[microreview]

Diamond Open Access

Group actions: a concise approach

Open Mathematics Collaboration^{*†}

May 10, 2020

Abstract

Some definitions and results of group actions are presented within the fewest words and symbols as possible.

keywords: group theory, abstract algebra, group actions

The most updated version of this paper is available at

<https://osf.io/9dt34/download>

Introduction

1. The purpose of this *microreview* is to provide some basic results of **group actions** using the fewest words and symbols as possible, without losing information.
2. We believe this *concise pedagogical approach* can be a very useful resource while one is studying and trying to solve pure mathematical problems.
3. These papers [1–4] can be read jointly.
4. [5–7].

^{*}All authors with their affiliations appear at the end of this paper.

[†]Corresponding author: mplobo@uft.edu.br | Join the Open Mathematics Collaboration

Notations & Definitions

#abelian

5. G_{Ab} Abelian group

#abelian #center #conjugation

6. (a) $Z(G) = \{z \in G : \forall g. zg = gz\}$ *center of G*
(b) $(zg = gz) \equiv (z = gzg^{-1})$

#action

7. (a) $(\alpha : X \times Y \rightarrow Z) \equiv (\alpha_x : Y \rightarrow Z)$
(b) $\alpha_x(y) = \alpha(x, y)$
(c) $\{\alpha_{x_i}(y)\} =$ one-parameter family of functions
8. $G \curvearrowright X$ G acts on X
9. $G \curvearrowright_t X$ G acts transitively on X
10. G acts on X if
- (a) $\exists \alpha : G \times X \rightarrow X$
 - (b) $\alpha =$ action
 - (c) $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$
 - (d) $\alpha_1 = 1_X =$ identity function
 - (e) $\alpha_g(x) \equiv gx$
 - (f) $g(g'x) = (gg')x$
11. $\alpha : G \times X \rightarrow X$ *action of G on X*

#action #centralizer #conjugacy #orbit

12. (a) $(G \curvearrowright G, \alpha_g^c(x)) \rightarrow (\mathcal{O}(g_1) = \{g^2 : g^2 = g_3 g_1 g_3^{-1}\})$
 (b) $g^G \equiv \mathcal{O}(g_1) = \text{conjugacy class of } g_1$
 (c) $C_G(g) \equiv G_g = \{g' : g' g g'^{-1} = g\}$
 (d) $C_G(g) \leq G, C_G(g) = \text{centralizer of } g \text{ in } G$

#action #orbit

13. $(\exists x \in X : \mathcal{O}(x) = X) \rightarrow (G \curvearrowright_t X)$

#action #orbit #stabilizer

14. $\mathcal{O}(x) = \text{orbit of } x, G_x = \text{stabilizer of } x$
 $(G \curvearrowright X, x \in X) \rightarrow (\mathcal{O}(x) = \{gx\} \subseteq X) \rightarrow (G_x = \{g : gx = x\} \leq G)$

#alternating

15. $A_n = \text{alternating group}$

#center #centralizer #index #order

16. *class equation*
 $|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$
 one x_i is selected from each conjugacy class having more than one element

#conjugacy

17. $\mathcal{O}(x) = x^G = \text{conjugacy class of } x$

#conjugation

18. $\alpha_g^c(x) \equiv g x g^{-1} \quad \text{conjugation}$

#coset

19. G/H family of *cosets* of $H \leq G$

#cyclic

20. $\langle g \rangle =$ *cyclic* subgroup of G *generated* by g

#equivalence

21. R equivalence relation

22. $[x] =$ equivalence class

#exists

23. $\exists!_2$ there are *exactly* two

#gcd

24. (m, n) gcd of m and n

#group

25. $g, g' \in G =$ group

26. $S \leq G$ subgroup

27. $S < G$ *proper* subgroup

28. $s \in S$

#homomorphism

29. $\simeq, \xrightarrow{h}, f^h, \varphi^h, \dots$ homomorphism

30. $f^h : G \rightarrow M$

#homomorphism #translation

31. $\tau_a = \text{translation}$

(a) $\tau_a : G \rightarrow G, \tau_a(g) = ag, \forall g$

(b) $(a \neq 1) \rightarrow (\tau_a \neq \tau^h)$

#index #order

32. $|G : S| = \frac{|G|}{|S|} = \text{index of } S \text{ in } G$

#integer

33. $i \in \mathbb{I} = \{1, 2, 3, \dots\}$

34. $k = \text{finite}$

35. $n \geq 0$

#integer modulo m

36. $\mathbb{I}_m$ integers modulo m

#isomorphism

37. $\cong, f^i, \varphi^i, \dots$ isomorphism

#normal

38. $S_{\triangleleft} \equiv (S \triangleleft G)$ S is a normal subgroup of G

#normal #simple

39. $G = \text{simple}$ if

(a) $G \neq \{1\}$

(b) $\exists!_2 S \triangleleft G : (S = \{1\}) \vee (S = G)$

#order

40. $|G| = \text{order} = \text{number of elements}$

41. wherever there is $|G|$, it means that G is *finite*

#order #prime

42. $|G_p| = p^n$

#prime

43. $p = \text{prime}$

44. $G_p = p\text{-group}$

#quaternion

45. $\mathbf{Q} = \text{quaternions}$

#set

46. $X = \text{set}$

47. $x \in X$

48. $\sqcup_i$ disjoint union

#simple

49. G_s or G^s G is *simple*

#symmetric

50. $S_G = \text{symmetric group}$ (all permutations of G)

