Credit Risk Valuation Model

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ABSTRACT

This article presents a new model for valuing financial contracts subject to credit risk correlation and concentration. The model characterizes default dependencies exogenously, and models collateral processes directly based on collateral agreements. For brevity we focus on CDS contracts, but many of the points we make are equally applicable to other derivatives. CDS has trilateral credit risk, where three parties – buyer, seller and reference entity – are defaultable. We show that default dependency has a significant impact on asset pricing. We also show that full collateralization cannot eliminate counterparty risk completely in some markets.

Key Words: credit risk modeling; collateralization; correlation, CDS.

1 Introduction

The main drawback of the conditionally independent assumption or the reduced-form models is that the range of default correlations that can be achieved is typically too low when compared with empirical default correlations. The responses to correct this weakness can be generally classified into two categories: endogenous default relationship approaches and exogenous default relationship approaches.

Given a default model, one can value a risky derivative contract and compute credit value adjustment (CVA) that is a relatively new area of financial derivative modeling and trading. CVA is the expected loss arising from the default of a counterparty (see Brigo and Capponi (2008), Lipton and Sepp (2009), Pykhtin and Zhu (2006), Gregory (2009), Bielecki et al (2013) and Crepey (2015), etc.)

This paper presents a new framework for valuing defaultable financial contracts with or without collateral arrangements. The framework characterizes default dependencies exogenously, and models collateral processes directly based on the fundamental principles of collateral agreements.

In general, a CDS contract is used to transfer the credit risk of a reference entity from one party to another. The risk circularity that transfers one type of risk (reference credit risk) into another (counterparty credit risk) within the CDS market is a concern for financial stability. Some people claim that the CDS market has increased financial contagion or even propose an outright ban on these instruments.

The standard CDS pricing model in the market assumes that there is no counterparty risk. Although this oversimplified model may be accepted in normal market conditions, its reliability in times of distress has recently been questioned. In fact, counterparty risk has become one of the most dangerous threats to the CDS market. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account (see ECB (2009)).

There is a significant increase in the use of collateral for CDS after the recent financial crises. Many people believe that, if a CDS is fully collateralized, there is no risk of failure to pay. Collateral posting regimes are originally designed and utilized for bilateral risk products, e.g., interest rate swap (IRS), but there are many reasons to be concerned about the success of collateral posting in offsetting the risk of CDS

contracts. First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movements of *IRS* contracts are far smoother and less volatile than *CDS* prices. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization *cannot* eliminate counterparty risk completely for a CDS contract.

2 Pricing Financial Contracts Subject to Three-way Credit Risk

In the reduced-form approach, the stopping (or default) time τ_i of firm i is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau_i = \inf \left\{ t : \int_0^t h_i(s, Z_s) ds \geq H_i \right\} \quad (1)$$

where $h_i(t)$ or $h_i(t, Z_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Z_t , and H_i is a unit exponential random variable independent of Z_t .

It is well-known that the survival probability from time t to s in this framework is defined by

$$p_i(t, s) := P_i(\tau > s \mid \tau > t, Z_t) = \exp\left(-\int_t^s h_i(u) du\right) \quad (2a)$$

The default probability for the period (t, s) in this framework is given by

$$q_i(t, s) := P_i(\tau \leq s \mid \tau > t, Z_t) = 1 - p_i(t, s) = 1 - \exp\left(-\int_t^s h_i(u) du\right) \quad (2b)$$

A CDS is normally used to transfer the credit risk of a reference entity between two counterparties. The contract reduces the credit risk of the reference entity but gives rise to another form of risk: *counterparty risk*. Since the dealers are highly concentrated within a small group, any of them may be too big to fail. The interconnected nature, with dealers being tied to each other through chains of OTC derivatives, results in increased contagion risk. Due to its concentration and interconnectedness, the CDS market seems to pose a systemic risk to financial market stability. In fact, the CDS is blamed for playing a pivotal role in the collapse of Lehman Brothers and the disintegration of AIG.

