Rippling 3-Riemannian structure describing gravity with dark matter effects

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Abstract

In an attempt to solve the missing mass problem, the paper introduces a probabilistic three-dimensional structure which is locally described by energy density, time density and a Riemannian metric. This proposition has its roots in the results of general relativity and quantum theory. On large scale, source mass binds energy density which causes curvature in the Riemannian manifold of space measure leading to variations in length and time *scales*. Additional gravitational effects are predicted for a source mass which are caused by the flow of bounded energy density and is proposed as a candidate for ‘dark matter’ model. The paper makes testable predictions some of which may have already been observed as ‘dark matter’ or ‘dark energy’.

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1 Introduction

The missing mass problem was first noted by Fritz Zwicky way back in 1930s [1] which led to the understanding that the most mass of galaxy or galaxy clusters is non-luminous eventually leading to the term ‘dark matter’. There is undeniable evidence for effects observed under the head ‘dark matter’ especially on largest scales. There have been two approaches to solve this problem, non-baryonic matter and modified laws of gravity. The hypothesis of existence of cold, weakly interacting non-baryonic particle [2, 3] is subject to the correctness of general relativity (GR). However, GR is well-established experimentally within the weak conditions of solar system [4] and predictions of existence of black holes and gravitational waves [5] have also been confirmed. So far, it cannot be tested beyond the gravitational field strength larger or smaller than scales within solar system and has remained incompatible with quantum theory. Therefore, validation of GR can be contested in conditions on largest scales or on smallest scales and because 95 percent of the universe consists of unknown matter or energy initially unaccounted for in the theory. MODified Newtonian Dynamics (MOND) [6, 7, 8] is presented as an alternative and proposes to replace ‘dark matter’ by explaining Mass Discrepancy Acceleration Relation (MDAR) [9, 10]. Recently, Super-fluid dark matter (SFDM) model was presented [11, 12] that describes a scalar particle that condenses into a superfluid. However, these models have limitations on largest scales.

According to GR, spacetime is featureless and has no intrinsic properties. It can curve and bend like a differentiable manifold or surface and is a dynamical stage on which physical phenomenon takes place. The theory has provided incredible development in our understanding of space and time where both are assumed to be dimensions with no physical rooting but this has served the theory well. Yet, there remains an enigma around understanding time and emergence of arrow of time [13]. Results in general relativity are limited to geometry of spacetime and its effects on matter. Consequently, any physical phenomenon has to be captured in terms of metric of spacetime. This is simple and elegant but limiting as well.

The mystery of the accelerated expansion [14, 15] of universe is mathematically resolved by non-zero cosmological constant but what is sourcing it and why it has unusually small value are still open questions. Negative mass spanning the interstellar space is discussed [16] as a source of ‘dark energy’ accelerating the universe and alternatively, SFDM models unify dark energy and dark matter. However, both models are found to have limitations. Other models propose dark matter-dark energy interactions [17, 18] as possible source.

In this paper we provide a novel theoretical framework towards solving the missing mass problem. The framework assigns energy to a three-dimensional structure, Kaal (“dark”), spanning space i.e. space is not featureless. This assumption leads to emergence of local length and time *scales* in correspondence with results of general relativity and further predicts common origin for the observed phenomenon attributed to dark energy and dark matter. In case of rotating source mass, additional gravitational effects are predicted to be caused by energy bounded by the source and the flow of energy from equatorial planes to the poles along axis of rotation. The model bears similarity with SFDM model in principle.

The central idea of the paper is that variations in local *scales* of space and time are symmetric

and orthogonal; emerge from flow and distribution of energy density in a local patch of Kaal structure. The part of spacetime metric in general relativity that reflects non-orthogonality of space and time coordinates i.e. motion of source mass is captured by dragging of *Kaal Fluid* of energy in the proposed framework. The paper relies on the distinction between scales and measurements; while locally scales are constant, measurements can be arbitrary. This paper is about emergence of local *scales* that enable measurements by an observer. Thus, the present framework does not override any existing theory rather may assist or complement in furtherance of current understanding and interpretation.

1.1 Outline of this work

The paper is organized as follows. In section 2, we state assumptions that govern the framework of Kaal structure. These find their roots in special and general theory of relativity and quantum theory. In section 3, using these assumptions, analysis is performed on a local patch of Kaal structure to establish correspondence with special relativity. This distinguishes the character of Kaal structure in inertial and non-inertial frames of reference. In section 4, generalization of local analysis is done on large scale and curvature of 3-Riemannian manifold of space measure is equated with the source mass energy. Subsequently, Einstein's Reduced Equation (ERE), $R = 4\pi G\rho_s$, is introduced that is used to locally define space and time *scales* at any point near a source mass. In section 5, to further establish correspondence with tests of general relativity, ERE is used to discuss the solutions in Schwarzschild conditions. For rotating source masses, general equations are presented and discussed in context of metric solutions in general relativity for rotating sources. In section 6, there is a brief discussion on how time is considered in the present framework. In section 7, key result, prediction of additional gravitational effects that are pushing-in towards the source is presented as a candidate for 'dark matter'. Finally, conclusion and additional results are discussed in section 8.

2 Definitions and Assumptions

2.1 Kaal

Kaal ($\check{\mathcal{N}}$) is a three-dimensional structure, each point of which is characterized by three entities: Kaal Energy Density (KED, κ), Metric of Riemannian 3-Manifold (g_{mn}) and Time Density (τ). All three entities co-exist and neither one is independent of the other.¹

2.1.1 Kaal Energy Density (KED)

In classical theories, energy is a scalar, abstract property associated with matter. Rest energy, kinetic energy, potential energy etc. - all forms of energy are known to be associated with matter. And in quantum theories, energy is described by wave-function of the system. We take

¹This is not a 3-Riemannian manifold. It is rather like a fluid filling 3-dimensions of flat space. Metric (g_{mn}) signifies variations in local length scales, elaborated later.

an alternative description though similar to quantum theory and postulate: *Energy is a property of Kaal, not of matter; matter interacts with Kaal to produce physical phenomenon.* What we conventionally consider as the energy of matter can now be thought of as energy of a structure spanning empty space. KED (κ) is a positive definite quantity.