51. $V = \{(1), (12)(34), (13)(24), (14)(23)\}$ = four permutations group = four-group = Klein group

52. $\alpha_r \in S_n$ α_r is an r -cycle

53. $|S_n|_r$ = the number of r -cycle permutations in S_n

54. $\sigma_r = r$ -cycle

55. τ_2 = transposition

56. $\alpha \in S_n$
 α^{S_n} = all the permutations in S_n with the same cycle structure as α

#translation

57. $\tau_a^x \equiv (\tau_a : x \mapsto ax)$

Results by (collection of) tags

#abelian #normal

$$58. (gg' = g'g) \rightarrow ((S \leq G) \rightarrow (S \triangleleft G))$$

#abelian #order

$$59. |gg'| = |g'g|$$

#abelian #order #prime

$$60. (|G| = p^2) \rightarrow (gg' = g'g)$$

#abelian #order #prime #simple

$$61. (G_{Ab} = G_s) \leftrightarrow (|G| = p)$$

#action #conjugation

$$62. G \curvearrowright G \text{ by } \textit{conjugation}, \forall g, \alpha_g : G \rightarrow G, \alpha_g(x) = gxg^{-1}.$$

#action #conjugation #coset #orbit #stabilizer
#translation

$$63. \begin{aligned} \text{(a)} \quad & (G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH) \\ \text{(b)} \quad & G_g H = gHg^{-1} \end{aligned}$$

#action #coset #translation

$$64. G \curvearrowright G/H \text{ by (left) } \textit{translation}.$$

#action #cyclic #orbit #order #stabilizer
#symmetric

$$65. X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$$

$$(a) \quad \mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$$

$$(b) \quad |\mathcal{O}(i)| = r$$

$$(c) \quad l = \text{symbol}$$

$$(\sigma \text{ fixes } l) \rightarrow (G_l = G)$$

$$(\sigma \text{ moves } l) \rightarrow (G_l < G)$$

#action #equivalence #orbit

$$66. (a) \quad (G \curvearrowright X) \rightarrow ((\exists g : y = gx) \rightarrow (xRy))$$

$$(b) \quad [x] = \{y \in X : yRx\} = \{gx\} = \mathcal{O}(x)$$

#action #homomorphism #symmetric

$$67. (a) \quad \alpha \rightarrow (G \xrightarrow{h} S_X, \quad g \mapsto \alpha_g)$$

$$(b) \quad (B : G \xrightarrow{h} S_X) \rightarrow (\beta : G \times X \rightarrow X, \quad \beta(g, x) = B(g)(x) = \text{action})$$

#action #index #orbit #order

$$68. (G \curvearrowright X) \rightarrow (|\mathcal{O}(x)| = [G : G_x])$$

#action #isomorphism #symmetric

$$69. (a) \quad \forall G \leq S_X : G \curvearrowright X$$

$$(b) \quad G \cong S_X$$

#action #orbit #order

$$70. (G \curvearrowright X) \rightarrow (X = \bigsqcup_i \mathcal{O}(x_i)) \rightarrow ((|X| = k) \rightarrow (|X| = \sum_i |\mathcal{O}(x_i)|))$$

one x_i is chosen from each orbit

$$71. (G \circ X, |G| = k) \rightarrow (|\mathcal{O}(x)| \mid |G|)$$

#action #translation

$$72. G \circ G \text{ by (left) translations } \tau_a : x \mapsto ax$$

#alternating #index #prime

$$73. \nexists S \leq A_6 : [A_6 : S] = p$$

#alternating #order #symmetric

$$74. (n \geq 3) \rightarrow (\exists ! A_n \leq S_n : |A_n| = \frac{1}{2}n!)$$

#alternating #simple

$$75. A_5 = A_5^s$$

$$76. A_6 = A_6^s$$

#center #conjugacy

$$77. (x^G = \{x\}) \leftrightarrow (x \in Z(G))$$

#center #order

$$78. (|G_p| > 1) \rightarrow (Z(G) \neq \{1\})$$

#centralizer #conjugacy #index #order

$$79. (a) C_S(s) = S \cap C_G(s)$$

$$(b) ([G : S] = 2) \rightarrow ((|s^S| = |s^G|) \vee (|s^S| = \frac{1}{2}|s^G|))$$

$s^S = \text{conjugacy class of } s \text{ in } S$

$$80. (x \in G) \rightarrow (|x^G| = [G : C_G(x)]) \rightarrow (|x^G| \mid |G|)$$

#conjugacy #normal

81. (a) $\forall S_{\triangleleft} : S_{\triangleleft} = \bigcup_i g_i^G$
(b) $g_1^G = \{1\}$

#coprime #isomorphism #integer modulo m

82. $((m, n) = 1) \rightarrow (\mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n)$

#cyclic #symmetric

83. $(\sigma_5, \tau_2 \in S_5) \rightarrow (\langle \sigma_5 \tau_2 \rangle = S_5)$

#homomorphism #kernel #order #symmetric

84. *Representation on Cosets*
 $(|M| = n) \rightarrow (\exists \varphi^h : G \rightarrow S_n, \ker \varphi^h \leq M)$

