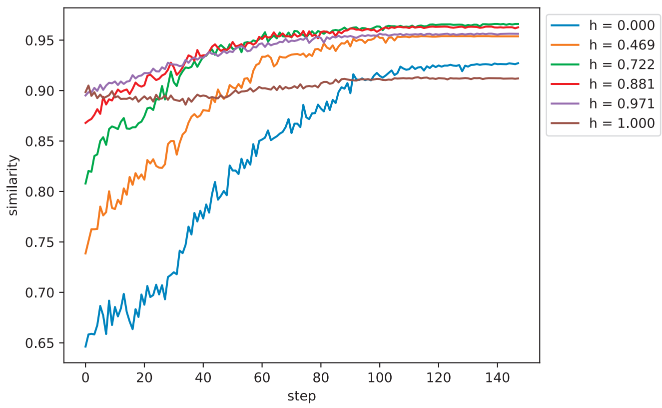


**Fig. 6** Change in average similarity for each entropy value of the simulation with uniform distribution ($\alpha = 1, \beta = 1, d = 0.7$, and $p = 0.3$)

Finally, we check the change in fuzziness in Figure 7. When moving from the initial state to the next step, the average fuzziness corresponding to each entropy value changes substantially. This is mainly because of cases with no or few neighbors in the randomly distributed initial state. After the next step, the average fuzziness decreases for the low entropy values $h = 0.000$ and $0.469$ but increases slowly for other values. Interestingly, the fuzziness of the agent with the strongest group identity with an entropy value of zero decreases significantly. This indicates that the movement and reallocation based on the modest desires of individual agents increases the clarity and distinctiveness of the group identity of the neighborhood of the agents with the strongest identity. This corresponds to a situation where strong identities are adjacent to each other, as shown earlier.

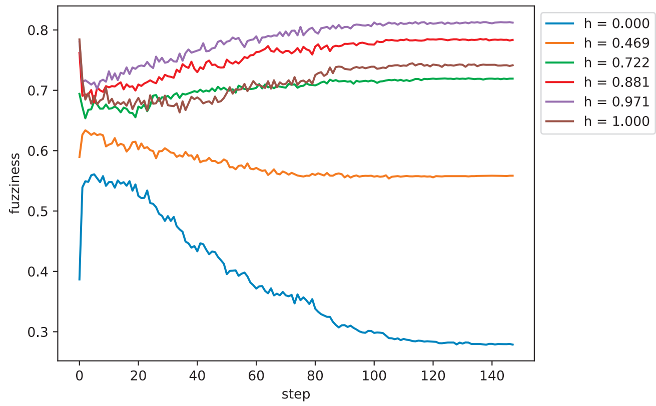


**Fig. 7** Change in average fuzziness for each entropy value of the simulation with uniform distribution ($\alpha = 1, \beta = 1, d = 0.7$, and $p = 0.3$)

## 3.2 Results of different settings

Next, we examine how the change in the three measures of the simulation vary with different initial conditions for the parameters of the distribution $\alpha$ and $\beta$, density $d$, and required similarity rate $p$. The distribution types are uniform ($\alpha = 1, \beta = 1$), unimodal ($\alpha = 2, \beta = 2$), and bimodal ($\alpha = 0.5, \beta = 0.5$). The density is $d = 0.3, 0.5$, and $0.7$. The required similarity is $p = 0.3, 0.5$, and $0.7$. The size of the space is $50 \times 50$. The simulation is run for up to 200 time steps or until equilibrium is reached.

Figure 8 shows the change in the three measures for a uniform distribution with different densities $d$ and required similarity rates $p$. For the same density, as the required similarity rate increases, a group of agents with lower entropy values is less likely to settle. Thus, the simulation as a whole is less likely to reach equilibrium. Furthermore, for the same required similarity rate, as the density increases, it becomes more difficult to find better available space. Thus, equilibrium is reached more slowly or even harder to reach.

Figures 9 and 10 show the results for the unimodal and bimodal distributions, respectively. In general, the bimodal distribution reaches equilibrium faster than the unimodal and uniform distributions, even when the density and required similarity are the same. Meanwhile, the unimodal distribution is less likely to reach equilibrium than the bimodal and uniform distributions. This is because the unimodal distribution has few neighbors whose identities are the same or stronger than one's own. This makes it harder to find a suitable location. Meanwhile, in a bimodal distribution, there are many such neighbors, making it easier to find them. The Schelling's original model assumes a crisp set, which can be regarded as a perfect bimodal membership distribution, hence it reaches equilibrium faster than the fuzzy extension models.

