Eppur ruota: modelling the ocean as a fluid with spin

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This note presents and motivates the idea of modelling the ocean on eddy-permitting and coarser scales as a continuum with spin.

The moment [Galileo] was set at liberty, he looked up to the sky and down to the ground, and, stamping with his foot, in a contemplative mood, said, *Eppur si move*; that is, *still it moves*, meaning the earth.

(Baretti 1757 p. 52)

1 Introduction

Modelling the behaviour of water at a resolution of, say, 4 mm – visible by the naked eye – employs quite a different mathematics than at a molecular-scale resolution of 0.2 nm. The very physical concepts used at these two resolutions are somewhat different. At millimetre resolutions, stress, temperature, and internal energy are extremely important, and long-range forces negligible. At such resolutions we use the Navier-Stokes equations. On the other hand, at nanometre resolutions, within a deterministic modelling, we don't use concepts like stress and temperature; sometimes we can do without internal energy; and non-contact forces dictate the dynamics. At such resolutions we use molecular dynamics.

This difference in mathematical and physical notions between the two models doesn't surprise us, because their resolutions are in a ratio of 20 000 000:1.

By the same token we shouldn't be surprised that modelling the behaviour of water at a resolution of, say, 80 km requires different mathematical *and* physical notions than at 4 mm. The two resolutions are also in a ratio of 20 000 000:1.

What's surprising, on the contrary, is that ocean modelling has so far managed to give a rough description of the behaviour of water at kilometre resolutions and above using the same fundamental balance

equations Navier-Stokes theory is built upon – linear momentum and energy – only with different constitutive (i.e., closure) equations for stress and internal energy (Gill 1982; Salmon 1998; Griffies 2004; Vallis 2006). Seeing the difference between the nanometre and millimetre cases, one would expect that *physical concepts not present in the Navier-Stokes theory* would be needed to describe water at geophysical resolutions.

In the present note I want to argue that modelling water at geophysical resolutions requires an additional physical notion and an additional prognostic equation: the notion of *internal rotational momentum*, or *spin* for short, and the *prognostic balance of rotational momentum*. The spin and its prognostic balance are not new: they are well-known in continuum mechanics for modelling particular materials and fluids, and have a sound physical motivation and derivation.

Rotational momentum is a fundamental physical quantity, and its balance a fundamental physical law, from subatomic to cosmological scales. This law is silently present in most fluid-dynamic models, including Navier-Stokes, as the requirement that the stress tensor be symmetric: $\mathbf{T}^\top = \mathbf{T}$. Combined with the balance of linear momentum, this requirement ensures the conservation of rotational momentum. For this reason it doesn't appear as a prognostic equation and we can simply forget about it.

The symmetry of the stress is a consequence of the fact that all rotatory motion of the fluid happens at the observed resolution, or that any unresolved rotatory motion does not affect the resolved motion, as we shall see shortly. Only in these two cases is the stress symmetry sufficient to guarantee the conservation of rotational momentum.

There are materials and fluids for which those two conditions do not hold, like liquid crystals, granular and porous materials, bone, blood flow (see e.g. de Gennes et al. 1993; Rey et al. 2002; and the dozens of references in Eremeyev et al. 2013 ch. 1). Most poignant to our discussion, those conditions often do not hold for turbulent flows either (e.g. Lanotte et al. 2016 and following papers). For some of these fluids the unresolved rotatory motion arises from the asymmetric shape of their molecules, but for others it is an eddying motion existing on microscales, the molecules themselves being roughly point-like. These unresolved eddies visibly affect the resolved motion, for example leading to an altered resolved viscosity or to enhancement of flows and jets (Peddieson 1972; Chaturani et al. 1984; Kirwan Jr. 1986; Siddiqui 2016; Ayano et al. 2018).