#index #isomorphism #order #simple
#symmetric

85. (a) $(S \leq G_s, [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$
(b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), [G_s : S] \text{ is finite}$

#integer modulo m #isomorphism

86. $\mathbb{I}_6 \cong \mathbb{I}_2 \times \mathbb{I}_3$

#integer modulo m #isomorphism #order
#symmetric

87. (a) $(\forall G : |G| = 4) \rightarrow ((G \cong \mathbb{I}_4) \vee (G \cong \text{four-group } V))$
(b) $\mathbb{I}_4 \not\cong V$
88. (a) $(|G| = 6) \rightarrow ((G \cong \mathbb{I}_6) \vee (G \cong S_3))$

(b) $\mathbb{I}_6 \not\cong S_3$

~~#isomorphism~~ ~~#order~~ ~~#prime~~

89. $((|G_i| = p_i) \wedge (|G_j| = p_j)) \rightarrow (G_i \cong G_j)$

~~#isomomorphism~~ ~~#order~~ ~~#symmetric~~

90. *Cayley*

(a) $\forall G : G \cong S \leq S_G$

(b) $(|G| = n) \rightarrow (G \cong S \leq S_n)$

~~#normal~~ ~~#order~~ ~~#prime~~

91. $(|G| = p^e) \rightarrow (S \triangleleft G, \quad \forall k \leq e : |S| = p^k)$

~~#normal~~ ~~#quaternion~~

92. $(q \in Q) \rightarrow ((Q' \leq Q) \rightarrow (Q' \triangleleft Q))$

~~#orbit~~ ~~#stabilizer~~

93. (a) $\mathcal{O}(g) = G$

(b) $G_g = \{1\}$

~~#order~~ ~~#prime~~

94. $(p \mid |G|) \rightarrow (|g| = p) \quad \text{Cauchy}$

~~#order~~ ~~#prime~~ ~~#simple~~

95. $(G_p = G_s) \leftrightarrow (|G| = p)$

96. $(|G| = mp, \quad 1 < m < p) \rightarrow (G \neq G_s)$

#order #prime #stabilizer

97. $(\forall g_i : |g_i| = p^n) \rightarrow (G_p)$

#order #symmetric

98. $(\alpha_r \in S_n) \rightarrow (|S_n|_r \mid n!)$

#permutation

99. permutations = functions

Final Remarks

100. We included some results of **group actions** organized by (collections of) **tags**.
101. In the **Appendix**, we listed the results per each **tag**.
102. One of the main purposes of these *concise series* of articles [1–4] is to *collect* some of the *mathematical pieces* of a given topic in order to *reassemble* them altogether in a more **complex** and **robust** result in future research.
103. *In summary, this article comprises the following two-fold strategy: **minimal notation plus tags classification** of some of the main results within a specific topic, namely, **group actions**.*

Open Invitation

Review, add content, and **co-author** this paper [8, 9].

Join the **Open Mathematics Collaboration**.

Send your contribution to mplobo@uft.edu.br.

Open Science

The **latex file** for this paper together with other *supplementary files* are available [10].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [11], please cite it accordingly [4]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

References

- [1] Lobo, Matheus P. “Groups: A Concise Approach.” *OSF Preprints*, 20 Apr. 2020. <https://doi.org/10.31219/osf.io/bfyhm>
- [2] Lobo, Matheus P. “Subgroups: A Concise Approach.” *OSF Preprints*, 22 Apr. 2020. <https://doi.org/10.31219/osf.io/wcaez>
- [3] Lobo, Matheus P. “Homomorphisms: A Concise Approach.” *OSF Preprints*, 26 Apr. 2020. <https://doi.org/10.31219/osf.io/7pgcs>
- [4] Lobo, Matheus P. “Group Actions: A Concise Approach.” *OSF Preprints*, 4 May 2020. <https://doi.org/10.31219/osf.io/9dt34>
- [5] Rotman, Joseph J. *A first course in abstract algebra*. Pearson College Division, 2000.
- [6] Dummit, David Steven, and Richard M. Foote. *Abstract Algebra*. Vol. 3. Hoboken: Wiley, 2004.
- [7] Warner, Steve. *Abstract Algebra for Beginners*. GET 800, 2018.
- [8] Lobo, Matheus P. “Microarticles.” *OSF Preprints*, 28 Oct. 2019. <https://doi.org/10.31219/osf.io/ejrct>

- [9] Lobo, Matheus P. “Simple Guidelines for Authors: Open Journal of Mathematics and Physics.” *OSF Preprints*, 15 Nov. 2019. <https://doi.org/10.31219/osf.io/fk836>
- [10] Lobo, Matheus P. “Open Journal of Mathematics and Physics (OJMP).” *OSF*, 21 Apr. 2020. <https://doi.org/10.17605/OSF.IO/6HZYP>
- [11] COS. *Open Science Framework*. <https://osf.io>

The Open Mathematics Collaboration

Matheus Pereira Lobo (lead author, mplobo@uft.edu.br)^{1,2}

¹Federal University of Tocantins (Brazil); ²Universidade Aberta (UAb, Portugal)