Kaal Ripples (KRs) is the flow of energy through Kaal structure and determine the amount of KED at any given point. Variations in KRs travel at speed of light (c). Presence of mass either drags² KED or slows down³ KRs. In absence of effects of matter, Kaal Ripples always flow from high to low KED regions.

No Rest, All Motion: Energy is always flowing i.e. Kaal Ripples exist at all times at all points. It will be discussed later, KRs correspond to motion of mass and thus this is a fair assumption to make in view of constant motion of the galaxies. Kaal is thus a dynamic structure; imagine space filled with *dark light*. The density and flow of KED correspond to time and space *scales* respectively.

2.1.2 Metric of Riemannian 3-Manifold

Kaal Ripples define the *scales* for length and time measurements at any point. The *scale* for space is described by three-dimensional Riemannian manifold (notation: $\mathcal{RIE}^3$) [19] with a metric (g_{mn}). The manifold is locally Euclidean and diffeomorphic to $\mathbb{R}^3$ i.e. length *scale* is constant locally.

Space is flat. The curvature is in local “measurement” of space.⁴

As Kaal Energy Density $\kappa \in (0, \infty)$, the measure of space also ranges from $(0, \infty)$ i.e. there exist infinite diffeomorphisms [20] of any patch $U \subset \mathcal{RIE}^3$.

2.1.3 Time Density: Rate of Passage of Time

The Kaal Energy Density (KED) at any point determines the Time Density which is the rate of passage of time, or vice-versa. *Definition:* Higher the KED is at any point, higher is the rate of passage of time i.e. clocks run slow or less ticks will read the same duration of time compared to say clocks at infinity. In comparison with results of general relativity, this means KED is higher near a gravitating source. This is analogous to and has its root in gravitational red shift phenomenon.

2.2 Coequals: Variations in Space and Time Scales

The *scales* for time and space measurements are coequals in Kaal theory i.e. variations in units of length and time are symmetric. Evaluating variations in space components can give us

²on large-scale

³on small scale, will not be discussed in this paper.

⁴Measurements can change either if the space being measured changes and scales remain constant or if the scales making the measurements change while space remains flat. If one use light rods and light clocks to measure lengths and time, the scales would vary giving an impression of stretching or squeezing of space and time measurement.

complete information about the variations in time scale, however inverse argument is not valid, as time cannot be resolved into three components. This also means, in regions free from matter, Kaal Ripples locally flow with equal speed with respect to space and time coordinates (in units $c = 1$). This assumption has roots in the symmetric nature of proper time and proper length across special and general relativity [21, 22].

2.3 Dynamics: Mass and Kaal

2.3.1 Curvature

Mass binds (collects, pulls, increases) KED in its neighborhood which results into curvature in the three-Riemannian manifold ($\mathcal{RIE}^3$) of space measure and increase in Time Density. If KED can be ignored, this is equivalent to curvature in static spacetime in general relativity. On large scale, KED contributes to gravitational effects along with the source mass and Kaal structure acts like a fluid in its neighborhood.

2.3.2 Motion

While mass binds KED in its neighborhood, motion of mass slows down Kaal Ripples with respect to space coordinates effects of which is most observable on small scales. On large scale, this is equivalent to considering dragging of Kaal ‘fluid’ in the neighborhood of source mass. The analysis is analogous to fluid dynamics, where the mass density of the fluid is equivalent to KED. This is discussed further in 5.4.

2.4 Minimizing (Repulsive) Nature

In regions free from influence of matter, KED tends to minimize itself i.e. Kaal structure expands⁵. This expansion is continuous and monotonically decreasing⁶. This is a naturally irreversible phenomenon unless intervened upon externally. This means a system tends to reduce its Time Density i.e. a system always tends to minimize energy. In sloppy terms, we say a system wishes to be free of time and tends to be time-less. This postulate has its roots in entropy and arrow of time.

2.5 Kaal - Mass Equivalence

This follows immediately from mass-energy equivalence in special relativity [23]. KED is equivalent to mass by the same relation, $E = mc^2$, where E corresponds to Kaal Energy Density (KED) at a point in $\tilde{\mathcal{N}}$. This equivalence is useful in large scale analysis where the KED bounded by a source mass is equivalent to additional mass that would contribute to gravitational effects

⁵As wavelength of Kaal Ripples will expand to minimize energy.

⁶If two particles have sufficient KED in between them such that in the process of minimization of KED, volume between them increases, the particles would find each other in relative motion.

apart from the energy/mass of the source. Further, this allows for considering the Kaal in the neighborhood of a source as a ‘fluid’ and is further discussed when evaluating rotations.

3 Local Analysis

Local analysis is discussed to establish correspondence with special relativity. It is abstract and is *not* a substitution to special relativity rather *complementary* to results of special relativity. This is not a proposal to replace any existing mathematics. One can say it is based upon the symmetric nature of proper length and proper time.

3.1 Local Kaal structure

As postulated, the Riemannian 3-Manifold ($\mathcal{RIE}^3$) of Kaal structure ($\check{\mathcal{N}}$) is locally Euclidean and diffeomorphic to $\mathbb{R}^3$. Here, it is convenient to consider Kaal Energy Density (KED, κ) as a scalar field over Euclidean flat three-space. As $\mathcal{RIE}^3$ is locally flat, this implies KED is a constant⁷ scalar field i.e. KED is not a function of any coordinates. This also implies Kaal Ripples are consistent and Time-Density is constant i.e. length and time *scales* are constant.

3.1.1 Inertial and Non-Inertial Frames

Consider regions $U, V \in \check{\mathcal{N}}$ such that KED in U and V is some constant function, κ_U and κ_V respectively with respect to some arbitrary observer in U region. This implies U, V are local regions in $\check{\mathcal{N}}$. Such regions in $\check{\mathcal{N}}$ with constant KED are equivalent to inertial frames of reference. Kaal Ripples in such regions flow with consistent structure i.e. with constant wavelengths and time period. Thus, time and space *scales* remain constant i.e. Time Density and metric of $\mathcal{RIE}^3$ are constant.