These kinds of materials and fluid cannot be satisfactorily modelled via standard continuum mechanics or fluid dynamics, not even with non-Newtonian and other general fluid models. They are modelled by the theory of polar materials or *polar fluids*, which has other appellations besides “polar”, like “micropolar”, “micromorphic”, “Cosserat”. (Cosserat et al. 1907; 1909)

Researchers in ocean modelling need not be reminded of the importance of eddying motion in ocean dynamics: it is the main actor in the transport of kinetic energy from resolved to unresolved scales, where it is dissipated, and from unresolved to resolved scales, where for example it reinforces jets (Holland et al. 1975; 1980; Vallis 2006 esp. ch. 10; Thorpe 2007). Obviously this rotatory motion exists at unresolved scales and affects resolved ones. It follows, from what we’ve said above, that to be continued

The ocean-modelling literature has been focusing on equations for stress and internal energy (Gill 1982; Salmon 1998; Griffies 2004; Vallis 2006), i.e. on the balances of linear momentum and energy. The balance of rotational momentum is rarely considered because for Navier-Stokes it reduces to the symmetry of the stress tensor, $\mathbf{T}^\top = \mathbf{T}$, usually automatically satisfied. Upon coarsening, however, this balance cannot be reduced to that symmetry condition (see e.g. Murdoch 2003; Pitteri 1990; Dahler et al. 1963). Coarsening is effected by a convolution, which does not commute with tensor products. This non-commutativity leads to an additional *intrinsic spin* in the balance for rotational momentum. It is the fine-scale rotational momentum that cannot be “seen” on the coarser scale.

2 Continua with spin

In the simplest continuum theories, the volume element at each location $\mathbf{r}$ of a fluid possesses mass (density) ρ , momentum $\rho \mathbf{v}$, temperature Θ , stress $\mathbf{T}$, external force $\mathbf{b}$, internal energy u , heat flux $\mathbf{q}$, entropy s , and any other quantity that can be derived from these, like stretching $\mathbf{D} := \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^\top)$, kinetic energy $\frac{1}{2}\rho \mathbf{v}^2$, vorticity $\mathbf{w}$. The volume elements are considered structureless, point-like. At each volume element the five fundamental balances of mass, momentum, energy, entropy, rotational momentum are satisfied:

$$\text{mass:} \quad \dot{\rho} = -\rho \nabla \cdot \mathbf{v}, \quad (1)$$

$$\text{momentum:} \quad \rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T}^\top + \rho \mathbf{b}, \quad (2)$$

$$\text{rotational momentum:} \quad \mathbf{T}^\top = \mathbf{T}, \quad (3)$$

$$\text{energy:} \quad \rho \dot{u} = \mathbf{T} : \mathbf{D} + \nabla \cdot \mathbf{q}, \quad (4)$$

$$\text{entropy:} \quad \rho \dot{s} \geq \nabla \cdot \frac{\mathbf{q}}{\Theta}. \quad (5)$$

Of these, those for mass (in the compressible case), momentum, and energy are prognostic, and the independent prognostic quantities are mass, velocity, and temperature or internal energy in one of its alter egos (e.g., available potential energy). The energy balance is sometimes replaced by an equation for the temperature, obtained from the former and the balance of entropy.

It is important to note that a fuller form of the balance of rotational momentum is

$$\mathbf{r} \wedge \rho \dot{\mathbf{v}} = \mathbf{r} \wedge \nabla \cdot \mathbf{T}^\top + \mathbf{r} \wedge \rho \mathbf{b} + \mathbf{T}^\top - \mathbf{T} \quad (6)$$

but the terms vector-multiplied by $\mathbf{r}$ vanish in virtue of the balance of momentum (2), yielding the reduced balance (3), which only requires the stress to be symmetric.

In the theories of continua with spin each volume element possesses an additional quantity: the *intrinsic rotational momentum* or *spin* $\mathbf{L}$, represented as a pseudo-vector or antisymmetric tensor. It expresses the angular momentum associated with unresolved rotatory motion occurring “within” the volume element. In the same fashion the *couple stress* $\mathbf{M}$ is also introduced, expressing the unresolved torques exerted within the volume element.

Spin and couple stress enter the equations of rotational momentum (6) and energy (4) on the same footing as the rotational momentum $\mathbf{r} \wedge \rho \mathbf{v}$ and the stretching $\mathbf{D}$, leading in the simplest case to new equations for rotational momentum and internal energy like these:

$$\text{rot. momentum:} \quad \rho k^2 \dot{\mathbf{L}} = \nabla \cdot \mathbf{M} - \boldsymbol{\varepsilon} : \mathbf{T}, \quad (3b)$$

$$\text{energy:} \quad \rho \dot{u} = \mathbf{T} : \mathbf{D} - \mathbf{T} : \boldsymbol{\varepsilon} \cdot (\mathbf{L} - \tfrac{1}{2} \mathbf{w}) + \mathbf{M} : \nabla \mathbf{L} + \nabla \cdot \mathbf{q}. \quad (4b)$$

where $\mathbf{w}$ is the vorticity.