Let A denote the protection buyer, B denote the protection seller and C denote the reference entity. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for firm j ($j = A$ or B or C) follows a Bernoulli distribution, which takes value 1 with default probability q_j , and value 0 with survival probability p_j . The marginal default distributions can be determined by the reduced-form models. The joint distributions of a multivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations. The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990)) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_C = 0) = p_A p_B p_C + p_C \sigma_{AB} + p_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3a)$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_C = 0) = q_A p_B p_C - p_C \sigma_{AB} - p_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3b)$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_C = 0) = p_A q_B p_C - p_C \sigma_{AB} + q_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3c)$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_C = 1) = p_A p_B q_C + q_C \sigma_{AB} - p_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3d)$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_C = 0) = q_A q_B p_C + p_C \sigma_{AB} - q_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3e)$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_C = 1) = q_A p_B q_C - q_C \sigma_{AB} + p_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3f)$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_C = 1) = p_A q_B q_C - q_C \sigma_{AB} - q_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3g)$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_C = 1) = q_A q_B q_C + q_C \sigma_{AB} + q_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3h)$$

where

$$\theta_{ABC} := E((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)) \quad (3i)$$

We introduce the concept of *comvariance* into credit risk modeling arena to exploit any statistical relationship among multiple random variables. Furthermore, we define a new statistic, *comrelation*, as a scaled version of comvariance (just like correlation is a scaled version of covariance) as follows:

Definition 1: For three random variables X_A , X_B , and X_C , let μ_A , μ_B , and μ_C denote the means of X_A , X_B , and X_C . The *comrelation* of X_A , X_B , and X_C is defined by

$$\zeta_{ABC} = \frac{E[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)]}{\sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3}} \quad (4)$$

According to the Holder inequality, we have

$$\begin{aligned} |E((X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C))| &\leq E|(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)| \\ &\leq \sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3} \end{aligned} \quad (5)$$

Obviously, the correlation is in the range of $[-1, 1]$. Given the correlation, Equation (3i) can be rewritten as

$$\begin{aligned} \theta_{ABC} &:= E((Y_A - q_A)(Y_B - q_B)(X_C - q_C)) = \zeta_{ABC} \sqrt[3]{E|Y_A - q_A|^3 \times E|Y_B - q_B|^3 \times E|Y_C - q_C|^3} \\ &= \zeta_{ABC} \sqrt[3]{p_A q_A (p_A^2 + q_A^2) p_B q_B (p_B^2 + q_B^2) p_C q_C (p_C^2 + q_C^2)} \end{aligned} \quad (6)$$

where $E(Y_j) = q_j$ and $E|Y_j - q_j|^3 = p_j q_j (p_j^2 + q_j^2)$, $j=A, B, \text{ or } C$.

If we have a series of n measurements of X_A , X_B , and X_C written as x_{Ai} , x_{Bi} and x_{Ci} where $i = 1, 2, \dots, n$, the sample correlation coefficient can be obtained as:

$$\zeta_{ABC} = \frac{\sum_{i=1}^n (x_{Ai} - \mu_A)(x_{Bi} - \mu_B)(x_{Ci} - \mu_C)}{\sqrt[3]{\sum_{i=1}^n |x_{Ai} - \mu_A|^3 \times \sum_{i=1}^n |x_{Bi} - \mu_B|^3 \times \sum_{i=1}^n |x_{Ci} - \mu_C|^3}} \quad (7)$$

More generally, we define the correlation in the context of n random variables as

Definition 2: For n random variables $X_1, X_2, \dots, X_n$, let μ_i denote the mean of X_i where $i=1, \dots, n$. The correlation of $X_1, X_2, \dots, X_n$ is defined as

$$\zeta_{12\dots n} = \frac{E[(X_1 - \mu_1)(X_2 - \mu_2) \cdots (X_n - \mu_n)]}{\sqrt[n]{E|X_1 - \mu_1|^n \times E|X_2 - \mu_2|^n \cdots E|X_n - \mu_n|^n}} \quad (8)$$

Correlation is just a specific case of correlation where $n = 2$. Again, the correlation $\zeta_{12\dots n}$ is in the range of $[-1, 1]$ according to the Holder inequality.

2.1 Risky valuation without collateralization

Let valuation date be t . Suppose that a CDS has m scheduled payments represented as $X_i = -sN\delta(T_{i-1}, T_i)$ with payment dates $T_1, \dots, T_m$ where $i=1, \dots, m$, $\delta(T_{i-1}, T_i)$ denotes the accrual factor for period (T_{i-1}, T_i) , N denotes the notional/principal, and s denotes the CDS premium. Party A pays the

premium/fee to party B if reference entity C does not default. In return, party B agrees to pay the protection amount to party A if reference entity C defaults before the maturity. We have the following proposition.