Then, inertial frames of references are defined as diffeomorphisms of a local region of Kaal structure that preserve the local structure. Because κ_U and κ_V are some constants i.e. U, V have local structure, these qualify as inertial frames of reference.

Similarly, if in a certain region of $\check{\mathcal{N}}$, KED is some non-constant function of space and time parameters then such regions correspond to non-inertial frames of reference. The structure of Kaal Ripples is also a function of these parameters and so are the local space and time *scales*.

3.2 Equations and Solutions

In this section, the effects on the Kaal structure due to mass of test particle/observer are neglected. In regions free from effects of matter and its motion, Kaal Ripples flow with equal speed with respect to space and time parameters (when $c = 1$). Locally, the description of Kaal

⁷Neglecting the effect of presence of matter and gravitating effects of KED.

Ripples is consistent with the equations,

$$|\nabla\Omega| = \left|\frac{\partial\Omega}{\partial t}\right| \quad (1)$$

$$\nabla^2\Omega = \frac{\partial^2\Omega}{\partial t^2} \quad (2)$$

where ∇ is the gradient operator in the adopted spatial coordinates. The relationship described by (1) is the fundamental local *invariant* identity in Kaal theory. Wave-like motion of Kaal Ripples is assumed in setting up the above equations as such a motion (flow) of energy can accommodate $(N+1)$ parameters in N dimensional space. Also, waves are ubiquitous in physics.

General solution of (2) is known to be a superposition of left and right travelling waves parameterized by (t, x^1, x^2, x^3)

$$\Omega = f(\vec{k} \cdot \vec{r} \pm \omega t) \quad (3)$$

where $(\vec{k}, \omega)$ indicate the wave-number ($\equiv$ metric) and frequency ($\equiv$ KED and Time Density) of Kaal Ripples at the point of interest. These are also equivalent to momentum and energy of Kaal Ripples respectively.

Ω is the *probability amplitude* of locating KED described in (3) at the point of observation. For inertial frames of reference i.e. regions in $\check{\mathcal{N}}$ with constant KED, the solution Ω is a plane-wave spanning such region. Thus, the probability of locating KED at any point in the region is unity as expected. On large scale, these ripples in Kaal structure generally do not affect the physical phenomenon but KED does; however on small scale, rippling plays a critical role not being discussed in this paper. ⁸

Based on above, (1) is equivalent to

$$|\vec{k}| = |\omega| \quad (4)$$

which is true for Kaal Ripples and light (classically) at all points in regions free from effects of matter. Kaal Energy Density (KED) is formulated as

$$\kappa = H\omega \quad (5)$$

where H is some constant.

This leads to considering a wave-function like entity beyond quantum scale which is not smooth like a sheet of paper but rippling like the ocean. It is like a corrugated sheet which on a large scale can be assumed smooth but can curve like a sheet of paper does; on smaller scales, corrugations affect the motion of particles.

⁸ Ω is not identical to quantum wave-function Ψ though they can be considered similar. Differences lie in mathematical treatment of the quantum phenomenon where Ψ can be considered as mathematical package of interactions between matter properties and Kaal structure. Also, because of multiple states a system can exhibit. Description of quantum systems is not discussed in this paper.

Consider the following solution for (3),

$$\Omega = \exp(i\vec{k} \cdot \vec{r} + i\omega t) \quad (6)$$

to work with hereon. This is a plane wave solution [24] with $|\Omega| = 1$ at all points as expected. In large scale analysis, this solution suffices as only wave-vector $\vec{k}$ and frequency ω are enough to define local space, time *scales* and Kaal Ripples are not significant.

3.3 Diffeomorphic Transformations

On the Kaal structure $(\check{\mathcal{N}})$ consider two local regions U, V with flow of Kaal Ripples defined as Ω_U, Ω_V respectively. U, V can be thought of as local patches on a manifold as in theory of differentiable manifolds. Clearly, Ω is parameterized by three spatial coordinates $(x^1, x^2, x^3) \in \mathbb{R}^3$ and one time-coordinate, $t \in \mathbb{R}$.

Let us consider two inertial frames of reference Σ, Σ' such that Σ' is moving with relative constant velocity, v with respect to Σ along one coordinate axis. Let the regions U, V correspond to these inertial frames Σ, Σ' respectively. Consider the solutions for Kaal Ripples in U, V as

$$(U, \Sigma) : \exp(i\vec{k}_U \cdot \vec{r} + i\omega_U t) \quad (7)$$

$$(V, \Sigma') : \exp(i\vec{k}_V \cdot \vec{r} + i\omega_V t) \quad (8)$$

with $|\vec{k}_U| = |\omega_U|$ and $|\vec{k}_V| = |\omega_V|$. From special relativity, clocks in Σ' run slower with respect to clocks in Σ i.e. Time Density is higher in (V, Σ') with respect to Time Density in (U, Σ) i.e. Kaal Energy Density (KED) is higher in V with respect to KED in U i.e.

$$\kappa_V > \kappa_U$$

Without loss of generality, using the time dilation (or length contraction) results from special relativity, it is evident time (or length) scales must be governed by

$$\kappa_V = \gamma \kappa_U \quad (9)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. This implies

$$\omega_V = \gamma \omega_U \quad (10)$$

For the ease of understanding, let us assume the Kaal Ripples are propagating in the same direction, r in U, V . Thus we can write

$$\kappa_V = \gamma \kappa_U \quad (11)$$

such that, in the units with $c = 1$,

$$\frac{\omega_V}{\kappa_V} = \frac{\omega_U}{\kappa_U} = 1 \quad (12)$$

and this is equivalent to two regions $U, V \in \tilde{\mathcal{N}}$ having a *diffeomorphic mapping* between them

$$N_{UV} : U \rightarrow V \quad (13)$$

$$N_{UV} : \exp(i\kappa_U r + i\omega_U t) \rightarrow \exp(i\kappa_V r + i\omega_V t) \quad (14)$$

The diffeomorphic map N_{UV} leads to Lorentz transformations between Σ and Σ' .