The spin $\mathbf{L}$ is an independent *prognostic* quantity; therefore *the balance of rotational momentum (3b) becomes a prognostic equation*. The couple stress $\mathbf{M}$ is given by a closure equation, just like the stress $\mathbf{T}$.

A quick look at the momentum and energy balances (2), (4b) shows that the spin affects both via its own presence, via the stress tensor $\mathbf{T}$, and via the couple stress $\mathbf{M}$. The spin is moreover present in the closure (i.e., constitutive) equations for stress, heating, couple stress. A simple example (Cowin 1974; Faria et al. 1997) of a closure equation for the stress is

$$\mathbf{T} = -p + \lambda \operatorname{tr} \mathbf{D} + 2\mu \mathbf{D} + 2\zeta \boldsymbol{\varepsilon} \cdot (\mathbf{L} - \tfrac{1}{2} \mathbf{w}), \quad (7)$$

$$\mathbf{M} = -\alpha \nabla \cdot \mathbf{L}, \quad (8)$$

where $\zeta \geq 0$ is called the coefficient of rotational viscosity.

If we put eqs (1), (2), (3b), (4b), (7) ✂ to be continued

The spin might look like an exotic quantity, but it is actually very akin to internal energy and temperature. The latter give us information about the invisible kinetic energy of unresolved motions, which cannot appear in the resolved kinetic energy $\frac{1}{2} \rho v^2$; their prognostic presence in the equations allows the unresolved kinetic energy to affect the resolved flow. Likewise, the spin gives us information about the invisible rotational momentum of unresolved eddying motions, which cannot appear in the resolved rotational momentum $\mathbf{r} \wedge \rho \mathbf{v}$; its prognostic presence in the equations allows the unresolved rotational momentum to affect the resolved flow. Most theories of continua on human-size scales do not need spin because unresolved eddying motions are often negligible: rotations either cancel out or appear only on resolved scales. But if there are unresolved, uncancelling rotations, the spin becomes a necessary and useful quantity just like temperature and internal energy.

A summary of theories of fluids with spin is given by Faria & Kremer (1997) and Cowin (1974; 1968). There are also several textbooks available (Stokes 1984; Beris et al. 1994; Łukaszewicz 1999).

3 Why is spin needed in ocean modelling?

There are compelling physical reasons to model the ocean as a continuum with spin on eddy-permitting and coarser scales. There also are purely pragmatic reasons. I present both below.

3.1 Physical reason: coarsening and loss of rotational momentum

In all of physics, from subatomic to galactic scales, we have four additive, conserved physical quantities: mass, momentum, energy, and rotational momentum. Their total is preserved in our physical theories as we bridge scales and resolve less and less space and time.

Let an overbar $\bar{\cdot}$ over a quantity denote its value on a coarser scale.

The total mass within every volume is always preserved as we bridge scales. For example, the mass of a body of water in a glass, described by Navier-Stokes, is the total of the masses of its molecules. We can express this as $\int \rho \, dV = \int \bar{\rho} \, d\bar{V}$.

The total momentum within every volume is also preserved across scales. For example, the total momentum of a still body of water is practically zero on molecular scales owing to chaotic motion, and Navier-Stokes also assigns zero momentum to each coarse volume element. Water flowing in a pipe has a non-zero momentum on molecular scales, owing to an incomplete cancelling out of the chaotic motion; and Navier-Stokes assigns this leftover momentum to each coarse volume element. The velocity of a volume element on a coarse scale is defined as a weighted average, $\bar{v} := \frac{\bar{\rho v}}{\bar{\rho}}$, exactly for the purpose of preserving the total momentum: $\int \rho v \, dV = \int \bar{\rho} \bar{v} \, d\bar{V}$.

A first consequence of the preservation of mass and momentum across scales is that the balance of mass (1) assumes exactly the same form on all scales – no matter whether the mass density is discontinuous, on molecular scales; or continuous, on kilometric scales.

The momentum flux $\rho v \otimes v$ cannot be completely resolved on the coarse scale: $\overline{\rho v \otimes v} \neq \bar{\rho} \bar{v} \otimes \bar{v}$. But the unresolved difference is not lost: it is preserved in the stress tensor. For example, for a gas the stress represents,

for the most part, unresolved molecular momentum flux. In coarse ocean models the Reynolds stress, which augments the stress of millimetric scales, represents the difference of resolved and unresolved momentum flux. As a consequence the balance of momentum (2) assumes the same form on all scales. The values of its individual terms will be different on two different scales, but the overall momentum and flux are preserved.

