TWO NEW EQUIVALENT STATEMENTS TO RIEMANN HYPOTHESIS

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Abstract

The distribution of such prime numbers among all natural numbers does not follow any regular pattern; however, the German mathematician G. F. B. Riemann (1826-1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function called the Riemann zeta function $\zeta(s)$. The nontrivial zeroes of zeta function have $1/2$ as their real part. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers. The celebrated Riemann hypothesis remains unsolved since it was introduced in 1859. Miscellaneous approaches have been considered without any exact and complete proof. Furthermore, some equivalent statements have been established.

In this work, we consider the famous Robin inequality and propound a connection to the theory of univalent functions by the means of Koebe function.

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1. Introduction and Statement of the Problem

Identified as one of the 7 Millennium Problems, see [1, 2, 4-9, 19, 33-35], the Riemann zeta hypothesis has successfully evaded mathematicians for over 100 years. Simply stated, Riemann conjectured that all of the nontrivial zeroes of the zeta function have real part equal to $1/2$. In his attempt to provide an accurate gap between prime numbers, Riemann has considered the Euler's combination of p -series and prime numbers:

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p\text{-prime}} \frac{1}{1 - p^{-s}}, \quad s > 1.$$

The left hand side of the equation above is simply called *zeta function*, which is convergent for $s > 1$. Riemann implemented complex analysis in order to expand the convergence by the means of complex field, i.e., he considered s as a complex number and provided an entire expansion by the means of Gamma function as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \int_0^{\infty} \frac{1}{1 - e^{-t}} t^{s-1} dt.$$

The above function is entire except a simple hole at $s = 1$. Furthermore, trivial zeros of the zeta function could be readily provided by the functional equations (see [1]). However, the non-trivial zeroes that Riemann has conjectured to lie on the critical strip and with real part equal to $1/2$ is what we call Riemann hypothesis (RH), simply saying:

If s is a zero of zeta function, then $Re(s) = 1/2$.

2. Robin's Inequality

Since it was introduced, RH inspired a lot of mathematicians to decode its ambiguity by the means of miscellaneous approach either through complex analysis, analytical number theory or even by elementary approach. Many equivalent statements to RH have been introduced which provide

alternative strategies and methodologies. References ([3] and [36]) provided a beautiful equivalent inequality to RH. Robin's inequality states that:

RH is equivalent to:

$$\frac{\sigma(n)}{n} < e^{\gamma} \ln \ln(n) \text{ for } n > 5041 \text{ and } \gamma = 0.57... \text{ (Euler's constant).}$$

Since then, some partial results have been proved, see, for example, [6, 7]. Mathematically, despite the huge number of zeta zeros that have $1/2$ as real part, none can assume that RH is proved. Unfortunately, all attempts to prove (disprove) RH remain incomplete, that is a different approach which may be considered in the future. Recently, in 2017, Durmagambetov [7] introduced a modified zeta function which allows the estimation of Riemann's zeta function and solving Riemann problem. Some authors used analytic method to deny the Riemann hypothesis [9]. Wünsche approached to Riemann hypothesis by means of second mean value theorem of calculus. However, our motivation towards Riemann problem is to involve fractional calculus through univalent functions, some advancement made in this connection can be studied in [5] and [6].

3. New Approach

The aim of this piece of work is to implement the theory of univalent functions in RH in accordance its wide connection with fractional calculus applications. The difficulty certainly arises from the chaotic and unpredictable pattern of the sum of divisors function. With a numerical approach and by manipulating the tolerance error, one may possibly wish to partially prove RH. This is due to the wide applications of fractional calculus that implies a crucial impact on the univalent functions along with its miscellaneous applications in mathematics, physics and economy, see [10-17, 22].

4. Main Propositions

In order to derive the main results, we need to apply the following lemmas. Further definitions and relevant results on special and univalent functions can be found in [18, 20, 21, 23-32].

Lemma 1. *Given that $f(z)$ of the form: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent, then:*

$$|a_n| \leq n, \text{ for every } n = 2, 3, 4, \dots \text{ and } |z| < 1.$$

Lemma 2. *Given that $f(z)$ of the form: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is convex, then:*

$$|a_n| \leq 1, \text{ for every } n = 2, 3, 4, \dots \text{ and } |z| < 1.$$

Next, two new criteria or statements that are equivalent to RH are readily proven:

Theorem 1.

$$RH \text{ is equivalent to: } f(z) = z + \sum_{n=5041}^{\infty} \frac{\sigma(n)}{e^{\gamma} \ln \ln(n)} z^n \text{ is univalent.}$$

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, let $a_n = \frac{\sigma(n)}{e^{\gamma} \ln \ln(n)}$ and $a_i = 0$, $i = 2, 3, 4, \dots, 5040$. If $f(z)$ is univalent, then by Lemma 1, $a_n = \frac{\sigma(n)}{e^{\gamma} \ln \ln(n)} < n$, $n \geq 5041$, that is Robin inequality which is equivalent to Riemann hypothesis.

Theorem 2.

$$RH \text{ is equivalent to: } g(z) = z + \sum_{n=5041}^{\infty} \frac{\sigma(n)}{e^{\gamma} n \ln \ln(n)} z^n \text{ is convex.}$$

Proof. For $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$, let $a_n = \frac{\sigma(n)}{e^{\gamma} n \ln \ln(n)}$ and $a_i = 0$, $i = 2, 3, 4, \dots, 5040$. If $f(z)$ is convex, then by Lemma 2, $a_n = \frac{\sigma(n)}{e^{\gamma} n \ln \ln(n)} < 1$, $n \geq 5041$, that is Robin inequality which is equivalent to Riemann hypothesis.

The assertions above follow from the coefficient bound of univalent (convex) functions.

5. Plausible Approach (Open Problems)

In order to achieve the objective, we consider the following steps:

- (a) Given $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$ is univalent, if $a_i = 0$ for some $i = 1, 2, 3, \dots$ is $f(z)$ still univalent?
- (b) Given $g(z) = z + \sum_{n=1}^{\infty} b_n z^n$ is convex, if $b_i = 0$ for some $i = 1, 2, 3, \dots$ is $g(z)$ still convex?
- (c) Given $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$. How to derive the general form of $f(z)$ without using Taylor series?
- (d) Curve fitting $(n : \sigma(n))$ by the means of convenient methods like Newton's, Lagrange interpolating and by using computer process.
- (e) Assuming that the function generated in (d) represents $f(z)$ and $g(z)$ by considering and manipulating the error.
- (f) Use graphics to show the univalence (convexity).
- (g) Control the error such that the generated functions satisfy Robin's condition.

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