Result. Local analysis relates the space, time *scales* of two arbitrary inertial reference frames which is captured by proper length, proper time in results of special relativity. The analysis in this section does not relate arbitrary space, time coordinates measurements between two reference frames that is already captured by Lorentz transformations rather the relation between length, time *scales* is discussed in the framework of Kaal structure.

4 Large Scale Analysis

4.1 Discussion

To completely determine the three entities of Kaal structure - Kaal Energy Density (KED), Metric, Time Density - understanding of metric of 3-Riemannian manifold ($\mathcal{RLE}^3$) is necessary and sufficient. The metric determines the local *scale* of space with respect to flat conditions and consequently, Time Density and KED can be arrived at.

Locally, KED and Time Density are constant. However, in the generalized description, the variations in Kaal structure are taken into consideration which leads to changes in KED, space and time scales. If KED is ignored in large-scale description and only the variations in space and time scales are considered, general relativity applies. However, if KED is to be considered, then general relativity is an approximation and it would require additional source of gravitation such as some unidentified source of matter or energy to explain the observations.

4.1.1 Einstein's Field Equations (EFEs)

Field equations of general relativity [25] are stated here for reference. Greek scripts are used for general relativity equations.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (15)$$

which reduce to the following form for vacuum conditions,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \quad (16)$$

4.1.2 Bounded Kaal Energy (BKE)

Bounded Kaal Energy (BKE) is the distinction between general relativity and Kaal theory on large scale. From section 2.3, we know that higher the KED, higher is the Time Density and

that mass binds KED in its neighborhood causing curvature in $\mathcal{RIE}^3$. If we compare these with results of general relativity, it is clear that KED increases as one moves towards the source mass i.e. Kaal structure is denser closer to the source.

Bounded Kaal Energy (BKE) is *defined* as the cumulative amount of KED enclosed between the source mass and the point of interest. Consider a spherical source mass, say a star. If an observer wishes to know the Kaal structure at radial distance, r from the star, the KED enclosed in the spherical shell volume extending from surface of the star to r is BKE.

In general relativity, if space-time curvature is to be determined at a radial distance r outside of the source mass, EFEs are solved for vacuum conditions i.e. $T_{\mu\nu} = 0$. However in the present framework, apart from the source energy, BKE is also the source of gravitation at r i.e. there are *no vacuum conditions* in Kaal theory.

By mass-energy i.e. Kaal-Mass equivalence, BKE can be treated as a fluid. At any given point in this *Kaal Fluid*, mass density is equivalent to KED. Kaal Fluid is dragged by motion of mass or fundamentally, the flow of Kaal fluid governs motion of mass on large scale.⁹

4.2 Einstein's Reduced Equation (ERE)

4.2.1 Generalization

The invariance that describes flow of Kaal Ripples in free regions locally ($|\vec{k}| = \omega = \text{constant}$) captured by (1) and (2) is reproduced here,

$$|\nabla\Omega| = \left|\frac{\partial\Omega}{\partial t}\right| \quad (17)$$

$$\nabla^2\Omega = \frac{\partial^2\Omega}{\partial t^2} \quad (18)$$

These equations are required to be satisfied and generalized [26] over large scale structure. On large scale, ($|\vec{k}| = \omega \neq \text{constant}$) i.e. the character of Kaal Ripples vary and so does the space and time *scales*. Because $\vec{k}$ is not a constant on large scale, the corresponding *scales* of lengths vary and correspond to the line element of 3-Riemannian manifold $\mathcal{RIE}^3$. Lengths vary not because space expands or contracts but because *scales* expand or contract. Space is flat and infinite; Kaal structure expands or contracts.

Clearly, there is strong correlation between $\vec{k}$ and the metric g_{mn} of $\mathcal{RIE}^3$. Thus, the problem hereon is to determine the metric g_{mn} of $\mathcal{RIE}^3$ and consequently arrive at KED and Time Density.

Ricci Curvature Tensor. For 3-Riemannian manifolds, *Ricci Curvature Tensor*, R_{ij} contains necessary and sufficient information about the curvature at any point and is defined as

$$R_{ij} = R_{ijk}^k = \frac{\partial\Gamma_{ik}^k}{\partial x^j} - \frac{\partial\Gamma_{ij}^k}{\partial x^k} + \Gamma_{ik}^r\Gamma_{rj}^k - \Gamma_{ij}^r\Gamma_{rk}^k \quad (19)$$

⁹Interactions with other properties of matter such as charge is not considered in this paper. On small scale, Kaal Ripples govern motion of particles. Not discussed in this paper.

$$R_j^i = g^{ik} R_{kj} \quad (20)$$

Contracting the Ricci Curvature Tensor, we obtain the *Ricci Scalar Curvature*, R as

$$R = R_i^i = g^{ij} R_{ij} \quad (21)$$

Source of Curvature. Kaal theory separates energy from matter and considers energy as property of Kaal structure $\tilde{\mathcal{N}}$. Mass can also be considered as “bounded energy” by the Kaal-Mass equivalence. Thus, in the present framework, source of curvature in $\mathcal{RLE}^3$ i.e. any change in KED (κ) or Time Density (τ) must be energy itself. Unlike the source in general relativity which is a rank-2 energy-momentum tensor $T_{\mu\nu}$, the *source in Kaal theory is a scalar quantity, energy*.

Coordinate-independent formulation is still part of this framework as the source of curvature is now a rank-0 tensor i.e. a scalar which is independent of coordinate system adopted.

Equations. From the postulates, mass binds energy i.e. KED increases in the neighborhood of a source mass. This causes curvature in $\mathcal{RLE}^3$ and increase in Time Density (τ). Therefore, curvature in $\mathcal{RLE}^3$ must be related with source mass. However, the source of curvature is a scalar (energy), thus the relation must involve Ricci Scalar Curvature, R and be of the form

$$R = a\rho_s \quad (22)$$

where ρ_s is the source energy and a is some constant.