The total internal + kinetic energy is also preserved across scales. For example, part of the kinetic energy of the molecular scale is retrieved in Navier-Stokes as internal energy; so the two parts are not separately preserved across scales, but their total is: $\int (u + \frac{1}{2}\rho v^2) dV = \int (\bar{u} + \frac{1}{2}\bar{\rho}\bar{v}^2) d\bar{V}$. In fact, for fluids the greatest part of the internal energy on millimetre scales represents the kinetic energy of molecular scales. In coarse ocean models the kinetic energy of unresolved motion is usually called “eddy kinetic energy” but it is actually a part of the internal energy (or of one of its alter egos like available potential energy).

The energy flux, like the momentum flux, cannot be completely resolved on a coarser scale, but the unresolved part is retrieved as heating flux and as work done by the stress tensor.

We have thus seen that coarse ocean models preserve the same total mass, momentum, and energy that are present in molecular and millimetric scales.

Is the same true for rotational momentum?

The total rotational momentum within every volume ought to be preserved across scales, too. Navier-stokes theory actually does *not* do this. The reason is that in most cases the total rotational momentum within millimetric volumes vanishes, owing to molecular chaotic motion. Even in cases with a net overall momentum, for example water flowing in a pipe, the rotational momentum will still cancel out. In these cases the total rotational momentum within a volume is therefore well represented by the momentum of the resolved motion: $\overline{\vec{r} \wedge \rho \vec{v}} \approx \vec{r} \wedge \bar{\rho} \bar{\vec{v}}$. The same holds true for the flux of rotational momentum. As a consequence, Navier-Stokes theory uses an approximate balance of rotational momentum that only consider resolved objects: so to speak

$$\vec{r} \wedge \frac{d}{dt}(\text{resolved momentum}) = \vec{r} \wedge (\text{resolved forces}), \quad (9)$$

which is eq. (6).

There are situations, however, in which non-zero rotatory motions appears on the finer scale, and the resolved rotational momentum

cannot capture the full rotational momentum: $\overline{\mathbf{r} \wedge \rho \mathbf{v}} \neq \bar{\mathbf{r}} \wedge \bar{\rho} \bar{\mathbf{v}}$. In these situations the Navier-Stokes balance of rotational momentum $\mathbf{T}^\top = \mathbf{T}$ does not reflect the physics, yielding incorrect predictions for the flow. This happens for example for particular fluids like polymers, or in very turbulent regimes. It is in these cases that the theory of continua with spin is used.

The spin $\mathbf{L}$ represents the difference between resolved and unresolved momentum: $\mathbf{L} := \overline{\mathbf{r} \wedge \rho \mathbf{v}} - \bar{\mathbf{r}} \wedge \bar{\rho} \bar{\mathbf{v}}$, and the couple stress $\mathbf{M}$ the difference between resolved and unresolved torques. Their presence guarantees the preservation of rotational momentum across scales. Conversely, their absence may mean that part of the physics has been lost upon bridging scales.

3.2 Pragmatic reason: loss of rotational momentum in ocean models

A fundamental characteristic of oceanic flow is the presence of eddying motions across all scales, from metres to kilometres. The rotational momentum of such eddying motions does *not* cancel out as in the molecular case. A net non-zero rotational momentum survives across coarser and coarser scales. This eddying motion is the main actor in the transport of kinetic energy from coarser to finer scales, where it is dissipated, and from finer to coarser scales, where for example it reinforces jets.

Unfortunately the modelling via Navier-Stokes equations does not preserve this net rotational momentum.

A pictorial illustration is worth a thousand words. Take a look at fig. 1: on the left we have water flow in a two-dimensional region represented by some finite-difference or finite-element numerical scheme. The flow has a net eddying motion; the net momentum in the dashed subregion has vector value $\rho \mathbf{v} = (5, 2)$ in some units, and rotational momentum $\mathbf{r} \wedge \rho \mathbf{v} = 19$ with respect to the centre represented by a thick dot (being a pseudo-vector, it has only one component in two dimensions).

