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[microreview]
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Counting with groups: a concise approach

Open Mathematics Collaboration*†

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Abstract

Some definitions and results for counting with groups are presented within the fewest words and symbols as possible.

keywords: group theory, abstract algebra, counting with groups

The most updated version of this paper is available at
https://osf.io/9hdwj/download

Introduction

1. The purpose of this *microreview* is to provide some basic results for **counting with groups** using the fewest words and symbols as possible, without losing information.

2. We believe this *concise pedagogical approach* can be a very useful resource while one is studying and trying to solve pure mathematical problems.

3. These papers [1–5] can be read jointly.

4. [6–8].

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Notations & Definitions

5. *Please check the full list of* ***notations*** *and* ***definitions*** [9] that we are using here.

6. One can easily *search* a *tag* by typing #tag.

Results by (collection of) tags

#action #color #orbit

7. (a) $(G \circlearrowleft (X = \{1, ..., n\})) \to (G \circlearrowleft \mathcal{C}, \forall g(c_1, ..., c_n))$
   (b) $\forall g : g(c_1, ..., c_n) = (c_{g1}, ..., c_{gn})$
   (c) $(c_1, ..., c_n) \in \mathcal{C}^n$
   (d) $\mathcal{O}(c_1, ..., c_n) = (q, G)$-coloring of X

#action #color #order #symmetric

8. $\tau \in S_n, \qquad F^{\mathcal{C}^n}(\tau)$ = number of elements in $\mathcal{C}^n$ fixed by τ
   $t(\tau)$ = number of cycles in the complete factorization of τ
   $N^X_{(q,G)}$ = number of (q, G)-colorings of X
   (a) $(F^{\mathcal{C}^n}(\tau) \wedge t(\tau)) \to (F^{\mathcal{C}^n}(\tau) = q^{t(\tau)})$
   (b) (finite $G \circlearrowleft (X = \{1, ..., n\})) \to \left(N^X_{(q,G)} = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}\right)$

#action #conjugation #orbit #order #stabilizer

9. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_x g^{-1})$
   (b) G, X = finite
   $((G \circlearrowleft X) \wedge (x, y$ lie same orbit$)) \to (|G_y| = |G_x|)$

#action #orbit #order

10. $G \circlearrowleft (X = \text{finite})$
    $N_{\mathcal{O}} = \frac{1}{|G|} \sum_g F^X(g)$ = number of orbits
    $F^X(g)$ = number of x fixed by g

#color

11. $\mathcal{C}$ = set of q colors

12. $\mathcal{C}^n$ = all n-tuples of colors

13. $N^X_{(q,G)}$ = number of (q,G)-colorings of X

#color #cycle index #order #symmetric

14. $(|X| = n, G \leq S_n) \to (N^X_{(q,G)} = P_G(q, ..., q))$

#color #order #symmetric

15. (a) $N^X_{(q,G)}$ = number of (q,G)-colorings of X
    (b) $(G \leq S_X,\ \ |X| = n,\ \ |\mathcal{C}| = q,\ \ \forall i \geq 1 : \sigma_i = c_1^i + ... + c_q^i) \to$
    $\to (N^X_{(q,G)}$ having f_r elements of color $c_r \forall r =$
    $=$ the coefficient of $c_1^{f_1} c_2^{f_2} \cdots c_q^{f_q})$

#cycle index #index #order #symmetric

16. $P_G(x_1, ..., x_n)$ = *cycle index* of G = polynomial in n variables with coefficients in $\mathbb{Q}$

17. (a) $\operatorname{ind}(\tau)$ = index of τ = monomial
    (b) $(G \leq S_n) \to (P_G(x_1, ..., x_n) = \frac{1}{|G|} \sum_g \operatorname{ind}(g))$

#index #symmetric

18. $\text{ind}(\tau)$ = index of τ = monomial

19. $(e_r(\tau) \geq 0) \to (\text{ind}(\tau) = x_1^{e_1(\tau)} \cdots x_n^{e_n(\tau)})$

#symmetric

20. $\tau \in S_n$

21. $t(\tau)$ = number of cycles in the complete factorization of τ

22. $e_r(\tau)$ = number of r-cycles in the complete factorization of τ

Final Remarks

23. We included some results for **counting with groups** organized by (collections of) **tags**.

24. In the **Appendix**, we listed the results per **tag**.

25. One of the main purposes of these *concise series* of articles [1–5] is to *collect* some of the *mathematical pieces* of a given topic in order to *reassemble* them altogether in a more **complex** and **robust** result for future research.

26. *In summary, this article comprised the following two-fold strategy:* **minimal notation** *plus* **tags classification** *of some of the main results within a specific topic, namely,* **counting with tags**.

Open Invitation

Review, *add* content, and **co-author** this paper [10, 11].
Join the **Open Mathematics Collaboration**.
Send your contribution to mplobo@uft.edu.br.