Appendix: Results by (individual) tags

#abelian

- 104. G_{Ab} Abelian group
- 105. (a) $Z(G) = \{z \in G : \forall g. zg = gz\}$ *center of G*
 (b) $(zg = gz) \equiv (z = gzg^{-1})$
- 106. $(gg' = g'g) \rightarrow ((S \leq G) \rightarrow (S \triangleleft G))$
- 107. $|gg'| = |g'g|$
- 108. $(|G| = p^2) \rightarrow (gg' = g'g)$
- 109. $(G_{Ab} = G_s) \leftrightarrow (|G| = p)$

#action

- 110. (a) $(\alpha : X \times Y \rightarrow Z) \equiv (\alpha_x : Y \rightarrow Z)$
 (b) $\alpha_x(y) = \alpha(x, y)$
 (c) $\{\alpha_{x_i}(y)\} = \text{one-parameter family of functions}$
- 111. $G \curvearrowright X$ G acts on X
- 112. $G \curvearrowright_t X$ G acts transitively on X
- 113. G acts on X if
 - (a) $\exists \alpha : G \times X \rightarrow X$
 - (b) $\alpha = \text{action}$
 - (c) $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$
 - (d) $\alpha_1 = 1_X = \text{identity function}$
 - (e) $\alpha_g(x) \equiv gx$
 - (f) $g(g'x) = (gg')x$

114. $\alpha : G \times X \rightarrow X$ *action of G on X*
115. (a) $(G \curvearrowright G, \alpha_g^c(x)) \rightarrow (\mathcal{O}(g_1) = \{g_2 : g_2 = g_3 g_1 g_3^{-1}\})$
 (b) $g^G \equiv \mathcal{O}(g_1) = \text{conjugacy class of } g_1$
 (c) $C_G(g) \equiv G_g = \{g' : g' g g'^{-1} = g\}$
 (d) $C_G(g) \leq G, \quad C_G(g) = \text{centralizer of } g \text{ in } G$
116. $(\exists x \in X : \mathcal{O}(x) = X) \rightarrow (G \curvearrowright_t X)$
117. $\mathcal{O}(x) = \text{orbit of } x, \quad G_x = \text{stabilizer of } x$
 $(G \curvearrowright X, x \in X) \rightarrow (\mathcal{O}(x) = \{gx\} \subseteq X) \rightarrow (G_x = \{g : gx = x\} \leq G)$
118. $G \curvearrowright G$ by *conjugation*, $\forall g, \alpha_g : G \rightarrow G, \alpha_g(x) = gxg^{-1}$.
119. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH)$
 (b) $G_g H = g H g^{-1}$
120. $G \curvearrowright G/H$ by (left) *translation*.
121. $X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$
 (a) $\mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$
 (b) $|\mathcal{O}(i)| = r$
 (c) $l = \text{symbol}$
 $(\sigma \text{ fixes } l) \rightarrow (G_l = G)$
 $(\sigma \text{ moves } l) \rightarrow (G_l < G)$
122. (a) $(G \curvearrowright X) \rightarrow ((\exists g : y = gx) \rightarrow (x R y))$
 (b) $[x] = \{y \in X : y R x\} = \{gx\} = \mathcal{O}(x)$
123. (a) $\forall G \leq S_X : G \curvearrowright X$
 (b) $G \cong S_X$
124. (a) $\alpha \rightarrow (G \xrightarrow{h} S_X, g \mapsto \alpha_g)$
 (b) $(B : G \xrightarrow{h} S_X) \rightarrow (\beta : G \times X \rightarrow X, \beta(g, x) = B(g)(x) = \text{action})$

125. $(G \curvearrowright X) \rightarrow (|\mathcal{O}(x)| = [G : G_x])$
126. $(G \curvearrowright X) \rightarrow (X = \sqcup_i \mathcal{O}(x_i)) \rightarrow ((|X| = k) \rightarrow (|X| = \sum_i |\mathcal{O}(x_i)|))$
 one x_i is chosen from each orbit
127. $(G \curvearrowright X, |G| = k) \rightarrow (|\mathcal{O}(x)| \mid |G|)$
128. $G \curvearrowright G$ by (left) *translations* $\tau_a : x \mapsto ax$

#alternating

129. $A_n = \text{alternating group}$
130. $\nexists S \leq A_6 : [A_6 : S] = p$
131. $(n \geq 3) \rightarrow (\exists! A_n \leq S_n : |A_n| = \frac{1}{2}n!)$
132. $A_5 = A_5^s$
133. $A_6 = A_6^s$

#center

134. (a) $Z(G) = \{z \in G : \forall g. zg = gz\}$ *center of G*
 (b) $(zg = gz) \equiv (z = gzg^{-1})$
135. *class equation*
 $|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$
 one x_i is selected from each conjugacy class having more than one element
136. $(x^G = \{x\}) \leftrightarrow (x \in Z(G))$
137. $(|G_p| > 1) \rightarrow (Z(G) \neq \{1\})$

#centralizer

138. (a) $(G \curvearrowright G, \alpha_g^c(x)) \rightarrow (\mathcal{O}(g_1) = \{g^2 : g^2 = g_3 g_1 g_3^{-1}\})$