Proposition 1: *The value of the CDS is given by*

$$V(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t \right] \quad (9a)$$

where $t = T_0$ and

$$O(T_j, T_{j+1}) = 1_{(V(T_{j+1}) + X_{j+1}) \geq 0} \phi_B(T_j, T_{j+1}) + 1_{(V(T_{j+1}) + X_{j+1}) < 0} \phi_A(T_j, T_{j+1}) \quad (9b)$$

$$\begin{aligned} \phi_A(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_A(T_{j+1}) \\ & + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\phi}_A(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_{AB}(T_{j+1}) \\ & + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \phi_A(T_{j+1}) - \bar{\phi}_A(T_{j+1}) + \phi_{AB}(T_{j+1})) \\ & + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \phi_A(T_{j+1})) + q_B(T_j, T_{j+1}) (\bar{\phi}_A(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \bar{\phi}_A(T_{j+1})) + q_A(T_j, T_{j+1}) (\phi_A(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \theta_{ABC}(T_j, T_{j+1}) (-1 + \bar{\phi}_A(T_{j+1}) - \phi_{AB}(T_{j+1}) + \phi_A(T_{j+1})) \} D(T_j, T_{j+1}) \end{aligned} \quad (9c)$$

$$\begin{aligned} \phi_B(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\phi}_B(T_{j+1}) \\ & + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_{AB}(T_{j+1}) \\ & + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \phi_B(T_{j+1}) - \bar{\phi}_B(T_{j+1}) + \phi_{AB}(T_{j+1})) \\ & + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\phi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\phi_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \phi_B(T_j)) + q_A(T_j, T_{j+1}) (\bar{\phi}_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \theta_{ABC}(T_j, T_{j+1}) (-1 + \bar{\phi}_B(T_{j+1}) - \phi_{AB}(T_{j+1}) + \phi_B(T_{j+1})) \} D(T_j, T_{j+1}) \end{aligned} \quad (9d)$$

$$\begin{aligned} \Omega(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \bar{\phi}_B(T_{j+1}) \\ & + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \phi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \phi_{AB}(T_{j+1}) \\ & + q_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \phi_B(T_{j+1}) - \bar{\phi}_B(T_{j+1}) + \phi_{AB}(T_{j+1})) \\ & - \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\phi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\phi_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & - \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \phi_B(T_{j+1})) + q_A(T_j, T_{j+1}) (\bar{\phi}_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \theta_{ABC}(T_j, T_{j+1}) (1 - \bar{\phi}_B(T_{j+1}) + \phi_{AB}(T_{j+1}) - \phi_B(T_{j+1})) \} D(T_j, T_{j+1}) \end{aligned} \quad (9e)$$

where $R(T_j, T_{j+1}) = (N(1 - \phi_C(T_{j+1})) - \alpha(T_j, T_{j+1}))$, $\alpha(T_j, T_{j+1}) = sN\delta(T_S, T)/2$, and $X_i = -sN\delta(T_j, T_{j+1})$.

Proof: See the Appendix.

Proposition 1 provides a general form for pricing a CDS. Applying it to a particular situation in which we assume that counterparties A and B are default-free, i.e., $p_j = 1$, $q_j = 0$, $\rho_{kl} = 0$, and $\varsigma_{ABC} = 0$, where $j=A$ or B and $k, l=A, B$, or C , we derive the following corollary.

Corollary 1: If counterparties A and B are default-free, the value of the CDS is given by

$$\begin{aligned} V(t) &= \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i | \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t \right] \\ &= \sum_{i=1}^m E \left[D(t, T_i) p_C(t, T_i) X_i | \mathcal{F}_t \right] + \sum_{i=1}^m E \left[D(t, T_i) p_C(t, T_{i-1}) q_C(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t \right] \end{aligned} \quad (10)$$

where $O(T_{i-1}, T_i) = D(T_{i-1}, T_i) p_C(T_{i-1}, T_i)$; $\Omega(T_{i-1}, T_i) = D(T_{i-1}, T_i) q_C(T_{i-1}, T_i)$.

The proof of this corollary becomes straightforward according to Proposition 1 by setting $\rho_{kl}=0$,

$$\varphi_{AB} = 0, \quad \varsigma_{ABC} = 0, \quad p_j = 1, \quad q_j = 0, \quad p_C(t, T_i) = \prod_{g=0}^{i-1} p(T_g, T_{g+1}), \quad \text{and} \quad D(t, T_i) = \prod_{g=0}^{i-1} D(T_g, T_{g+1}).$$

If we further assume that the discount factor and the default probability of the reference entity are uncorrelated and the recovery rate φ_C is constant, we have