Open Science

The **latex file** for this paper together with other *supplementary files* are available [12].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [13], please cite it accordingly [5]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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Appendix: Results by individual tags

#action

27. (a) $(G \circlearrowleft (X = \{1, ..., n\})) \to (G \circlearrowleft \mathcal{C}, \forall g(c_1, ..., c_n))$
    (b) $\forall g : g(c_1, ..., c_n) = (c_{g1}, ..., c_{gn})$
    (c) $(c_1, ..., c_n) \in \mathcal{C}^n$
    (d) $\mathcal{O}(c_1, ..., c_n) = (q, G)$-coloring of X

28. $\tau \in S_n, \qquad F^{\mathcal{C}^n}(\tau)$ = number of elements in $\mathcal{C}^n$ fixed by τ
    $t(\tau)$ = number of cycles in the complete factorization of τ
    $N^X_{(q,G)}$ = number of (q, G)-colorings of X
    (a) $(F^{\mathcal{C}^n}(\tau) \wedge t(\tau)) \to (F^{\mathcal{C}^n}(\tau) = q^{t(\tau)})$
    (b) $(\text{finite } G \circlearrowleft (X = \{1, ..., n\})) \to \left(N^X_{(q,G)} = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}\right)$

29. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_x g^{-1})$
    (b) G, X = finite
    $((G \circlearrowleft X) \wedge (x, y \text{ lie same orbit})) \to (|G_y| = |G_x|)$

30. $G \circlearrowleft (X = \text{finite})$
    $N_{\mathcal{O}} = \frac{1}{|G|} \sum_g F^X(g)$ = number of orbits
    $F^X(g)$ = number of x fixed by g

#color

31. (a) $(G \circlearrowleft (X = \{1, ..., n\})) \to (G \circlearrowleft \mathcal{C}, \forall g(c_1, ..., c_n))$
    (b) $\forall g : g(c_1, ..., c_n) = (c_{g1}, ..., c_{gn})$
    (c) $(c_1, ..., c_n) \in \mathcal{C}^n$
    (d) $\mathcal{O}(c_1, ..., c_n) = (q, G)$-coloring of X

32. $\tau \in S_n, \quad F^{\mathcal{C}^n}(\tau)$ = number of elements in $\mathcal{C}^n$ fixed by τ
    $t(\tau)$ = number of cycles in the complete factorization of τ
    $N^X_{(q,G)}$ = number of (q,G)-colorings of X
    (a) $(F^{\mathcal{C}^n}(\tau) \wedge t(\tau)) \to (F^{\mathcal{C}^n}(\tau) = q^{t(\tau)})$
    (b) $(\text{finite } G \circlearrowleft (X = \{1,...,n\})) \to \left(N^X_{(q,G)} = \frac{1}{|G|}\sum_{\tau \in G} q^{t(\tau)}\right)$

33. $\mathcal{C}$ = set of q colors

34. $\mathcal{C}^n$ = all n-tuples of colors

35. $N^X_{(q,G)}$ = number of (q,G)-colorings of X

36. (a) $N^X_{(q,G)}$ = number of (q,G)-colorings of X
    (b) $(G \le S_X,\ |X| = n,\ |\mathcal{C}| = q,\ \forall i \ge 1 : \sigma_i = c_1^i + ... + c_q^i) \to$
    $\to (N^X_{(q,G)}$ having f_r elements of color $c_r \forall r =$
    $=$ the coefficient of $c_1^{f_1} c_2^{f_2} \cdots c_q^{f_q})$

37. $(|X| = n, G \le S_n) \to (N^X_{(q,G)} = P_G(q,...,q))$

#conjugation

38. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_xg^{-1})$
    (b) G, X = finite
    $((G \circlearrowleft X) \wedge (x, y \text{ lie same orbit})) \to (|G_y| = |G_x|)$

#cycle index

39. $(|X| = n, G \le S_n) \to (N^X_{(q,G)} = P_G(q,...,q))$

40. $P_G(x_1,...,x_n)$ = *cycle index* of G = polynomial in n variables with coefficients in $\mathbb{Q}$

41. (a) $\text{ind}(\tau)$ = index of τ = monomial
    (b) $(G \le S_n) \to (P_G(x_1,...,x_n) = \frac{1}{|G|}\sum_g \text{ind}(g))$

#index

42. $P_G(x_1, ..., x_n)$ = *cycle index* of G = polynomial in n variables with coefficients in $\mathbb{Q}$

43. (a) $\text{ind}(\tau)$ = index of τ = monomial
    (b) $(G \leq S_n) \to (P_G(x_1, ..., x_n) = \frac{1}{|G|} \sum_g \text{ind}(g))$

44. $\text{ind}(\tau)$ = index of τ = monomial

45. $(e_r(\tau) \geq 0) \to (\text{ind}(\tau) = x_1^{e_1(\tau)} \cdots x_n^{e_n(\tau)})$

#orbit

46. (a) $(G \circlearrowleft (X = \{1, ..., n\})) \to (G \circlearrowleft \mathcal{C}, \forall g(c_1, ..., c_n))$
    (b) $\forall g : g(c_1, ..., c_n) = (c_{g1}, ..., c_{gn})$
    (c) $(c_1, ..., c_n) \in \mathcal{C}^n$
    (d) $\mathcal{O}(c_1, ..., c_n) = (q, G)$-coloring of X

47. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_x g^{-1})$
    (b) G, X = finite
    $((G \circlearrowleft X) \wedge (x, y \text{ lie same orbit})) \to (|G_y| = |G_x|)$

48. $G \circlearrowleft (X = \text{finite})$
    $N_{\mathcal{O}} = \frac{1}{|G|} \sum_g F^X(g)$ = number of orbits
    $F^X(g)$ = number of x fixed by g

#order

49. $\tau \in S_n$, $\quad F^{\mathcal{C}^n}(\tau)$ = number of elements in $\mathcal{C}^n$ fixed by τ
    $t(\tau)$ = number of cycles in the complete factorization of τ
    $N^X_{(q,G)}$ = number of (q, G)-colorings of X

    (a) $(F^{\mathcal{C}^n}(\tau) \wedge t(\tau)) \to (F^{\mathcal{C}^n}(\tau) = q^{t(\tau)})$

(b) $(\text{finite } G \circlearrowleft (X = \{1, ..., n\})) \to \left(N^X_{(q,G)} = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}\right)$

50. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_x g^{-1})$

    (b) $G, X = \text{finite}$

    $((G \circlearrowleft X) \wedge (x, y \text{ lie same orbit})) \to (|G_y| = |G_x|)$

51. $G \circlearrowleft (X = \text{finite})$

    $N_{\mathcal{O}} = \frac{1}{|G|} \sum_g F^X(g) = \text{number of orbits}$

    $F^X(g) = \text{number of } x \text{ fixed by } g$

52. (a) $N^X_{(q,G)} = \text{number of } (q, G)\text{-colorings of } X$

    (b) $(G \leq S_X,\ \ |X| = n,\ \ |\mathcal{C}| = q,\ \ \forall i \geq 1 : \sigma_i = c_1^i + ... + c_q^i) \to$

    $\to (N^X_{(q,G)} \text{ having } f_r \text{ elements of color } c_r \forall r =$

    $= \text{the coefficient of } c_1^{f_1} c_2^{f_2} \cdots c_q^{f_q})$

53. $(|X| = n, G \leq S_n) \to (N^X_{(q,G)} = P_G(q, ..., q))$

54. $P_G(x_1, ..., x_n)$ = *cycle index* of G = polynomial in n variables with coefficients in $\mathbb{Q}$

55. (a) $\text{ind}(\tau) = \text{index of } \tau = \text{monomial}$

    (b) $(G \leq S_n) \to (P_G(x_1, ..., x_n) = \frac{1}{|G|} \sum_g \text{ind}(g))$

#stabilizer

56. (a) $(G \circlearrowleft X) \to (G_{gx} = gG_x g^{-1})$

    (b) $G, X = \text{finite}$

    $((G \circlearrowleft X) \wedge (x, y \text{ lie same orbit})) \to (|G_y| = |G_x|)$

#symmetric

57. $\tau \in S_n, \qquad F^{\mathcal{C}^n}(\tau) = \text{number of elements in } \mathcal{C}^n \text{ fixed by } \tau$

    $t(\tau) = \text{number of cycles in the complete factorization of } \tau$

    $N^X_{(q,G)} = \text{number of } (q, G)\text{-colorings of } X$

(a) $(F^{\mathcal{C}^n}(\tau) \wedge t(\tau)) \to (F^{\mathcal{C}^n}(\tau) = q^{t(\tau)})$

(b) (finite $G \circlearrowleft (X = \{1, ..., n\})) \to \left(N^X_{(q,G)} = \frac{1}{|G|} \sum_{\tau \in G} q^{t(\tau)}\right)$

58. (a) $N^X_{(q,G)}$ = number of (q, G)-colorings of X

(b) $(G \leq S_X,\ \ |X| = n,\ \ |\mathcal{C}| = q,\ \ \forall i \geq 1 : \sigma_i = c_1^i + ... + c_q^i) \to$
$\to (N^X_{(q,G)}$ having f_r elements of color $c_r \forall r$ =
= the coefficient of $c_1^{f_1} c_2^{f_2} \cdots c_q^{f_q})$

59. $(|X| = n, G \leq S_n) \to (N^X_{(q,G)} = P_G(q, ..., q))$

60. $P_G(x_1, ..., x_n)$ = *cycle index* of G = polynomial in n variables with coefficients in $\mathbb{Q}$

61. (a) $\mathrm{ind}(\tau)$ = index of τ = monomial

(b) $(G \leq S_n) \to (P_G(x_1, ..., x_n) = \frac{1}{|G|} \sum_g \mathrm{ind}(g))$

62. $\tau \in S_n$

63. $t(\tau)$ = number of cycles in the complete factorization of τ

64. $e_r(\tau)$ = number of r-cycles in the complete factorization of τ

65. $\mathrm{ind}(\tau)$ = index of τ = monomial

66. $(e_r(\tau) \geq 0) \to (\mathrm{ind}(\tau) = x_1^{e_1(\tau)} \cdots x_n^{e_n(\tau)})$