Meagerness of Lipschitz Functions

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Abstract

We show that the class of Lipschitz real-valued functions on a compact set K in $\mathbb{R}$ is meager in $C(K)$ with respect to the supremum metric. As these Lipschitz functions are dense in $C(K)$, this result complements the classical surprising result for the meagerness of somewhere differentiable functions in $C([0, 1])$.

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Throughout, every function is taken as real-valued; every space of continuous functions on a compact space is equipped with the supremum metric. For reference: A Lipschitz function on a subset of $\mathbb{R}$ is by definition precisely a function f such that $|f(x) - f(y)| \leq M|x - y|$ for some real $M \geq 0$ and all x, y in the domain; a meager set in a topological space is by definition precisely a countable union of nowhere dense sets, i.e. a countable union of sets whose closure has empty interior.

It is well-known that somewhere differentiable functions in $C([0, 1])$ are “scarce” in the sense that they form a meager set, which is surprising in contrast with both the denseness of differentiable functions in $C([0, 1])$ and a belief known ([1]) to be once generally held.

Also unexpected is the Morgenstern’s result that somewhere (real-)analytic functions in $C^\infty([0, 1])$ form a meager set.

Since Lipschitz functions on a compact set K in $\mathbb{R}$ are dense in $C(K)$, it would be interesting to know if Lipschitz functions in $C(K)$ are also meager. The question is affirmed by the following

Theorem 1. *If $K \subset \mathbb{R}$ is compact, then the class of Lipschitz functions $K \rightarrow \mathbb{R}$ is meager in $C(K)$.*

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Proof. Fix any $x \in K$, and let

$$A_r := \{f \in C(K) \mid |f(x) - f(y)| \leq r|x - y| \text{ for all } y \in K\}$$

for all rational $r \geq 0$.

Denote by $C_L(K)$ the class of Lipschitz functions $K \rightarrow \mathbb{R}$. Evidently, if $f \in C_L(K)$ then $f \in A_r$ for some rational $r \geq 0$; in other words, we have

$$C_L(K) \subset \bigcup_{r \in \mathbb{Q} \cap [0, +\infty[} A_r.$$

Denote by d the supremum metric of $C(K)$. We claim that every A_r is closed in $C(K)$. Indeed, we have

$$A_r = \bigcap_{y \in K} \{f \in C(K) \mid |f(x) - f(y)| \leq r|x - y|\} =: \bigcap_{y \in K} A_{r,y}$$

for all r . If $f \in A_{r,y}^c$, so that $|f(x) - f(y)| > r|x - y|$, we choose

$$\varepsilon := \frac{1}{2}(|f(x) - f(y)| - r|x - y|).$$

Then, since $|g(x) - g(y)| > |f(x) - f(y)| - 2d(f, g)$ for all $g \in C(K)$, it holds that $d(f, g) < \varepsilon$ implies $g \in A_{r,y}^c$; the claim follows.

To show that every A_r is nowhere dense, let $f \in A_r$. If $\varepsilon > 0$, we consider the function

$$g : z \mapsto \left(\frac{\varepsilon}{\sup_{z \in K} z^2 + 1} \right) z^2 + f(z) =: az^2 + f(z)$$

on K . Then $d(f, g) < \varepsilon$. But since $|g(x) - g(y)| = |a(x - y)(x + y) + f(x) - f(y)|$ for all $y \in K$, if $g \in A_r$ then by triangle inequality the function $y \mapsto a|x + y|$ is majorized everywhere by a constant; so we have $g \in A_r^c$, and the desired nowhere denseness of A_r follows. Since a subset of a meager set is meager, the proof is complete. $\square$

Remark. The proof of Theorem 1 indeed asserts more: Since K is compact, upon choosing a countable dense subset K_0 of K we see that the set

$$\bigcup_{\substack{x \in K_0; \\ r \in \mathbb{Q} \cap [0, +\infty[}} \{f \in C(K) \mid |f(x) - f(y)| \leq r|x - y| \text{ for all } y \in K\}$$

is also meager in $C(K)$.

It would be informative to add that, for suitable choices of g (whenever possible), Theorem 1 is readily generalized for subclasses of the class of compact metric spaces. $\square$

For reference: a Baire space is by definition precisely a topological space where every countable intersection of open dense sets is dense. Then we have

Corollary 1. *If $K \subset \mathbb{R}$ is compact, and if $C_L(K)$ is the class of all Lipschitz functions $K \rightarrow \mathbb{R}$, then the interior of $C_L(K)$ in $C(K)$ is empty.*

Proof. Since $C(K)$ is a complete metric space, it follows that $C(K)$ is Baire by a celebrated (standard) result of Baire's. Upon observing that, in any given topological space, if A is nowhere dense then the complement $(\text{cl}(A))^c$ of the closure of A is dense, we see that every meager set in a Baire space has empty interior; this is also a result of Baire's. Therefore, as $C_L(K)$ is meager in $C(K)$ by Theorem 1, the desired conclusion is obtained. $\square$

References

- [1] S.G. Krantz and H.R. Parks, *The Geometry of Domains in Space*, Springer, 1999.