Continuous Probability Distributions over Second-Countable Spaces are Perfectly Supported

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Abstract

We prove that every Borel probability measure over an arbitrary second-countable space vanishing at any singletons has support being a perfect set and being included in some co-countable perfect set. Thus the support of a continuous probability distribution over a second-countable space turns out to admit a richer structure.

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Throughout, a second-countable space is always Hausdorff. By slightly abusing language, a Borel probability measure (or simply a probability distribution) over a second-countable space, in particular over a separable metric space, is called *continuous* if and only if its value is zero at every singleton. In a sense, working with second-countable spaces in the present context is the most general possible scenario. The theory of continuous probability distributions over Euclidean spaces $\mathbb{R}^n$ is well-developed and applied.

It is well-known (Theorem 2.1, Parthasarathy [2]) that every Borel probability measure $\mathbb{P}$ over a given separable metric space Ω may be uniquely assigned a “well-behaved” closed set, called the support $\text{supp}(\mathbb{P})$ of $\mathbb{P}$, i.e. a unique closed subset C of Ω such that i) $\mathbb{P}(C) = 1$, ii) $C = \cap\{F \subset \Omega \mid F \text{ is closed, } \mathbb{P}(F) = 1\}$, and iii) C equals the set of all $x \in \Omega$ such that $G \in \mathcal{T}_\Omega(x)$ implies $\mathbb{P}(G) > 0$; here $\mathcal{T}_\Omega(x)$ denotes the collection of all open subsets of Ω that contain x , that is, the collection of all neighborhoods of x . We extend the definition of support of Borel probability measures to second-countable spaces.

By a countable set we mean precisely a set that may be injected into $\mathbb{N} \equiv \{1, 2, \dots\}$. We show that the support of a continuous probability distribution over a given second-

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countable space is in fact also a perfect set, namely a closed set without isolated points, and is a subset of some co-countable perfect set:

Theorem. *If Ω is second-countable, in particular if Ω is a separable metric space, and if $\mathbb{P}$ is a continuous probability distribution on the Borel sigma-algebra of Ω , then $\text{supp}(\mathbb{P})$ exists, is perfect, and is included in some co-countable perfect set.*

Proof. We claim that Ω includes a co-countable perfect set. Indeed, the idea of Cantor-Bendixson construction (e.g. Theorem 6.4, Kechris [1]) applies. Upon taking a countable basis $\mathcal{T}_\Omega^0$ for the given topology $\mathcal{T}_\Omega$ of Ω , let

$$\mathcal{V} := \{V \in \mathcal{T}_\Omega^0 \mid V \text{ is countable}\}.$$

Then $\cup \mathcal{V}$ is countable, and $F := \Omega \setminus \cup \mathcal{V}$ is closed- $\mathcal{T}_\Omega$.

Since Ω is Hausdorff from assumption, every countable subset of Ω is a Borel subset of Ω . On the other hand, the probability measure $\mathbb{P}$ is by assumption a continuous probability distribution, and hence $\mathbb{P}(A) = 0$ for all countable $A \subset \Omega$. Since $\mathbb{P}(F) = 1$, it follows that F is an uncountable set. But then F has no isolated points, i.e. F is included in its derived set $\text{cl}'(F)$; for, if $x \in F \setminus \text{cl}'(F)$ then there is some uncountable $G \in \mathcal{T}_\Omega(x)$ such that $G \cap F = \{x\}$ and $G \cap (\Omega \setminus F) \subset \cup \mathcal{V} \setminus \{x\}$, which is impossible. Since F is then perfect, the claim follows.

The proof of Theorem 2.1 in Parthasarathy [2] also serves second-countable spaces without changes, and so $\text{supp}(\mathbb{P})$ exists (uniquely) and is such that

$$\text{supp}(\mathbb{P}) = \Omega \setminus \cup \{G \in \mathcal{T}_\Omega \mid \mathbb{P}(G) = 0\}.$$

Since $\mathbb{P}(V) = 0$ for all $V \in \mathcal{V}$, we have

$$\text{supp}(\mathbb{P}) \subset F.$$

There remains to show that $\text{supp}(\mathbb{P})$ is perfect. But if $x \in \text{supp}(\mathbb{P})$ and $G \in \mathcal{T}_\Omega(x)$ with $G \cap \text{supp}(\mathbb{P}) = \{x\}$, then

$$\mathbb{P}(G) = 0 + \mathbb{P}(G \cap (\Omega \setminus \text{supp}(\mathbb{P}))) = 0$$

by the assumed continuity of $\mathbb{P}$, which contradicts a defining property of $\text{supp}(\mathbb{P})$. Now that $\text{supp}(\mathbb{P})$ is perfect, the proof is complete. $\square$

References

- [1] Kechris, A. S. (1995). *Classical Descriptive Set Theory*, Springer.
- [2] Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces*, Academic Press.