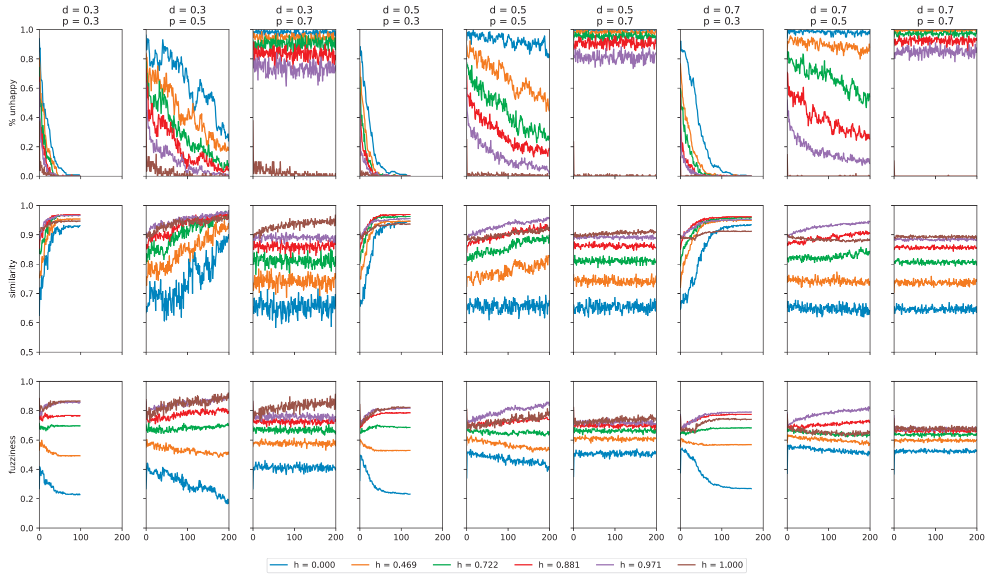


**Fig. 8** Change in the three measures for the uniform distribution ($\alpha = 1$ and $\beta = 1$) with different densities $d$ and required similarity rates $p$

# 4 Discussion

We introduce and examine the fuzzy Schelling model. Here, we discuss the implications of the simulation results. First, although the fuzzy Schelling model is more complex and less likely to reach equilibrium than the original crisp Schelling model, it reproduces the most important property of segregation at the macro level formed by the accumulation of agents' modest desires and actions.

The advantage of this extended model is that it shows that the behavior and neighborhood of agents leading to equilibrium differ depending on the fuzziness of their membership. In particular, agents with strong identities, represented by low entropy values, have more difficulty finding a suitable location, and therefore, wander for longer. They are forced to avoid high-entropy neighborhoods in which they have already settled. Consequently, even when they reach equilibrium, they settle in places with relatively low similarity and neighborhood fuzziness. Typically, this is a situation in which members of different groups with strong identities are adjacent to each other. This result predicts the case where strong identities are paradoxically adjacent and high tensions arise with highly diverse identity distributions. Overall, it reveals one potential mechanism for the emergence of ethnic conflict.

Furthermore, the tendency to reach equilibrium differs depending on the initial distribution of the membership values. In a unimodal distribution with many intermediate identities, the small number of agents with strong identities is a disturbing factor, and the system reaches equilibrium more slowly. However, in a bimodal distribution with many identities at both extremes, each agent is more likely to find a neighborhood that meets their requirements, resulting in a relatively fast equilibrium.