(b) $g^G \equiv \mathcal{O}(g_1) = \text{conjugacy class of } g_1$

(c) $C_G(g) \equiv G_g = \{g' : g' g g'^{-1} = g\}$

(d) $C_G(g) \leq G, C_G(g) = \text{centralizer of } g \text{ in } G$

139. *class equation*

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

one x_i is selected from each conjugacy class having more than one element

140. (a) $C_S(s) = S \cap C_G(s)$

(b) $([G : S] = 2) \rightarrow ((|s^S| = |s^G|) \vee (|s^S| = \frac{1}{2}|s^G|))$
 $s^S = \text{conjugacy class of } s \text{ in } S$

141. $(x \in G) \rightarrow (|x^G| = [G : C_G(x)]) \rightarrow (|x^G| \mid |G|)$

#conjugacy

142. (a) $(G \curvearrowright G, \alpha_g^c(x)) \rightarrow (\mathcal{O}(g_1) = \{g^2 : g^2 = g_3 g_1 g_3^{-1}\})$

(b) $g^G \equiv \mathcal{O}(g_1) = \text{conjugacy class of } g_1$

(c) $C_G(g) \equiv G_g = \{g' : g' g g'^{-1} = g\}$

(d) $C_G(g) \leq G, C_G(g) = \text{centralizer of } g \text{ in } G$

143. $\mathcal{O}(x) = x^G = \text{conjugacy class of } x$

144. $(x^G = \{x\}) \leftrightarrow (x \in Z(G))$

145. (a) $C_S(s) = S \cap C_G(s)$

(b) $([G : S] = 2) \rightarrow ((|s^S| = |s^G|) \vee (|s^S| = \frac{1}{2}|s^G|))$
 $s^S = \text{conjugacy class of } s \text{ in } S$

146. $(x \in G) \rightarrow (|x^G| = [G : C_G(x)]) \rightarrow (|x^G| \mid |G|)$

147. (a) $\forall S_{\triangleleft} : S_{\triangleleft} = \bigcup_i g_i^G$
 (b) $g_1^G = \{1\}$

#conjugation

148. (a) $Z(G) = \{z \in G : \forall g. zg = gz\}$ *center of G*
 (b) $(zg = gz) \equiv (z = gzg^{-1})$
149. $\alpha_g^c(x) \equiv gxg^{-1}$ *conjugation*
150. $G \curvearrowright G$ by *conjugation*, $\forall g, \alpha_g : G \rightarrow G, \alpha_g(x) = gxg^{-1}$.
151. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH)$
 (b) $G_g H = gHg^{-1}$

#coprime

152. $((m, n) = 1) \rightarrow (\mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n)$

#coset

153. G/H family of *cosets* of $H \leq G$
154. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH)$
 (b) $G_g H = gHg^{-1}$
155. $G \curvearrowright G/H$ by (left) *translation*.

#cyclic

156. $\langle g \rangle =$ *cyclic* subgroup of G *generated* by g
157. $X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$
 (a) $\mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$
 (b) $|\mathcal{O}(i)| = r$

- (c) l = symbol
 $(\sigma \text{ fixes } l) \rightarrow (G_l = G)$
 $(\sigma \text{ moves } l) \rightarrow (G_l < G)$

158. $(\sigma_5, \tau_2 \in S_5) \rightarrow (\langle \sigma_5 \tau_2 \rangle = S_5)$

#equivalence

159. R equivalence relation

160. $[x]$ = equivalence class

161. (a) $(G \curvearrowright X) \rightarrow ((\exists g : y = gx) \rightarrow (xRy))$

(b) $[x] = \{y \in X : yRx\} = \{gx\} = \mathcal{O}(x)$

#exists

162. $\exists!_2$ there are *exactly* two

#gcd

163. (m, n) gcd of m and n

#group

164. $g, g' \in G$ = group

165. $S, H \leq G$ subgroups

166. $S < G$ *proper* subgroup

167. $s \in S$

#homomorphism

168. $\simeq, \xrightarrow{h}, f^h, \varphi^h, \dots$ homomorphism

169. $f^h : G \rightarrow M$

170. $\tau_a = \text{translation}$

$$(a) \quad \tau_a : G \rightarrow G, \quad \tau_a(g) = ag, \quad \forall g$$

$$(b) \quad (a \neq 1) \rightarrow (\tau_a \neq \tau^h)$$

171. $\tau_a^x \equiv (\tau_a : x \mapsto ax)$

172. (a) $\alpha \rightarrow (G \xrightarrow{h} S_X, g \mapsto \alpha_g)$

$$(b) \quad (B : G \xrightarrow{h} S_X) \rightarrow (\beta : G \times X \rightarrow X, \beta(g, x) = B(g)(x) = \text{action})$$

173. *Representation on Cosets*

$$(|M| = n) \rightarrow (\exists \varphi^h : G \rightarrow S_n, \ker \varphi^h \leq M)$$

#index

174. *class equation*

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

one x_i is selected from each conjugacy class having more than one element

175. $|G : S| = \frac{|G|}{|S|} = \text{index of } S \text{ in } G$

176. $(G \curvearrowright X) \rightarrow (|\mathcal{O}(x)| = [G : G_x])$

177. $\nexists S \leq A_6 : [A_6 : S] = p$

178. (a) $C_S(s) = S \cap C_G(s)$

$$(b) \quad ([G : S] = 2) \rightarrow ((|s^S| = |s^G|) \vee (|s^S| = \frac{1}{2}|s^G|))$$