In weak conditions, where the contribution of Bounded Kaal Energy (BKE) to the source term can be ignored (Schwarzschild solution), present results must correspond with the results in general relativity when spatial hyperspace is orthogonal to time vector in (1+3) spacetime [27]. In such conditions, the equation (22) relating scalar curvature and source energy must take the form¹⁰ as

$$R = \frac{4\pi G}{c^4} \rho_s \quad (23)$$

where c is speed of light. In units with $c = 1$, this is

$$R = 4\pi G \rho_s \quad (24)$$

The numerical part in this equation is inconsequential as it can be shifted to left hand side and made part of the definition of the metric. This equation (24) is termed as *Einstein's Reduced Equation (ERE)* for Kaal theory.

4.3 Dynamics

The present framework describes the relationship between energy, space and time. Therefore, any description of motion of a particle or source mass is essentially a description of *distribution* and *flow* of Kaal Energy Density (KED). There are two conditions in which motion should be analyzed-

¹⁰Also satisfies Dimensional Analysis.

1. **Rest Energy Dominates:** In such conditions, space and time *scales* vary, KED is not constant i.e. variations in Kaal structure contribute significantly in mass dynamics. Motion of source mass through $\mathcal{N}$ must consider (κ, τ, g_{mn}) at each point in $\mathcal{N}$. If BKE is ignored and only variations in space and time *scales* are considered, equivalent model is (1+3) pseudo-Riemannian spacetime.
2. **Kinetic Energy Dominates:** In these conditions (generally small scale), the structure of Kaal Ripples effects the motion of the particle; distribution and flow of KED determines particle dynamics. These conditions are not discussed in this paper.

Condition 1 is partially considered in general relativity where gravitational effects of BKE are ignored. If these are considered, it implies the right hand side of the ERE (24) never vanishes. Thus, large scale experimental results done with general relativity as complete framework will lead to existence of either additional form of matter or energy.

5 Solutions in Large Scale Analysis

5.1 From Metric to KED and Time Density

How does the metric g_{mn} lead to evaluating KED and Time Density at a point in Kaal structure? Consider a metric solution to ERE (24) given by the following line element in some locally orthogonal coordinates (x^1, x^2, x^3) on $\mathcal{RIE}^3$ near some source mass,

$$ds^2 = p(x^1, x^2)(dx^1)^2 + q(x^1, x^2)(dx^2)^2 + r(x^1, x^2)(dx^3)^2 \quad (25)$$

where p, q, r are functions of (x^1, x^2) . It can be thought of having symmetry in x^3 -plane. The metric g_{mn} then looks like

$$g_{mn} \equiv \begin{pmatrix} p(x^1, x^2) & 0 & 0 \\ 0 & q(x^1, x^2) & 0 \\ 0 & 0 & r(x^1, x^2) \end{pmatrix} \quad (26)$$

and clearly, metric is also a function of (x^1, x^2) and independent of x^3 -coordinate. At a distance far from the source mass, say $x^1 \rightarrow \infty$, say the metric corresponds to flat space metric.

Since metric is a function of (x^1, x^2) only, so when an observer moves through $\mathcal{N}$ any variation is expected only in (x^1, x^2) plane locally with respect to flat conditions. Because variations can only be carried by variations in Kaal Ripples i.e. flow of energy, it can be assumed Kaal Ripples flow only in (x^1, x^2) plane locally. This further implies that KED (κ) and Time Density (τ) are function of (x^1, x^2) only and independent of x^3 -coordinate. Thus, the variations in measurement of space and time locally must be a function of (x^1, x^2) only. This discussion implies

$$dt \rightarrow z(x^1, x^2)dt \quad (27)$$

where z is some function of (x^1, x^2) .

Now, it is typical physics of waves. If the direction of propagation of a wave is along some line in x^1x^2 plane, the wave vector $\vec{k}$ can be resolved along x^1, x^2 coordinates say as k_1, k_2 such that $|\vec{k}| = k = \sqrt{k_1^2 + k_2^2}$. The wave frequency (ω) is related with wave number (k) by wave velocity. Since, Kaal Ripples are propagating along a line in (x^1, x^2) plane locally, the wave vector $\vec{k}$ of Kaal Ripples must only depend on changes in metric along (x^1, x^2) directions. Thus, $\vec{k}$ depends on $g_{11} = p(x^1, x^2)$ and $g_{22} = q(x^1, x^2)$. Consequently, frequency (Time Density and KED) depends on them, as $\omega = |\vec{k}|$ in $c = 1$ units. This is all *relative* to flat conditions.

Because of inverse relation between energy and wavelength, we have Kaal Energy Density relative to flat conditions as,

$$\kappa \equiv \left(\frac{1}{p(x^1, x^2)} + \frac{1}{q(x^1, x^2)} \right)^{-\frac{1}{2}} \quad (28)$$

where conveniently it can be considered $|\vec{k}|_{flat} = \omega_{flat} \equiv 1$ in (x^1, x^2) plane. Consequently,

$$z(x^1, x^2) \equiv \left(\frac{1}{p(x^1, x^2)} + \frac{1}{q(x^1, x^2)} \right)^{\frac{1}{2}} \quad (29)$$

i.e. time scale changes by the factor $z(x^1, x^2)$.

Clearly, above is a general discussion using a particular coordinate system. The exact nature of metric, KED and Time Density depend on the solution to ERE (24) in a suitable coordinate system.