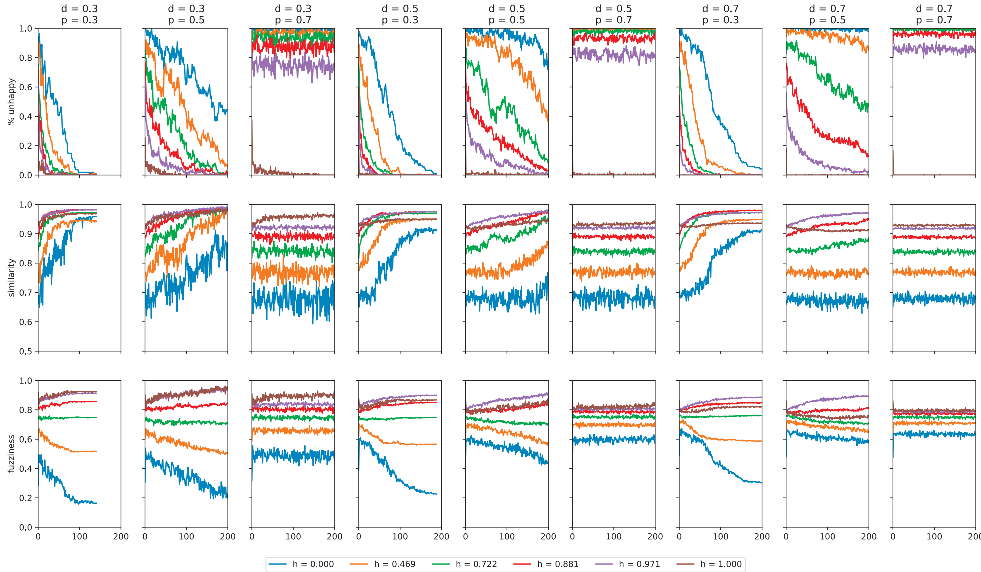

**Fig. 9** Change in the three measures for the unimodal distribution ($\alpha = 2$ and $\beta = 2$) with different densities $d$ and required similarity rates $p$

The ultimate situation is the original Schelling model with crisp sets. That is, the low fuzziness of the system makes it stable, whereas high fuzziness destabilizes it.

Finally, we discuss the further developments and potential applications of this model. This study introduces the basic idea of a fuzzy extension of the Schelling model under basic assumptions. Future works can extend this model to address a wider variety of social phenomena by adding further elements and assumptions. For example, the model assumes that the initial membership value does not change throughout the steps; future works can change this value depending on the neighborhood's situations.

Furthermore, the model can be applied not only to the actual issues of residential segregation but also to the issue of separation and coherence of opinions in an abstract opinion space, such as political polarization.

In recent years, political polarization has been observed in many democratic societies, and various social science studies have explored its mechanisms [11]. Some argue that the recent rapid development of the Internet and social networking services, and their algorithms has created a filter bubble that facilitates political polarization [15, 20]. Meanwhile, the fuzzy extension of the Schelling model can be used to examine a more general mechanism by which group identity or opinion is reinforced by a general tendency toward network homophily.

**Supplementary information.** The animation of the simulation in Section 3.1 is provided as Supplement1.mp4.

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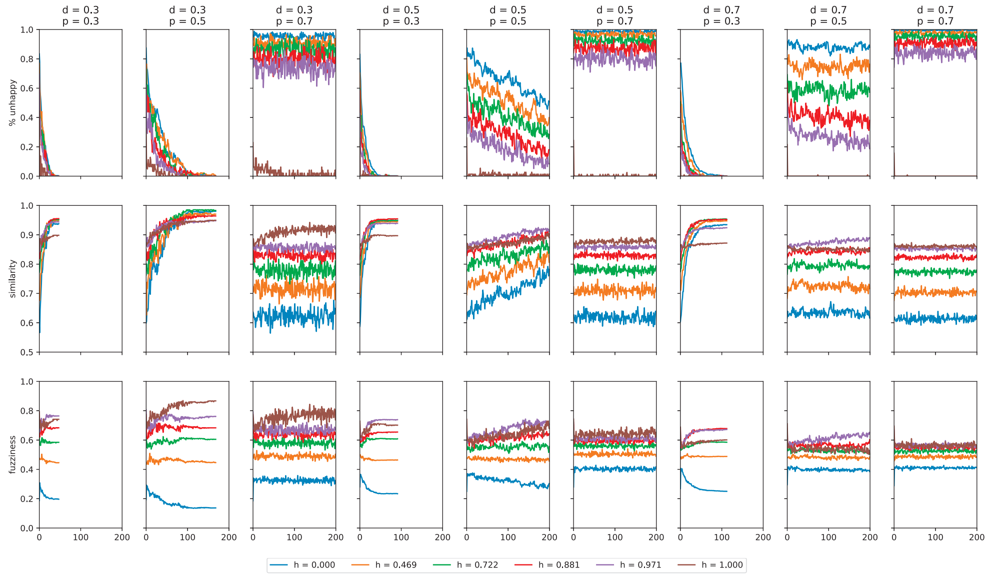


**Fig. 10** Change in the three measures for the bimodal distribution ($\alpha = 0.5$ and $\beta = 0.5$) with different densities $d$ and required similarity rates $p$

**Data availability.** This study does not use any empirical data. The python code for the simulation is available at the author's GitHub repository after acceptance.

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