Corollary 2: Assume that i) counterparties A and B are default-free, ii) the discount factor and the default probability of the reference entity are uncorrelated; iii) the recovery rate φ_C is constant; the value of the CDS is given by

$$V(t) = \sum_{i=1}^m P(t, T_i) \bar{p}_C(t, T_{i-1}) \bar{q}_C(T_{i-1}, T_i) (N(1 - \varphi_C) - \alpha(T_{i-1}, T_i)) - \sum_{i=1}^m P(t, T_i) \bar{p}_C(t, T_i) sN\delta(T_{i-1}, T_i) \quad (11)$$

where $P(t, T_i) = E[D(t, T_i) | \mathcal{F}_t]$ denotes the bond price, $\bar{p}_C(t, T_i) = E[p_C(t, T_i) | \mathcal{F}_t]$, $\bar{q}_C(t, T_i) = 1 - \bar{p}_C(t, T_i)$, $\bar{p}(t, T_{i-1}) \bar{q}(T_{i-1}, T_i) = \bar{p}(t, T_{i-1}) - \bar{p}(t, T_i)$.

This corollary is easily proved according to Corollary 1 by setting $E[XY | \mathcal{F}_t] = E[X | \mathcal{F}_t] E[Y | \mathcal{F}_t]$ when X and Y are uncorrelated. Corollary 2 is the formula for pricing CDS in the market.

Our methodology can be extended to the cases where the number of parties $n \geq 4$. A generating function for the (probability) joint distribution (see details in Teugels (1990)) of n -variate Bernoulli can be expressed as

$$p^{(n)} = \begin{bmatrix} p_n & -1 \\ q_n & 1 \end{bmatrix} \otimes \begin{bmatrix} p_{n-1} & -1 \\ q_{n-1} & 1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} p_1 & -1 \\ q_1 & 1 \end{bmatrix} \sigma^{(n)} \quad (12)$$

where $\otimes$ denotes the Kronecker product; $p^{(n)} = \{p_k^{(n)}\}$ and $\sigma^{(n)} = \{\sigma_k^{(n)}\}$ are vectors containing 2^n

components: $p_k^{(n)} = p_{k_1, k_2, \dots, k_n}$, $k = 1 + \sum_{i=1}^n k_i 2^{i-1}$, $k_i \in \{0, 1\}$; $\sigma_k^{(n)} = \sigma_{k_1, k_2, \dots, k_n} = E \left(\prod_{i=1}^n (Y_i - q_i)^{k_i} \right)$.

2.2 Risky valuation with collateralization

The main role of collateral should be viewed as an improved recovery in the event of a counterparty default. According to Bankruptcy law, if there has been no default, the collateral is returned to the collateral giver by the collateral taker. If a default occurs, the collateral taker possesses the collateral. In other words, collateral does not affect the survival payment; instead, it takes effect on the default payment only.

We assume that a CDS is fully collateralized, i.e., the posting of collateral is equal to the amount of the current *MTM* value: $C(t) = V(t)$. For a discrete one-period (t, u) economy, there are several possible states at time u : i) A , B , and C survive with probability p_{000} . The instrument value is equal to the market value $V(u)$; ii) A and B survive, but C defaults with probability p_{001} . The instrument value is the default payment $R(u)$; iii) For the remaining cases, either or both counterparties A and B default. The instrument value is the future value of the collateral $V(t) / D(t, u)$ (Here we consider the time value of money only). The value of the collateralized instrument at time t is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E\{D(t, u)[p_{000}(t, u)V(u) + p_{001}(t, u)R(u) \\ &\quad + (p_{100}(t, u) + p_{010}(t, u) + p_{110}(t, u) + p_{101}(t, u) + p_{011}(t, u) + p_{111}(t, u))V(t) / D(t, u)] | \mathcal{F}_t\} \quad (14a) \\ &= E\{[D(t, u)(p_{000}(t, u)V(s) + p_{001}(t, u)R(u)) + (1 - p_{000}(t, u) - p_{001}(t, u))V(t)] | \mathcal{F}_t\} \end{aligned}$$

or

$$\begin{aligned} &E[(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)) | \mathcal{F}_t]V(t) \\ &= E\{[D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)) \\ &\quad + D(t, u)(V(u) - R(u))(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u))] | \mathcal{F}_t\} \quad (14b) \end{aligned}$$

If we assume that $(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u))$ and $D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))$ are uncorrelated, we have

$$V(t) = V^F(t) + \xi_{ABC}(t, u) / \psi_{AB}(t, u) \quad (15a)$$

where

$$V^F(t) = E\{D(t, u)[p_C(t, u)V(u) + q_C(t, u)R(u)] | \mathcal{F}_t\} \quad (15b)$$

$$\psi_{AB}(t, u) = E\left\{\left[p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)\right] \middle| \mathcal{F}_t\right\} \quad (15c)$$

$$\xi_{ABC}(t, u) = E\left\{\left[D(t, u)\left(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u)\right)(V(u) - R(u))\right] \middle| \mathcal{F}_t\right\} \quad (15d)$$

The first term $V^F(t)$ in equation (15) is the counterparty-risk-free value of the CDS and the second term is the exposure left over under full collateralization, which can be substantial.