On the right the same flow is represented on a slightly coarser scale, which cannot resolve motions in the previous dashed subregion. It is easy to calculate that the total momentum of the dashed area is nonetheless preserved: $\bar{\rho} \bar{\mathbf{v}} = (5, 2)$. The rotational momentum, however, is not: $\bar{\mathbf{r}} \wedge \bar{\rho} \bar{\mathbf{v}} = 14$, which means that 25 % of it has been lost on the coarser scale. If we coarsen the flow even further, so that the motion in

Figure 1 Loss of rotational momentum upon coarsening. The arrows indicate the total momentum within each cell. Upon coarsening, the total momentum is the dashed region is preserved, but 25 % of the rotational momentum, calculated with respect to the thick dot, is lost. A further coarsening would still preserve the (vanishing) total momentum but completely lose the (non-vanishing) total rotational momentum.

the whole region is unresolved, the total momentum is still preserved (vanishing in this case), but the rotational momentum is *completely lost*.

The situation is even worse when time evolution is taken into account: there is a build-up of loss of rotational momentum, because eddies are known to survive, with the same chirality, for times longer than the temporal resolution of numerical models.

It is important to realize that the rotational momentum that is lost on coarser scales is *not* and *cannot* be retrieved by the balances of mass, momentum, or energy of Navier-Stokes theory. The spin balance (3b) shows that the use of a symmetric stress is equivalent to saying that no rotational momentum is lost from the unresolved scales – a clearly unphysical assumption. The use of a non-symmetric stress (e.g., Marshall et al. 2012; Maddison et al. 2013) can recover part of the lost rotational momentum, because it implies that the spin tendency on the left side of the balance (3b) does not vanish. Yet, models with a non-symmetric stress relegate the spin-balance equation to a *diagnostic* role only, in which the momentum decides, via the stress, how much spin is lost or gained. Not even the energy can directly feedback on the spin, because only the symmetric part of the stress appears in its balance (4). But it's the opposite feedback that's physically more important: the spin determines new kinds of transfers of momentum and energy. This is possible only

if it is included as an additional prognostic variable and in closures equations like (7).

Figure 2 shows the rotational momentum lost upon coarsening a double-gyre flow from 7.5 km to 30 km in a square domain size of about 4000 km. The lost rotational momentum is of the same order of magnitude as the large-scale relative vorticity – this is quite telling. The greatest loss occurs along the jet.

In its prognostic role, the spin makes allowance for new mechanism of transport of momentum and energy across space and across scales via the balances (2), (3b), (4b).

For example, the resolved momentum in an eddying motion can be converted into spin, corresponding to unresolved eddying motion – unresolved “filaments” of vorticity – and in this form be transported by the flow to other spatial regions. There it can be either dissipated onto molecular scales (temperature) or re-transferred to resolved scales. Likewise, the kinetic energy of resolved eddying motion can be converted into spin energy – the term $\mathbf{T} : \mathbf{L} + \mathbf{M} : \nabla \mathbf{L}$ in the energy balance (4b) – and transported to other regions. There it can be dissipated as heat but also be partially re-converted into resolved kinetic energy. Analytical and numerical studies have shown that fluids with spin have enhanced velocities and jets with respect to Navier-Stokes fluids

4 Advantages of an ocean model with spin

The idea of an analytic and especially numerical implementation of an ocean model with spin seems to be virtually unexplored in ocean modelling. The only studies vaguely related to ocean modelling I know of are Peddieson’s (1972) and Iovieno et al.’s (2002). The idea of an ocean model with spin has several attractive points:

- The physics behind the idea is physically compelling, as shown in the preceding sections.
- Theories of fluids with spin have already been developed, from very simple to very complex, so their prognostic equations and closures are already out there.
- Numerical codes for fluids with spin have also been developed (Chen et al. 2010; 2012), so there is already a literature explaining, for

Figure 2 Snapshot of spin, i.e. lost angular momentum, in coarsening a double-gyre flow (with eastward jet) from 7.5 km to 30 km resolution (Porta Mana et al. 2014); domain size is 3840 km. The spin has an order of magnitude of $500 \text{ m}^2/\text{s}$ corresponding to an angular velocity of $3.5 \times 10^{-6} \text{ Hz}$, comparable to the large-scale relative vorticity. The greatest loss is along the jet; this gives strength to the hypothesis that the spin would have an important role in jet formation and maintenance.

example, the kinds of boundary conditions required by a numerical simulation with spin.

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(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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