$s^S = \text{conjugacy class of } s \text{ in } S$

179. $(x \in G) \rightarrow (|x^G| = [G : C_G(x)]) \rightarrow (|x^G| \mid |G|)$

180. (a) $(S \leq G_s, [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$
 (b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), \quad [G_s : S] \text{ is finite}$

#integer

181. $i \in \mathbb{I} = \{1, 2, 3, \dots\}$

182. $k = \text{finite}$

183. $n \geq 0$

#integer modulo m

184. $\mathbb{I}_m$ integers modulo m

185. $((m, n) = 1) \rightarrow (\mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n)$

186. $\mathbb{I}_6 \cong \mathbb{I}_2 \times \mathbb{I}_3$

187. (a) $(\forall G : |G| = 4) \rightarrow ((G \cong \mathbb{I}_4) \underline{\vee} (G \cong \text{four-group } V))$

(b) $\mathbb{I}_4 \not\cong V$

188. (a) $(|G| = 6) \rightarrow ((G \cong \mathbb{I}_6) \underline{\vee} (G \cong S_3))$

(b) $\mathbb{I}_6 \not\cong S_3$

#isomorphism

189. $\cong, f^i, \varphi^i, \dots$ isomorphism

190. (a) $\forall G \leq S_X : G \curvearrowright X$

(b) $G \cong S_X$

191. $((m, n) = 1) \rightarrow (\mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n)$

192. (a) $(S \leq G_s, [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$

(b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), \quad [G_s : S] \text{ is finite}$

193. $\mathbb{I}_6 \cong \mathbb{I}_2 \times \mathbb{I}_3$
194. (a) $(\forall G : |G| = 4) \rightarrow ((G \cong \mathbb{I}_4) \vee (G \cong \text{four-group } V))$
 (b) $\mathbb{I}_4 \not\cong V$
195. (a) $(|G| = 6) \rightarrow ((G \cong \mathbb{I}_6) \vee (G \cong S_3))$
 (b) $\mathbb{I}_6 \not\cong S_3$
196. $((|G_i| = p_i) \wedge (|G_j| = p_j)) \rightarrow (G_i \cong G_j)$
197. *Cayley*
- (a) $\forall G : G \cong S \leq S_G$
 (b) $(|G| = n) \rightarrow (G \cong S \leq S_n)$

#kernel

198. *Representation on Cosets*
 $(|H| = n) \rightarrow (\exists \varphi^h : G \rightarrow S_n, \ker \varphi^h \leq H)$

#normal

199. $S_{\triangleleft} \equiv (S \triangleleft G)$ S is a *normal subgroup* of G
200. $G = \text{simple}$ if
- (a) $G \neq \{1\}$
 (b) $\exists!_2 S \triangleleft G : (S = \{1\}) \vee (S = G)$
201. $(gg' = g'g) \rightarrow ((S \leq G) \rightarrow (S \triangleleft G))$
202. (a) $\forall S_{\triangleleft} : S_{\triangleleft} = \bigcup_i g_i^G$
 (b) $g_1^G = \{1\}$
203. $(|G| = p^e) \rightarrow (S \triangleleft G, \forall k \leq e : |S| = p^k)$
204. $(q \in Q) \rightarrow ((Q' \leq Q) \rightarrow (Q' \triangleleft Q))$

#orbit

205. (a) $(G \curvearrowright G, \alpha_g^c(x)) \rightarrow (\mathcal{O}(g_1) = \{g^2 : g^2 = g_3 g_1 g_3^{-1}\})$
 (b) $g^G \equiv \mathcal{O}(g_1) = \text{conjugacy class of } g_1$
 (c) $C_G(g) \equiv G_g = \{g' : g' g g'^{-1} = g\}$
 (d) $C_G(g) \leq G, C_G(g) = \text{centralizer of } g \text{ in } G$
206. $(\exists x \in X : \mathcal{O}(x) = X) \rightarrow (G \curvearrowright_t X)$
207. $\mathcal{O}(x) = \text{orbit of } x, G_x = \text{stabilizer of } x$
 $(G \curvearrowright X, x \in X) \rightarrow (\mathcal{O}(x) = \{gx\} \subseteq X) \rightarrow (G_x = \{g : gx = x\} \leq G)$
208. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH)$
 (b) $G_g H = g H g^{-1}$
209. $X = \{1, 2, \dots, n\}, \sigma \in S_n, (G = \langle \sigma \rangle) \curvearrowright X, i \in X$
 (a) $\mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$
 (b) $|\mathcal{O}(i)| = r$
 (c) $l = \text{symbol}$
 $(\sigma \text{ fixes } l) \rightarrow (G_l = G)$
 $(\sigma \text{ moves } l) \rightarrow (G_l < G)$
210. (a) $(G \curvearrowright X) \rightarrow ((\exists g : y = gx) \rightarrow (x R y))$
 (b) $[x] = \{y \in X : y R x\} = \{gx\} = \mathcal{O}(x)$
211. $(G \curvearrowright X) \rightarrow (|\mathcal{O}(x)| = [G : G_x])$
212. $(G \curvearrowright X) \rightarrow (X = \bigsqcup_i \mathcal{O}(x_i)) \rightarrow ((|X| = k) \rightarrow (|X| = \sum_i |\mathcal{O}(x_i)|))$
 one x_i is chosen from each orbit
213. $(G \curvearrowright X, |G| = k) \rightarrow (|\mathcal{O}(x)| \mid |G|)$
214. (a) $\mathcal{O}(g) = G$
 (b) $G_g = \{1\}$