Proposition 2: *If a CDS is fully collateralized, the risky value of the CDS is NOT equal to the counterparty-risk-free value, as shown in equation (15).*

Proposition 2 or equation (15) provides a theoretical explanation for the failure of full collateralization in the CDS market. It tells us that under full collateralization the risky value is in general not equal to the counterparty-risk-free value except in one of the following situations: i) the market value is equal to the default payment, i.e., $V(u) = R(u)$; ii) firms A , B , and C have independent credit risks, i.e., $\rho_{ij} = 0$ and $\varsigma_{ABC} = 0$; or iii) the following relationship holds $p_B\sigma_{AC} + p_A\sigma_{BC} = \theta_{ABC}$.

3 Numerical Results

Our goal in this section is to study the quantitative relationship between CDS premia and the credit quality of counterparties and reference entities, including the default correlations and comrelations.

Since the payoffs of a CDS are mainly determined by credit events, we need to characterize the evolution of the hazard rates. Here we choose the *Cox-Ingersoll-Ross* (CIR) model. The CIR process has been widely used in the literature of credit risk and is given by

$$dh_t = a(b - h_t)dt + \sigma\sqrt{h_t}dW_t \quad (16)$$

where a denotes the mean reversion speed, b denotes the long-term mean, and σ denotes the volatility.

Table 1: Current/spot market data

This table displays the current (spot) market data used for all calculations in this paper, including the term structure of continuously compounded interest rates, the term structure of A-rated breakeven CDS spreads, and the curve of at-the-money caplet volatilities.

Term (days)	31	91	182	365	548	730	1095	1825	2555	3650	5475
Interest Rate	0.002	0.002	0.002	0.004	0.007	0.010	0.016	0.024	0.030	0.035	0.040
	8	7	9	3	1	2		9	6	5	5
Credit Spread	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.007	0.007	0.009	0.010
	2	2	2	5	9	2	8		9	1	6
Caplet Volatility	0.326	0.331	0.337	0.350	0.364	0.377	0.308	0.247	0.214	0.167	0.163
	7		6	9	1	3		3	1	8	4

Table 2: Risk-neutral parameters for CIR model

This table presents the risk-neutral parameters that are calibrated to the current market shown in Table 1. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads.

Credit Quality	A	A+100bps	A+200bps	A+300bps
Long-Term Mean α	0.035	0.056	0.077	0.099
Mean Reverting Speed b	0.14	0.18	0.25	0.36
Volatility σ	0.022	0.028	0.039	0.056

The calibrated parameters are shown in table 2. We assume that interest rates are deterministic and select the regression-based Monte-Carlo simulation (see Longstaff and Schwartz (2001)) to perform risky valuation.

We first assume that counterparties A , B , and reference entity C have independent default risks, i.e., $\rho_{AB} = \rho_{AC} = \rho_{BC} = \varphi_{AB} = \zeta_{ABC} = 0$, and examine the following cases: i) B is risk-free and A is risky; and ii) A is risk-free and B is risky. We simulate the hazard rates using the CIR model and then determine the appropriate discount factors according to Proposition 1. Finally we calculate the prices via the regression-based Monte-Carlo method. The results are shown in Table 3 and 4.

Table 3: Impact of the credit quality of the protection buyer on CDS premia

This table shows how the CDS premium increases as the credit quality of party *A* decreases. The 1st data column represents the counterparty-risk-free results. For the remaining columns, we assume that party *B* is risk-free and party *A* is risky. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads. The results in the row ‘Difference from Risk-Free’ = current CDS premium – counterparty-risk-free CDS premium.

Credit Quality	Party A	-	A	A+100bps	A+200bps	A+300bps
	Party B	-	-	-	-	-
CDS premium		0.027	0.02703	0.02708	0.02713	0.02717
Difference from Risk-Free		0	0.003%	0.008%	0.013%	0.017%

Table 4: Impact of the credit quality of the protection seller on CDS premia

This table shows the decrease in the CDS premium with the credit quality of party *B*. The 1st data column represents the counterparty-risk-free results. For the remaining columns, we assume that party *A* is risk-free and party *B* is risky. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads. The results in the row ‘Difference from Risk-Free’ = current CDS premium – counterparty-risk-free CDS premium.