5.2 Conditions

On large scale, the rest energy of the source mass dominates. Because of this simplicity, two sets of conditions are discussed basis the contribution of Bounded Kaal Energy (BKE) as a source of gravitational effects,

1. **Weak BKE Conditions:** (eg. solar system) Rest energy of source mass dominates over the contribution from BKE and energy from motion (source is moving slowly), thus effects from these can be ignored for analysis. This will give results in correspondence with general relativity.
2. **Strong BKE Conditions:** (eg. supermassive black holes, galactic scales) There are two sub-conditions here,
 - **Large Mass:** Larger the source mass, more is the curvature in $\mathcal{RIE}^3$ and thus higher the KED near the source. In such conditions, contribution of BKE to cause gravitational effects' cannot be ignored and results will differ from general relativity.
 - **Relativistic Motion:** Higher the momentum, larger is the neighboring KED i.e. kinetic energy. Such effects are already considered in special relativity if effects of source mass are ignored. Considering the BKE by source mass, the results will differ from general relativity.

5.3 Case of Schwarzschild's Solution

Consider a slowly rotating equivalently static (this approximation is easy to solve for) isotropic source mass such as our Sun. The objective is to determine how local scales of space and time are affected near such source from their measurements in flat conditions i.e. at infinite distance. The approach is similar to how one solves for Schwarzschild's solution [28] in general relativity. Two points are to be considered -

1. Since, we are considering vacuum conditions i.e. $\rho_s = 0$, this means contribution of BKE is ignored and it is a case of weak BKE conditions.
2. Kaal Ripples exist due to slow rotation, however the effects of rotation are ignored ¹¹. Thus, static case implies evaluating KED bounded by source mass.

Borrowing assumptions and conditions from Schwarzschild's solution in general relativity [4], consider the following as the line element of $\mathcal{RIE}^3$ in the spherical coordinates (r, θ, ϕ) near the source mass,

$$ds^2 = e^{2X(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (30)$$

where function $X(r)$ needs to be determined. The metric in matrix form is

$$g_{mn} \equiv \begin{pmatrix} e^{2X(r)} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (31)$$

Solving for

$$R = 0 \quad (32)$$

or

$$g^{mn} R_{mn} = 0 \quad (33)$$

where R_{mn} is Ricci Curvature Tensor.

Using the metric in (31), Christoffel symbols, Γ_{st}^r can be derived which can then be used to evaluate the Ricci Curvature Tensor, R_{mn} . It turns out, if

$$e^{2X(r)} = \frac{1}{1 + b/r} \quad (34)$$

we have,

$$R_{mn} = \begin{pmatrix} \frac{b}{r^2(b+r)} & 0 & 0 \\ 0 & \frac{-b}{2r} & 0 \\ 0 & 0 & \frac{-b \sin^2 \theta}{2r} \end{pmatrix} \quad (35)$$

¹¹Will be considered in next section

Consequently, we get

$$g^{mn}R_{mn} = \left(1 + \frac{b}{r}\right) \left(\frac{b}{r^2(b+r)}\right) + \left(\frac{1}{r^2}\right) \left(\frac{-b}{2r}\right) + \left(\frac{1}{r^2 \sin^2 \theta}\right) \left(\frac{-b \sin^2 \theta}{2r}\right) \quad (36)$$

$$= 0 \quad (37)$$

i.e.

$$R = 0 \quad (38)$$

The manifold describing the variation in scales of space in vacuum region near a source mass is *scalar flat* but not Ricci flat ($R_{mn} \neq 0$). The line element from the solution is then

$$ds^2 = \frac{dr^2}{(1 + b/r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (39)$$

This line element describes exactly the same information as the Schwarzschild Solution to EFEs in the same conditions. This is not surprising as Schwarzschild metric is a static spacetime.

Considering the flow of Kaal Ripples to be entirely radial¹², only the radial component contributes to the wave vector $\vec{k}$ and thus to frequency ω . Therefore, *relative* to Kaal Ripples in flat conditions ($\omega_{flat} = 1$)

$$\omega = k = \left(1 + \frac{b}{r}\right)^{-\frac{1}{2}} \quad (40)$$

and consequently,

$$dt \rightarrow \left(1 + \frac{b}{r}\right)^{\frac{1}{2}} dt \quad (41)$$

This implies KED at a radial distance r from the source mass is $\left(1 + \frac{b}{r}\right)^{-1/2}$ times larger than KED at infinity (flat conditions).

Without any loss of generality, complete correspondence with general relativity results can be established if the constant $b = (-2GM/c^2)$, leading to the line element of the form

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (42)$$

and explicitly the metric g_{mn} of $\mathcal{RIE}^3$ is

$$g_{mn} \equiv \begin{pmatrix} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (43)$$

As discussed earlier, KED is higher nearer to the source mass i.e. it binds (collects, increases, pulls) KED in its neighborhood. In ideal conditions, this Bounded Kaal Energy (BKE) also contributes to the RHS of ERE (24) i.e. BKE is equivalent to additional source of gravitation.

¹²This is incorrect however it can be assumed for slow rotations. Rotations cause Kaal Ripples to flow radially and towards the poles along axis of rotation. Discussed later.

5.3.1 Newtonian Limit

Kaal structure gives further meaning to gravitational potential (or other potential types, depends on property of matter under study) and potential energy in terms of KED. In Newtonian conditions, (40) with $b = (-2GM/c^2)$, can be used to write relative KED (κ) as

$$\frac{\kappa_r}{\kappa_{flat}} = \left(1 - \frac{2GM}{c^2 r}\right)^{-\frac{1}{2}} \approx \left(1 + \frac{GM}{c^2 r}\right) \quad (44)$$

or with $\phi = -GM/r$,

$$\frac{\kappa_r}{\kappa_{flat}} = \left(1 - \frac{\phi}{c^2}\right) \quad (45)$$

$$\frac{\kappa_r - \kappa_{flat}}{\kappa_{flat}} = -\frac{\phi}{c^2} \quad (46)$$

i.e. gravitational potential at radial distance r is the (minus the) gain in KED coming from infinity.¹³

5.3.2 Schwarzschild Black Hole

From the metric (43) derived for Schwarzschild's condition it is evident that at Schwarzschild's radius $r_s = 2GM/c^2$, Kaal Ripples tend to vanish. Consequently, the space measurement tends to infinity, time measurements tend to zero and the KED approaches infinity.

$$r \rightarrow r_s \implies \kappa \rightarrow \infty \quad (47)$$

Not only does the results agree with the analysis of Schwarzschild's black hole in general relativity, but there is additional information relating the structure of black holes and energy density. Further analysis on Schwarzschild's black hole with respect to the present framework is deferred for now.