#order

215. *class equation*

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

one x_i is selected from each conjugacy class having more than one element

216. $|G : S| = \frac{|G|}{|S|} = \text{index of } S \text{ in } G$

217. $|G| = \text{order} = \text{number of elements}$

218. wherever there is $|G|$, it means that G is *finite*

219. $|G_p| = p^n$

220. $|gg'| = |g'g|$

221. $(|G| = p^2) \rightarrow (gg' = g'g)$

222. $(G_{Ab} = G_s) \leftrightarrow (|G| = p)$

223. $X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$

(a) $\mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$

(b) $|\mathcal{O}(i)| = r$

(c) $l = \text{symbol}$

$(\sigma \text{ fixes } l) \rightarrow (G_l = G)$

$(\sigma \text{ moves } l) \rightarrow (G_l < G)$

224. $(G \curvearrowright X) \rightarrow (|\mathcal{O}(x)| = [G : G_x])$

225. $(G \curvearrowright X) \rightarrow (X = \bigsqcup_i \mathcal{O}(x_i)) \rightarrow ((|X| = k) \rightarrow (|X| = \sum_i |\mathcal{O}(x_i)|))$

one x_i is chosen from each orbit

226. $(G \curvearrowright X, |G| = k) \rightarrow (|\mathcal{O}(x)| \mid |G|)$

227. $(n \geq 3) \rightarrow (\exists! A_n \leq S_n : |A_n| = \frac{1}{2}n!)$

228. $(|G_p| > 1) \rightarrow (Z(G) \neq \{1\})$
229. (a) $C_S(s) = S \cap C_G(s)$
 (b) $([G : S] = 2) \rightarrow ((|s^S| = |s^G|) \underline{\vee} (|s^S| = \frac{1}{2}|s^G|))$
 $s^S = \text{conjugacy class of } s \text{ in } S$
230. $(x \in G) \rightarrow (|x^G| = [G : C_G(x)]) \rightarrow (|x^G| \mid |G|)$
231. (a) $(S \leq G_s, [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$
 (b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), [G_s : S] \text{ is finite}$
232. (a) $(\forall G : |G| = 4) \rightarrow ((G \cong \mathbb{I}_4) \underline{\vee} (G \cong \text{four-group } V))$
 (b) $\mathbb{I}_4 \not\cong V$
233. (a) $(|G| = 6) \rightarrow ((G \cong \mathbb{I}_6) \underline{\vee} (G \cong S_3))$
 (b) $\mathbb{I}_6 \not\cong S_3$
234. *Representation on Cosets*
 $(|H| = n) \rightarrow (\exists \varphi^h : G \rightarrow S_n, \ker \varphi^h \leq H)$
235. $((|G_i| = p_i) \wedge (|G_j| = p_j)) \rightarrow (G_i \cong G_j)$
236. *Cayley*
 (a) $\forall G : G \cong S \leq S_G$
 (b) $(|G| = n) \rightarrow (G \cong S \leq S_n)$
237. $(|G| = p^e) \rightarrow (S \triangleleft G, \forall k \leq e : |S| = p^k)$
238. $(p \mid |G|) \rightarrow (|g| = p) \quad \text{Cauchy}$
239. $(G_p = G_s) \leftrightarrow (|G| = p)$
240. $(|G| = mp, 1 < m < p) \rightarrow (G \neq G_s)$
241. $(\forall g_i : |g_i| = p^n) \rightarrow (G_p)$
242. $(\alpha_r \in S_n) \rightarrow (|S_n|_r \mid n!)$

#permutation

243. permutations = functions

#prime

244. p = prime

245. G_p = p -group

246. $|G_p| = p^n$

247. $(|G| = p^2) \rightarrow (gg' = g'g)$

248. $(G_{Ab} = G_s) \leftrightarrow (|G| = p)$

249. $\nexists S \leq A_6 : [A_6 : S] = p$

250. $((|G_i| = p_i) \wedge (|G_j| = p_j)) \rightarrow (G_i \cong G_j)$

251. $(|G| = p^e) \rightarrow (S \triangleleft G, \quad \forall k \leq e : |S| = p^k)$

252. $(p \mid |G|) \rightarrow (|g| = p) \quad \text{Cauchy}$

253. $(G_p = G_s) \leftrightarrow (|G| = p)$

254. $(|G| = mp, \quad 1 < m < p) \rightarrow (G \neq G_s)$

255. $(\forall g_i : |g_i| = p^n) \rightarrow (G_p)$

#quaternion

256. $\mathbf{Q}$ = quaternions

257. $(q \in Q) \rightarrow ((Q' \leq Q) \rightarrow (Q' \triangleleft Q))$

#set

258. $X = \text{set}$

259. $x \in X$

260. $\sqcup_i$ disjoint union

#simple

261. $G = \text{simple}$ if

(a) $G \neq \{1\}$

(b) $\exists!_2 S \triangleleft G : (S = \{1\}) \vee (S = G)$

262. G_s or G^s G is *simple*

263. $(G_{Ab} = G_s) \leftrightarrow (|G| = p)$

264. $A_5 = A_5^s$

265. $A_6 = A_6^s$

266. (a) $(S \leq G_s, [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$

(b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), [G_s : S] \text{ is finite}$