Credit Quality	Party A	-	-	-	-	-
	Party B	-	A	A+100bps	A+200bps	A+300bps
CDS premium		0.027	0.02695	0.02687	0.0268	0.02672
Difference from Risk-Free		0.00%	-0.005%	-0.013%	-0.020%	-0.028%

From table 3 and 4, we find that a credit spread of about 100 basis points maps into a CDS premium of about 0.4 basis points for counterparty *A* and about -0.7 basis points for counterparty *B*. The credit impact on the CDS premia is approximately linear. As would be expected, i) the dealer’s credit quality has a larger impact on CDS premia than the investor’s credit quality; ii) the higher the investor’s credit risk, the higher

the premium that the dealer charges; iii) the higher the dealer's credit risk, the lower the premium that the dealer asks. Without considering default correlations and correlations

Figure 1: Impact of default correlations and comrelation on CDS premia

Each curve in this figure illustrates how CDS premium changes as default correlations and comrelation move from -1 to 1. For instance, the curve 'cor_BC' represents the sensitivity of the CDS premium to changes in the correlation ρ_{BC} when $\rho_{AB} = \rho_{AC} = \zeta_{ABC} = 0$.

Next, we are going to examine the impacts of the default correlations ρ_{AB} , ρ_{AC} , ρ_{BC} , and the comrelation ζ_{ABC} separately. Assume that party A has an 'A+100bps' credit quality and party B has an 'A' credit quality. The 5-year risky CDS premium is calculated as 0.02703.

Assume $\phi_{AB}=0.5$. The impact diagrams of the default correlations and comrelation are shown in Figure 1. From this graph, we can draw the following conclusions: First, the CDS premium and the default correlations/comrelation have a negative relation. Intuitively, a protection seller who is positively correlated with the reference entity (a wrong way risk) should charge a lower premium for selling credit protection. Next, the impacts of the default correlations and comrelation are approximately linear. Finally, the

sensitivity slopes of the CDS premium to the default correlations and comrelation are -0.06 to ρ_{AB} ; -0.09 to ρ_{AC} ; -53 to ρ_{BC} ; and -14 to ζ_{ABC} . Slope measures the rate of change in the premium as a result of a change in the default dependence. For instance, a slope of -53 implies that the CDS premium would have to decrease by 53 basis points when a default correlation/comrelation changes from 0 to 1.

4 Conclusion

This article presents a new valuation framework for pricing financial instruments subject to credit risk. In particular, we focus on modeling default relationships.

To capture the default relationships among more than two defaultable entities, we introduce a new statistic: *comrelation*, an analogue to correlation for multiple variables, to exploit any multivariate statistical relationship.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the comrelation, the correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

The model shows that a fully collateralized CDS is not equivalent to a risk-free one. Therefore, we conclude that collateralization designed to mitigate counterparty risk works well for financial instruments subject to bilateral credit risk, but fails for ones subject to three-way credit risk.

Appendix

Proof of Proposition 1. Let $t = T_0$. On the first payment date T_1 , let $V(T_1)$ denote the market value of the CDS excluding the current cash flow X_1 . There are a total of eight ($2^3 = 8$) possible states shown in Table A_1.

Table A_1. Payoffs of a trilaterally defaultable CDS

This table shows all possible payoffs at time T_1 . In the case of $V(T_1) + X_1 \geq 0$ where $V(T_1)$ is the market value excluding the current cash flow X_1 , there are a total of eight ($2^3 = 8$) possible states: i) A , B , and C survive with probability p_{000} . The instrument value equals the market value: $V(T_1) + X_1$. ii) A defaults, but B and C survive with probability p_{100} . The instrument value is a fraction of the market value: $\bar{\varphi}_B(T_1)(V(T_1) + X_1)$ where $\bar{\varphi}_B$ represents the non-default recovery rate of party B ¹. $\bar{\varphi}_B = 0$ represents the one-way settlement rule, while $\bar{\varphi}_B = 1$ represents the two-way settlement rule. iii) A and C survive, but B defaults with probability p_{010} . The instrument value is given by $\varphi_B(T_1)(V(T_1) + X_1)$ where φ_B represents the default recovery rate of defaulting party B . iv) A and B survive, but C defaults with probability p_{001} . The instrument value is the default payment: $R(T_0, T_1)$. v) A and B default, but C survives with probability p_{110} . The instrument value is given by $\varphi_{AB}(T_1)(V(T_1) + X_1)$ where φ_{AB} denotes the joint recovery rate when both parties A and B default simultaneously. vi) A and C default, but B survives with probability p_{101} . The instrument value is a fraction of the default payment: $\bar{\varphi}_B(T)R(T_0, T_1)$. vii) B and C default, but A survives with probability p_{011} . The instrument value is given by $\varphi_B(T)R(T_0, T_1)$. viii) A , B , and C default with probability p_{111} . The instrument value is given by $\varphi_{AB}(T)R(T_0, T_1)$. A similar logic applies to the case of $V(T_1) + X_1 < 0$.