5.4 Discussion: Moving Source

Before assessing cases that involve moving sources, following two points need review:

1. From the postulates, it is evident motion is fundamental to Kaal theory. Kaal Ripples continuously carry energy through Kaal structure and the structure of Kaal Ripples determine the local *scales* of space and time. Therefore, there are *no static conditions* in Kaal theory.
2. Because energy is not the property of matter but of Kaal structure, what is conventionally understood as kinetic energy¹⁴ needs to be evaluated in terms of KED.

¹³This possibly leads to understanding why the Lagrangians for problems in mechanics is $T - V$.

¹⁴Other energy forms are not being considered for simplicity.

5.4.1 Kaal Fluid

KED in the neighborhood of a source mass in Schwarzschild conditions is found to be

$$\kappa_r = \left(1 - \frac{2GM}{c^2 r}\right)^{-\frac{1}{2}} \kappa_{flat} \quad (48)$$

i.e. KED is higher near the source compared to KED in flat conditions at infinity. This distribution of energy surrounding the source mass can be equivalently considered as *fluid* and the source mass is analogous to a submerged sphere in a fluid. The mass density of this *Kaal Fluid* is equivalent to the KED by Kaal-Mass equivalence. Now, this is a typical fluid dynamics problem that considers Euclidean background space. Once again, it is reiterated, space is flat but scales/measurements is not.

5.4.2 Kinetic Energy

Kinetic energy of a moving source mass is the *incremental KED* ($\Delta\kappa$) in its neighborhood due to virtue of its motion. For this, results from local analysis are used to determine the incremental KED.

Rotations. Consider, in Schwarzschild conditions, KED in the neighborhood of the source mass M is κ_r as in (48); the radius of source is say, R_M and it rotates with some constant angular velocity, ω_M . Surface of the source mass presents one boundary condition i.e. KED in infinitesimal volume at surface is dragged by linear velocity $\omega_M R_M \sin\theta$ in the direction of rotation. Consequently, KED in the neighborhood increases by the factor γ as

$$(\kappa_r)_{rotation} = \gamma (\kappa_r)_{Schwarzschild} \quad (49)$$

$$\gamma = \left(1 - \frac{(\omega_M R_M \sin\theta)^2}{c^2}\right)^{-\frac{1}{2}} \quad (50)$$

This implies angular velocity of KED in infinitesimal volume element falls as radial distance increases. Now, what is understood as kinetic energy of the rotating source mass is the incremental KED in the neighborhood Kaal structure such that

$$Kinetic\ Energy = \int \int \int_{R_M}^r (\gamma - 1) \kappa_r dVol \quad (51)$$

Because, LHS is known quantity based on conventional physics, the unknown is the upper limit r of the integral on RHS. The amount of KED a source mass binds in its neighborhood is dragged just as a submerged rotating sphere drags fluid in its neighborhood.¹⁵

Translation. Following the discussion for rotations, the case for translation is very similar with the exception of no angular dependence, as all elements on the surface of the source mass

¹⁵Fundamentally, rotation of Kaal Fluid sustains rotation of the source mass.

move with same linear velocity. This makes translations less interesting. Rotations are vital from quantum to cosmic scales to observe gravity. Translatory motion is equivalent to dragging a certain block of Kaal structure with velocity same as source velocity.¹⁶ The above general equation (51) for kinetic energy also applies for translations.

5.5 Rotating Spherical Source

The most relevant case in large scale analysis is of rotating source mass. Generally, the source is considered to be spheroidal, however for simplicity let us consider a spherically isotropic source mass. The result can then be extended to spheroidal rotating mass through appropriate spheroidal coordinates as in Kerr's solution to EFEs. A correspondence will also be established with cross-terms in (1+3) spacetime metric signifying rotations in solutions to EFEs.

From the previous discussion on KED in the neighborhood of a rotating source, the equation (49) for $(\kappa_r)_{rotation}$ was

$$(\kappa_r)_{rotation} = \gamma(\kappa_r)_{Schwarzschild} \quad (52)$$

$$\gamma = \left(1 - \frac{(\omega_M R_M \sin\theta)^2}{c^2}\right)^{-\frac{1}{2}} \quad (53)$$

expanding (52), we have

$$(\kappa_r)_{rotation} = \left(1 - \frac{(\omega_M R_M \sin\theta)^2}{c^2}\right)^{-\frac{1}{2}} \left(1 - \frac{2GM}{c^2 r}\right)^{-\frac{1}{2}} \kappa_{flat} \quad (54)$$

KED is now a function of (r, θ) . Consequently, wave-vector ($\vec{k}$), frequency ($\equiv$ Time Density) (ω) and metric (g_{mn}) of $\mathcal{RIE}^3$ near such rotating source mass are also function of (r, θ) . This implies, locally Kaal Ripples flow in (r, θ) plane.

There continues to be symmetry in azimuthal plane and thus $\vec{k}, \omega, g_{mn}$ are all independent of azimuthal angle ϕ . However, by virtue of rotation of the source (or inversely) and evidently from equation (54), Kaal Fluid is *dragged* along the direction of rotation i.e. in azimuthal plane and this is equivalent to *frame-dragging effect* from solutions to EFEs for rotating source. The KED of an infinitesimal volume element at radial distance r due to rotation of source mass is equivalent to the rotation of that element by angular velocity $\omega_M R_M \sin\theta / r$. Clearly, as $r \rightarrow \infty$, rotation of Kaal Fluid slows down. However, the range of radial distance r is limited by the kinetic energy relation (51).