267. $(G_p = G_s) \leftrightarrow (|G| = p)$

268. $(|G| = mp, 1 < m < p) \rightarrow (G \neq G_s)$

#stabilizer

269. $\mathcal{O}(x) = \text{orbit of } x, G_x = \text{stabilizer of } x$

$(G \curvearrowright X, x \in X) \rightarrow (\mathcal{O}(x) = \{gx\} \subseteq X) \rightarrow (G_x = \{g : gx = x\} \leq G)$

270. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = gH)$

(b) $G_g H = gH g^{-1}$

$$271. X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$$

$$(a) \quad \mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$$

$$(b) \quad |\mathcal{O}(i)| = r$$

$$(c) \quad l = \text{symbol}$$

$$(\sigma \text{ fixes } l) \rightarrow (G_l = G)$$

$$(\sigma \text{ moves } l) \rightarrow (G_l < G)$$

$$272. \quad (a) \quad \mathcal{O}(g) = G$$

$$(b) \quad G_g = \{1\}$$

$$273. \quad (\forall g_i : |g_i| = p^n) \rightarrow (G_p)$$

#symmetric

$$274. \quad S_G = \text{symmetric group (all permutations of } G)$$

$$275. \quad V = \{(1), (12)(34), (13)(24), (14)(23)\} = \text{four permutations group} = \text{four-group} = \text{Klein group}$$

$$276. \quad \alpha_r \in S_n \quad \alpha_r \text{ is an } r\text{-cycle}$$

$$277. \quad |S_n|_r = \text{the number of } r\text{-cycle permutations in } S_n$$

$$278. \quad \sigma_r = r\text{-cycle}$$

$$279. \quad \tau_2 = \text{transposition}$$

$$280. \quad \alpha \in S_n$$

$$\alpha^{S_n} = \text{all the permutations in } S_n \text{ with the same cycle structure as } \alpha$$

$$281. X = \{1, 2, \dots, n\}, \quad \sigma \in S_n, \quad (G = \langle \sigma \rangle) \curvearrowright X, \quad i \in X$$

$$(a) \quad \mathcal{O}(i) = \{\sigma^k(i) : k \in \mathbb{Z}\}$$

$$(b) \quad |\mathcal{O}(i)| = r$$

- (c) $l = \text{symbol}$
 $(\sigma \text{ fixes } l) \rightarrow (G_l = G)$
 $(\sigma \text{ moves } l) \rightarrow (G_l < G)$

282. (a) $\forall G \leq S_X : G \circlearrowleft X$

(b) $G \cong S_X$

283. (a) $\alpha \rightarrow (G \xrightarrow{h} S_X, \ g \mapsto \alpha_g)$

(b) $(B : G \xrightarrow{h} S_X) \rightarrow (\beta : G \times X \rightarrow X, \ \beta(g, x) = B(g)(x) = \text{action})$

284. $(n \geq 3) \rightarrow (\exists! A_n \leq S_n : |A_n| = \frac{1}{2}n!)$

285. $(\sigma_5, \tau_2 \in S_5) \rightarrow (\langle \sigma_5 \tau_2 \rangle = S_5)$

286. (a) $(S \leq G_s, \ [G_s : S] = n) \rightarrow (G \cong S' \leq S_n)$

(b) $(|G_s| = \infty) \rightarrow (\nexists S : [G_s : S] = n > 1), \quad [G_s : S] \text{ is finite}$

287. (a) $(\forall G : |G| = 4) \rightarrow ((G \cong \mathbb{I}_4) \underline{\vee} (G \cong \text{four-group } V))$

(b) $\mathbb{I}_4 \not\cong V$

288. (a) $(|G| = 6) \rightarrow ((G \cong \mathbb{I}_6) \underline{\vee} (G \cong S_3))$

(b) $\mathbb{I}_6 \not\cong S_3$

289. *Representation on Cosets*

$(|H| = n) \rightarrow (\exists \varphi^h : G \rightarrow S_n, \ \ker \varphi^h \leq H)$

290. *Cayley*

(a) $\forall G : G \cong S \leq S_G$

(b) $(|G| = n) \rightarrow (G \cong S \leq S_n)$

291. $(\alpha_r \in S_n) \rightarrow (|S_n|_r \mid n!)$

#translation

292. $\tau_a = \text{translation}$

$$(a) \quad \tau_a : G \rightarrow G, \quad \tau_a(g) = ag, \quad \forall g$$

$$(b) \quad (a \neq 1) \rightarrow (\tau_a \neq \tau^h)$$

293. $\tau_a^x \equiv (\tau_a : x \mapsto ax)$

294. (a) $(G \curvearrowright G/H, \tau_a^{gH}) \rightarrow (\mathcal{O}(gH) = GH)$

$$(b) \quad G_g H = gH g^{-1}$$

295. $G \curvearrowright G/H$ by (left) *translation*.

296. $G \curvearrowright G$ by (left) *translations* $\tau_a : x \mapsto ax$