Status	Probability	Payoff if $V(T_1) + X_1 \geq 0$	Payoff if $V(T_1) + X_1 < 0$
$Y_A = 0, Y_B = 0, Y_C = 0$	p_{000}	$V(T_1) + X_1$	$V(T_1) + X_1$

¹ There are two default settlement rules in the market. The *one-way payment rule* was specified by the early ISDA master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the instrument is positive for the defaulting party. The *two-way payment rule* is based on current ISDA documentation. The non-defaulting party will pay the full market value of the instrument to the defaulting party if the contract has positive value to the defaulting party.

$Y_A = 1, Y_B = 0, Y_C = 0$	p_{100}	$\bar{\varphi}_B(T_1)(V(T_1) + X_1)$	$\varphi_A(T_1)(V(T_1) + X_1)$
$Y_A = 0, Y_B = 1, Y_C = 0$	p_{010}	$\varphi_B(T_1)(V(T_1) + X_1)$	$\bar{\varphi}_A(T_1)(V(T_1) + X_1)$
$Y_A = 0, Y_B = 0, Y_C = 1$	p_{001}	$R(T_0, T_1)$	$R(T_0, T_1)$
$Y_A = 1, Y_B = 1, Y_C = 0$	p_{110}	$\varphi_{AB}(T_1)(V(T_1) + X_1)$	$\varphi_{AB}(T_1)(V(T_1) + X_1)$
$Y_A = 1, Y_B = 0, Y_C = 1$	p_{101}	$\bar{\varphi}_B(T)R(T_0, T_1)$	$\bar{\varphi}_B(T)R(T_0, T_1)$
$Y_A = 0, Y_B = 1, Y_C = 1$	p_{011}	$\varphi_B(T)R(T_0, T_1)$	$\varphi_B(T)R(T_0, T_1)$
$Y_A = 1, Y_B = 1, Y_C = 1$	p_{111}	$\varphi_{AB}(T)R(T_0, T_1)$	$\varphi_{AB}(T)R(T_0, T_1)$

The risky price is the discounted expectation of the payoffs and is given by

$$\begin{aligned}
V(t) = E\{ & \mathbb{1}_{(V(T_1)+X_1) \geq 0} \langle (p_{000}(T_0, T_1) + p_{100}(T_0, T_1)\bar{\varphi}_B(T_1) + p_{010}(T_0, T_1)\varphi_B(T_1) + p_{110}(T_0, T_1)\varphi_{AB}(T_1))(V(T_1) + X_1) | \mathcal{F}_t \rangle \\
& + \mathbb{1}_{(V(T_1)+X_1) < 0} \langle (p_{000}(T_0, T_1) + p_{100}(T_0, T_1)\varphi_A(T_1) + p_{010}(T_0, T_1)\bar{\varphi}_A(T_1) + p_{110}(T_0, T_1)\varphi_{AB}(T_1))(V(T_1) + X_1) | \mathcal{F}_t \rangle \\
& + (p_{001}(T_0, T_1) + p_{101}(T_0, T_1)\bar{\varphi}_B(T_1) + p_{011}(T_0, T_1)\varphi_B(T_1) + p_{111}(T_0, T_1)\varphi_{AB}(T_1))R(T_s, T_1) | \mathcal{F}_t \rangle D(t, T) \} \\
= E\{ & [O(T_0, T_1)(V(T_1) + X_1) + \Omega(T_0, T_1)R(T_s, T_1)] | \mathcal{F}_t \}
\end{aligned} \tag{A1a}$$

where

$$O(T_0, T_1) = \mathbb{1}_{(V(T_1)+X_1) \geq 0} \phi_B(T_0, T_1) + \mathbb{1}_{(V(T_1)+X_1) < 0} \phi_A(T_0, T_1) \tag{A1b}$$