Open Question: The analysis for moving sources assumes that Kaal Fluid is equivalent to zero-viscosity superfluid. However, superfluids are known to exhibit some viscosity when in motion, say rotating. Therefore, alternatively Kaal Fluid can be assumed to exhibit some viscosity. In such a scenario, as a simple case (corresponding with fluid dynamics) this leads to velocity profile to fall by $1/r^3$ of the infinitesimal volume element of Kaal structure near a rotating spherical source mass. In both models, with or without viscosity, there is theoretical

¹⁶On microscopic scale, it is equivalent to slowed down Kaal Ripples extending over a certain region.

correspondence with frame-dragging effects. This can be termed as an open area still under investigation though there is an inclination towards non-zero viscosity model as energy is now a property of Kaal structure only and dissipation is insignificant.

5.5.1 Flow of KED: Equator to Poles

The equation (54) tells about the KED at location (r, θ) in the neighborhood of a rotating source mass. At certain fixed radial distance r , KED has a *maxima* at the equator ($\theta = \pi/2$) and *minima* at the poles ($\theta = 0, \pi$) along the axis of rotation.

From the postulates, Kaal Ripples flow from high KED region to low KED region of Kaal structure¹⁷. Therefore, Kaal Ripples must flow from equatorial plane to the poles and away from the rotating source along the axis of rotation.

This flow of Kaal Ripples from the equatorial plane to poles results in flow of Kaal Ripples radially pushing inwards towards the source analogous to how a fluid will flow in similar situation. Thus, in Schwarzschild conditions, the Kaal Ripples exist for slowly rotating source however these are ignored for simplicity in the problem.

5.5.2 Slow Rotations in Newtonian Conditions

Expanding the equation (54) for weak conditions,

$$(\kappa_r)_{rotation} \approx \left(1 + \frac{(\omega_M R_M \sin\theta)^2}{2c^2}\right) \left(1 + \frac{GM}{c^2 r}\right) \kappa_{flat} \quad (55)$$

$$\approx \left(1 + \frac{(\omega_M R_M \sin\theta)^2}{2c^2} + \frac{GM}{c^2 r}\right) \kappa_{flat} \quad (56)$$

ignoring the cross-term with c^4 in the denominator. Using ω_M, R_M for Earth, the contribution from second term is negligible. Thus, at a point outside of source mass, the radiation pressure from flow of KED towards the poles (along rotational axis) can be ignored. It however exists on near or on the boundary i.e. on the surface of earth. Thus the gravitational effects can be conveniently considered completely radial in such weak conditions.

6 Discussion: Time

There are three objects that define the notion of time - Rate of passage of time (Time Density), Measure of time, Time - within the framework of observed, observation and observer. *Rate of passage of time* is very well understood from general relativity and has been discussed previously in this paper. In Kaal theory, every particle is in some state of motion, so is a clock and its constituents always in motion; thus each tick of the clock measures the rate of passage of time. The number of ticks registered by the clock is the *measure of time*. If the Time Density is constant over a certain region of Kaal structure, notion of time gives the impression of being a

¹⁷Radially KED is bounded by mass.

parameter otherwise when it is not constant and varies over a certain region, it is equivalent to curvature in $\mathcal{RIE}^3$ of Kaal structure or in (1+3) spacetime of general relativity. *Time* is the total number of ticks registered by a clock for a phenomenon under observation. Thus, if phenomenon ends, so does Time for it. It is the concept of ‘rest’ which muddles the understanding of time vis a vis space (‘rest’ does not exist in present framework, Kaal structure is dynamical). There is no concept of time reversal in the present framework by the very way it considers measure of time. Past and future can be determined but cannot be accessed as ticking of clock is not independent of flow of KED.

Clocks enable measure of time using an object in motion such as light or a particle. Fundamentally, clocks compare one motion with another and thus motion is fundamental for the existence of measurement of time. However, at any point Time Density must not be dependent on an object; it must exist independent of light or particle.

7 Result

The description of gravitational effects as flow of energy that locally defines length and time *scales* corresponds with the EFEs description of gravity as curvature in spacetime but in weak conditions. On large scale, the energy bounded by source mass is an additional source of gravitation and its flow causes the gravitational effects as pushing-in, flowing from high to low energy density regions. The flow is analogous to the flow of super-fluid but is in deviation to SFDM models. Therefore, the present theoretical model predicts the existence of additional gravity i.e. ‘dark matter’ beyond what is predicted by EFEs. This is achieved without modifying EFEs and is presented as a model that assists general relativity and its results.

As part of the continuing work, the next steps are to resolve open questions and perform numerical analysis based on this theoretical framework that corroborates the data corresponding to current observations. Further, more types of coordinate systems can be evaluated with ERE to understand the space, time *scales* in various conditions.

8 Conclusions

The theoretical framework presented in this paper describes the physical structure that locally defines length and time *scales*. The energy density of Kaal Ripples bounded by source mass is an additional source of gravitational effects, pushing-in and flowing from equatorial plane to poles along axis of rotation. This must be most evident in strong conditions or large (galactic) scales.

The results evaluated in the paper theoretically correspond with the verified results in general relativity (solar system tests, black holes, gravitational waves). Additionally, it predicts the nature of ‘dark matter’ and source of ‘dark energy’ without introducing new particles and negative mass to describe repulsive gravity respectively. Further, the probabilistic nature of Kaal structure though on Newtonian or large scale is of no significance, it leads to a possible interpretation of source of wave-particle duality observed in quantum mechanics. The energy

minimizing (hence repulsive) character of Kaal structure can further lead to understanding entropy both on small or large scales and is a possible step towards deciphering the cause of arrow of time. Another consequence of minimizing character is that accelerated expansion of universe may not be isotropic and monotonically decreasing over a given region. As a result of the framework presented in this paper, it is expected the curvature of universe must correspond to flat space geometry as distinction has been made between energy and matter.

Curvature of spacetime in general relativity causes matter to move, and in the present framework flow and distribution of energy causes motion. Locally, the equation of motion for a free (falling) particle is a wave-function describing the Kaal structure however on large scale, pseudo-Riemannian spacetime of general relativity is still the best model to describe physical phenomenon mathematically.

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