$$\begin{aligned}
\phi_A(T_0, T_1) = & \{ p_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1) + q_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1)\varphi_A(T_1) \\
& + p_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\bar{\varphi}_A(T_1) + q_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_{AB}(T_1) \\
& + p_C(T_0, T_1)\sigma_{AB}(T_0, T_1)(1 - \varphi_A(T_1) - \bar{\varphi}_A(T_1) + \varphi_{AB}(T_1)) \\
& + \sigma_{AC}(T_0, T_1)[p_B(T_0, T_1)(1 - \varphi_A(T_1)) + q_B(T_0, T_1)(\bar{\varphi}_A(T_1) - \varphi_{AB}(T_1))] \\
& + \sigma_{BC}(T_0, T_1)[p_A(T_0, T_1)(1 - \bar{\varphi}_A(T_1)) + q_A(T_0, T_1)(\varphi_A(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T_1)(-1 + \bar{\varphi}_A(T_1) - \varphi_{AB}(T_1) + \varphi_A(T_1)) \} D(T_0, T_1)
\end{aligned} \tag{A1c}$$

$$\begin{aligned}
\phi_B(T_0, T_1) = & \{ p_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1) + q_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1)\bar{\varphi}_B(T_1) \\
& + p_A(T_1, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_B(T_1) + q_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_{AB}(T_1) \\
& + p_C(T_0, T_1)\sigma_{AB}(T_0, T_1)(1 - \varphi_B(T_1) - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1)) \\
& + \sigma_{AC}(T_0, T_1)[p_B(T_0, T_1)(1 - \bar{\varphi}_B(T_1)) + q_B(T_0, T_1)(\varphi_B(T_1) - \varphi_{AB}(T_1))] \\
& + \sigma_{BC}(T_0, T_1)[p_A(T_0, T_1)(1 - \varphi_B(T_1)) + q_A(T_0, T_1)(\bar{\varphi}_B(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T_1)(-1 + \bar{\varphi}_B(T_1) - \varphi_{AB}(T_1) + \varphi_B(T_1)) \} D(T_0, T_1)
\end{aligned} \tag{A1d}$$

$$\begin{aligned}
\Omega(T_0, T) = & \{p_A(T_0, T)p_B(T_0, T)q_C(T_0, T) + q_A(T_0, T)p_B(T_0, T)q_C(T_0, T)\bar{\varphi}_B(T_1) \\
& + p_A(T_0, T)q_B(T_0, T)q_C(T_0, T)\varphi_B(T_1) + q_A(T_0, T)q_B(T_0, T)q_C(T_0, T)\varphi_{AB}(T_1) \\
& + q_C(T_0, T)\sigma_{AB}(T_0, T)(1 - \varphi_B(T_1) - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1)) \\
& - \sigma_{AC}(T_0, T)[p_B(T_0, T)(1 - \bar{\varphi}_B(T_1)) + q_B(T_0, T)(\varphi_B(T_1) - \varphi_{AB}(T_1))] \\
& - \sigma_{BC}(T_0, T)[p_A(T_0, T)(1 - \varphi_B(T_1)) + q_A(T_0, T)(\bar{\varphi}_B(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T)(1 - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1) - \varphi_B(T_1))\}D(T_0, T)
\end{aligned} \tag{A1e}$$

Similarly, we have

$$V(T_1) = E\left\{[O(T_1, T_2)(X_2 + V(T_2)) + \Omega(T_1, T_2)R(T_1, T_2)] \middle| \mathcal{F}_{T_1}\right\} \tag{A2}$$

Note that $O(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. By definition, an $\mathcal{F}_{T_1}$ -measurable random variable is a random variable whose value is known at time T_1 . According to *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned}
V(t) = & E\left\{[O(T_0, T_1)(X_1 + V(T_1)) + \Omega(T_0, T_1)R(T_0, T_1)] \middle| \mathcal{F}_t\right\} = E[O(T_0, T_1)X_1 | \mathcal{F}_t] \\
& + E[\Omega(T_0, T_1)R(T_0, T_1) | \mathcal{F}_t] + E\left\{O(T_0, T_1)E\left\{[O(T_1, T_2)(X_2 + V(T_2)) + \Omega(T_1, T_2)R(T_1, T_2)] \middle| \mathcal{F}_{T_1}\right\} \middle| \mathcal{F}_t\right\} \\
= & \sum_{i=1}^2 E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right)X_i\right] + \sum_{i=1}^2 E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right)\Omega(T_{i-1}, T_i)R(T_{i-1}, T_i)\right] \\
& + E\left[\left(\prod_{j=0}^1 O(T_j, T_{j+1})\right)V(T_2) \middle| \mathcal{F}_t\right]
\end{aligned} \tag{A3}$$

By recursively deriving from T_2 forward over T_m , where $V(T_m) = X_m$, we have

$$V(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right)X_i \middle| \mathcal{F}_t\right] + \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right)\Omega(T_{i-1}, T_i)R(T_{i-1}, T_i) \middle| \mathcal{F}_t\right] \tag{A4}$$

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